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On Auto–Engel Polygroups

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Abstract

This paper introduces the concept of auto–Engel polygroups via the heart of hypergroups and investigates the relation between of auto–Engel polygroups and auto–nilpotent polygroups. Indeed, we show that the concept of heart of hypergroups plays an important role on construction of auto–Engel polygroups. This study considers the notation of characteristic set in hypergroups with respect to automorphism of hypergroups and shows that the heart of hypergroups is a characteristic set in hypergroups.

Keywords:Auto–Engel polygroup, characteristic(-closure) set, general fundamental relation.

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1. Introduction

In group theory, autocommutators and automorphisms play an important role in construct of especial groups such as autocommutator groups, *auto*-Engel groups and n-auto-Engel groups, where $n \in \mathbb{N}$. Moghaddam et al. introduced the concept of autocommutator group, auto-Kappe group, 2-*auto*-Engel group, auto-Bell group, via autocommutator and investigated their properties [12, 13]. The hyperstructure theory as a extension of classical structures, was firstly introduced, by F. Marty in 1934 [11]. In hyperalgebraic system, the hyperproduct of elements is a set and so any algebraic system is a hyperalgebraic system. Marty extended

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the concept of groups to hypergroups and other researchers presented the hyperalgebraic concepts such as hyperring, hypermodule, hyperfield, hypergraph, polygroup, multiring, etc. Hyperstructures are applied in several branches of sciences such as artificial intelligence, chemistry and (hyper)complex network [4, 5]. Recently, Hamidi et al. introduced the concept of very thin polygroups, autonilpotent polygroups and showed that under some conditions very thin polygroups are autonilpotent polygroups and investigate the connection between autonilpotent polygroups and nilpotent polygroups [2, 3]. More related content about groups and hypergroups are available in [1, 8, 9, 10].

We introduce a novel fundamental relation on any given hypergroup in such a way that all fundamental relations are a special case of this relation, so obtain famous fundamental relations from this novel fundamental relation under some conditions. The motivation to introduce this relation is obtained from the algebraic connection between of hypergroups and groups. This study applies the concept of characteristic set and investigates some of its properties in polygroups. Indeed, we apply the novel fundamental relation and automorphisms of hypergroups to construct a characteristic set. A main result of this paper is introduce to some classes of auto-Engel polygroups via autocommutators and automorphisms of hypergroups, thus defines the notation of autoEngel polygroups with respect to the hearts of hypergroups and consequently some class of auto-Engel groups. The characteristic set has an essential role in construction of auto-Engel polygroups, especially the heart of every hypergroup as a helpful characteristic set is a base for definition of auto-Engel polygroups. In final, we obtained a connection between class of polygroups and auto-Engel groups.

2. Preliminaries

In what follow, we recall some results from [14], that need in our work.

Assume that $H \neq \emptyset$ be an arbitrary set and $P^*(H) = \{G \mid \emptyset \neq G \subseteq H\}$. Each map $\varrho : H^2 \longrightarrow P^*(H)$ is said to be a *hyperoperation,* hyperstructure (H, ϱ) is called a *hypergroupoid* and for every $\emptyset \neq A, B \subseteq H, \varrho(A, B) = \bigcup_{a \in A, b \in B} \varrho(a, b)$. A *hypergroupoid* (H, ϱ) together with an associative binary hyperoperation is said a *semihypergroup* and a semihypergroup (H, ϱ) is called a *hypergroup* if for any $x \in H, \varrho(x, H) = \varrho(H, x) = H$(*reproduction axiom*). A semihypergroup (H, ϱ) is said to be a *polygroup*, if (i) $\exists\ e \in H, \forall\ x \in H,$ in a way $\varrho(e, x) = \varrho(x, e) = \{x\}$, (ii) $x \in \varrho(y, z)$ concludes that $y \in \varrho(x, \vartheta(z))$ and $z \in \varrho(\vartheta(y), x)$, where ϑ is an unitary operation on H(it follows that $\forall\ x \in H,\ \exists!\ \vartheta(x) \in H$ i.e $e \in (\varrho(x, \vartheta(x)) \cap (\varrho(\vartheta(x), x)), \vartheta(e) = e, \vartheta(\vartheta(x)) = x$) and is denoted by $(H, \varrho, e, \vartheta)$. A set $\emptyset \neq K \subseteq H$ is said to be a subpolygroup of H, if $\forall\ x, y \in K, \varrho(x, \vartheta(y)) \subseteq K$ and it is denoted by $K \leq H$. Suppose that (H, ϱ) is a hypergroup. For any given an equivalence relation ω on H, a hyperoperation σ on $\frac{H}{\omega}$ is defined by $\sigma(\omega(a), \omega(b)) = \{\omega(c) \mid c \in \varrho(\omega(a), \omega(b))\}$. It is shown that $(\frac{H}{\omega}, \sigma)$ is a hypergroup if and only if ω is a regular equivalence relation and $(\frac{H}{\omega}, \sigma)$ is a group if and if only

ω is a strongly regular equivalence relation [6]. One of famous algebraic relation on any given hypergroup is β which is defined by $a\beta b \Longleftrightarrow \exists\ u \in \mathcal{U}(H)$ s.t $\{a,b\} \subseteq u$, where $\mathcal{U}(H)$ is denoted by the set of all finite product of elements of H. The smallest transitive relation in a way contains β is denoted by β^* and it means the *transitive closure* of β and $(\frac{H}{\beta^*}, \sigma)$ is said the *fundamental group* of (H, ϱ). A map $f : H_1 \to H_2$ is called a homomorphism of hypergroups if $\forall\ x, y \in H_1$, we have $f(\varrho_1(x,y)) = \varrho_2(f(x), f(y))$ and it is said to be an isomorphism if it is a one to one and onto homomorphism. In similar to algebraic system, $Aut(H) = \{f : H \to H \mid f$ is an isomorphism on hypergroup $H\}$ is defined. Assume that $\varphi : H \longrightarrow H/\beta^*$ by $\varphi(x) = \beta^*(x)$ is the *canonical* homomorphism, then $w_H = \{x \in H \mid \varphi(x) = 1\}$ means *heart* of H. For any semihypergroup (H, ϱ) and $\emptyset \neq A \subseteq H$, A is said to be a *complete part* of H if for all $n \in \mathbb{N}$ and $a_1, \ldots, a_n$ of H, $A \cap \varrho(a_1, \ldots, a_n) \neq \emptyset$ implies that $\varrho(a_1, \ldots, a_n) \subseteq A$. For each $\emptyset \neq X \subseteq H$, a subpolygroup generated by X is the intersection of all subpolygroups of H which contain X and is denoted by $< X >$. In every hypergroup H, a *commutator* of $x, y \in H$ is shown by $[x, y] = \{h \in H \mid \varrho(x, y) \cap \varrho(h, y, x) \neq \emptyset\}$ and $H = L_0(H) \supseteq L_1(H) \supseteq \cdots$ is called a *lower series* of H, where for any $n \in \mathbb{N}^*$, $L_{n+1}(H) = \{h \in [x, y] \mid x \in L_n(H), y \in H\}$. Also $H = \Gamma_0(H) \supseteq \Gamma_1(H) \supseteq \cdots$ is called a *derived series* of H, where for each $n \in \mathbb{N}^*$, $\Gamma_{n+1}(H) = \{h \in [x, y] \mid x, y \in \Gamma_n(H)\}$. A polygroup $(H, \varrho, e, \vartheta)$ means a *nilpotent polygroup*, if for some given integer $n \in \mathbb{N}, \varrho(l_n(H), w_H) = w_H$, where $l_{n+1}(H) = \langle\{h \in [x, y] \mid x \in l_n(H), y \in H\}\rangle$ and $l_0(H) = H$ (if there exists a smallest integer c in a way that $\varrho(l_c(H), w_H) = w_H$, and c is called the *nilpotency class* for H). Also for each given $n \in \mathbb{N}$, get $H' = H^{(1)} = \langle\Gamma_1(H)\rangle$ and $H^{(n+1)} = (H^{(n)})'$.

3. Characteristic Set in Hypergroups

In this section, we introduce a fundamental relation on hypergroups in a way it is a generalization of famous fundamental relations such as β^* and γ^* and preserves their properties . Also the concept of characteristic set in hypergroups is defined and is obtained some characteristic sets with respect to fundamental relations on hypergroups and automorphism on hypergroups. In final, we show that the heart of every hypergroup is a characteristic set.

Definition 3.1. Assume H is a hypergroup and $K \subseteq H$. Define $R_{1,K} = \{(x, x) \mid x \in H)\}$ and $\forall\ 2 \leq n \in \mathbb{N}$:

$$(x, y) \in R_{n,K} \Leftrightarrow \exists\ (z_1, \ldots, z_n) \in H^n, u = \varrho(z_1, \ldots, z_n),\ \exists\ \sigma \in S_n, \text{s.t } x \in u, y \in u_\sigma$$

and $\forall\ 1 \leq i \leq n, z_i \in K$ implies that $\sigma(i) = i$, where $u_\sigma = \varrho(z_{\sigma(i)}, z_{\sigma(i)}, \ldots, z_{\sigma(i)})$.

Obviously, $R_K = \bigcup_{n \geq 1} R_{n,K}$ is a reflexive and symmetric relation. Let R_K^* be the *transitive closure* of R_K (the smallest transitive relation in a way that contains R_K). Then we have the following results.

Example 3.2. Consider the hypergroup (H, ϱ) as follows:

ϱ	-2	-4	-6
-2	$\{-4, -6\}$	$\{-2\}$	$\{-2\}$
-4	$\{-2\}$	$\{-4\}$	$\{-6\}$
-6	$\{-2\}$	$\{-4, -6\}$	$\{-4, -6\}$

.

If $K = \{-2, -4\}$, then $R_K = R_{1,K} \cup \{(-4, -6), (-6, -4)\} = R^*_K = \beta^*$.

Proposition 3.3. *Assume H is a hypergroup and $K \subseteq H$.*

(*i*) *If $K = H$, then $R_K = \beta^*$.*

(*ii*) *If $K = \emptyset$, then $R_K = \gamma^*$.*

Proof. Clearly $\beta^* \subseteq R_K$. Let $(x, y) \in R_K$, then $\exists\ (z_1, \ldots, z_n) \in H^n, u = \varrho(z_1, z_2, \ldots, z_n)$ and $\sigma \in S_n$ in a way that $x \in u, y \in u_\sigma$ and $\forall\ 1 \leq i \leq n, z_i \in K$ implies that $\sigma(i) = i$. Now, if $K = H$, we consider $\sigma = id$(identity map) and so $u = u_\sigma$. It follows that $(x, y) \in \beta^*$.

(ii) It is similar to (i). □

The Example 3.2, shows that the inverse of Proposition 3.3, is not necessarily true.

Theorem 3.4. *Assume H is a hypergroup and $K \subseteq H$. Then R^*_K is a strongly regular relation on H.*

Proof. Assume $(x, y) \in R_K$ and $z \in H$. Then $\exists\ (z_1, \ldots, z_n) \in H^n, u = \varrho(z_1, \ldots, z_n)$ and $\sigma \in S_n$ so that $x \in u, y \in u_\sigma$ and $\forall\ 1 \leq i \leq n, z_i \in K$ implies that $\sigma(i) = i$. So we have $\varrho(x, z) \subseteq \varrho(u, z)$, $\varrho(y, z) \subseteq \varrho(u_\sigma, z)$ and $z_i \in K$ implies that $\sigma(i) = i$. Consider $z_{n+1} = z$, $\alpha(i) = \sigma(i)$, whence $i \in \{1, \ldots, n\}$ and $\alpha(n+1) = n+1$. Thus $\varrho(x, z) \subseteq \varrho(z_1, z_2, \ldots, z_n)$ and $\varrho(y, z) \subseteq v_\alpha$ in a way $z_i \in K$ implies $\underline{\alpha(i)} = i$. It follows that $\varrho((x, z))\ \overline{\overline{R^*_K}}\ \varrho((y, z))$. In a similar way, we have $(\varrho(z, x))\ \overline{\overline{R^*_K}}\ (\varrho(z, y))$. Hence R^*_K is a strongly regular relation on H. □

Lemma 3.5. *Assume (H, e) is a polygroup and $\alpha \in Aut(H)$. Then $\alpha(e) = e$.*

Proof. Let $x \in H$ and $\alpha(x) = e$, hence we have $\{e\} = \{\alpha(x)\} = \alpha(\varrho(e, x)) = \varrho(\alpha(e), \alpha(x)) = \varrho(\alpha(e), e) = \{\alpha(e)\}$. □

Definition 3.6. Assume H is a hypergroup and $K \subseteq H$. Then K is called a characteristic subset of H, if $\forall\ \alpha \in Aut(H)$, we have $\alpha(K) \subseteq K$ and will denote it by $K \prec H$. Facility, $\emptyset \prec H$ and $H \prec H$.

Example 3.7. Let $H = \{0, -1, -2, -3, -4, -5, -6\}$. Then $(H, \varrho, 0, \vartheta)$ is a polygroup in Table 1, where $T = \{0, -4, -5, -6\}$. Simple computations show that

$$Aut(H) = \{xy \mid x \in S_3, y \in S_X, \text{ where } X = \{-4, -5, -6\}\},$$
$$S_3 \trianglelefteq Aut(H), S_X \trianglelefteq Aut(H), S_3 \cap S_X = \{e\}, Aut(H, \varrho) \cong S_3 \times S_3,$$

and $\forall\ \alpha \in Aut(H)$, we have $\alpha(T) = T$ and so $T \prec H$.

Table 1: Polygroup H.

ϱ	0	-1	-2	-3	-4	-5	-6
0	0	-1	-2	-3	-4	-5	-6
-1	-1	T	-3	-2	-1	-1	-1
-2	-2	-3	T	-1	-2	-2	-2
-3	-3	-2	-1	T	-3	-3	-3
-4	-4	-1	-2	-3	T	$T\setminus\{0\}$	$T\setminus\{0\}$
-5	-5	-1	-2	-3	$T\setminus\{0\}$	T	$T\setminus\{0\}$
-6	-6	-1	-2	-3	$T\setminus\{0\}$	$T\setminus\{0\}$	T

Example 3.8. [7] Let $H = \{1, 2, 4, 5, 7, 9, -1\}$. Then $(H, \varrho, 1, \vartheta)$ is a non–commutative polygroup as follows.

ϱ	1	2	4	5	7	9	-1
1	$\{1\}$	$\{2\}$	$\{4\}$	$\{5\}$	$\{7\}$	$\{9\}$	$\{-1\}$
2	$\{2\}$	$\{4\}$	$\{1,-1\}$	$\{9\}$	$\{5\}$	$\{7\}$	$\{2\}$
4	$\{4\}$	$\{1,-1\}$	$\{2\}$	$\{7\}$	$\{9\}$	$\{5\}$	$\{4\}$
5	$\{5\}$	$\{7\}$	$\{9\}$	$\{1,-1\}$	$\{2\}$	$\{4\}$	$\{5\}$
7	$\{7\}$	$\{9\}$	$\{5\}$	$\{4\}$	$\{1,-1\}$	$\{2\}$	$\{7\}$
9	$\{9\}$	$\{5\}$	$\{7\}$	$\{2\}$	$\{4\}$	$\{1,-1\}$	$\{9\}$
-1	$\{-1\}$	$\{2\}$	$\{4\}$	$\{5\}$	$\{7\}$	$\{9\}$	$\{1\}$

Thus $K = \{1, -1\} \prec H$.

Theorem 3.9. *Assume H is a hypergroup and $n \in \mathbb{N}$. Then*

(i) $L_n(H)$ *is a characteristic subset of* H*,*

(ii) $\Gamma_n(H)$ *is a characteristic subset of* H*.*

Proof. (i) We prove by induction on n. Let $h \in L_{n+1}(H)$, then $\exists\ x \in L_n(H)$ and $y \in H$ so that $h \in [x, y]$. It concludes that $z \in (\varrho(x, y)) \cap \varrho((h, y, x)$ and so $\alpha(z) \in (\varrho(\alpha(x), \alpha(y))) \cap (\varrho(\alpha(h), \alpha(y), \alpha(x)))$. Hence for every $\alpha \in Aut(H)$, we have $\alpha(h) \in [\alpha(x), \alpha(y)]$ and using hypotheses of induction, $\alpha(h) \in L_{n+1}(H)$ is obtained.

(ii) In a similar way of (i), one can show that $\Gamma_n(H)$ is a characteristic subset of H. □

Corollary 3.10. *Assume H is a hypergroup, $K \prec H$ and $n \in \mathbb{N}$. Then*

(i) $H \setminus K \prec H$,

(ii) $H \setminus L_n(H) \prec H$,

(iii) $H \setminus \Gamma_n(H) \prec H$,

(*iv*) *if* (H, e) *is a polygroup, then* $H \setminus \{e\} \prec H$.

Proof. We prove just (i) and other options are obtained from (i) and Theorem 3.9.

(i) Let $x \in H \setminus K$ and $\alpha \in Aut(H)$. If $\alpha(x) \in K$, then $K \prec H$ implies that $x = \alpha^{-1}(\alpha(x)) \in K$, which is a contradiction. Hence $\alpha(x) \in H \setminus K$ and so $\alpha(H \setminus K) \subseteq H \setminus K$. □

Theorem 3.11. *Assume* H *is a polygroup,* $K \prec H$ *and* $n \in \mathbb{N}$. *Then*

(*i*) $\langle K \rangle \prec H$,

(*ii*) $L_n(K), \Gamma_n(K) \prec H$,

(*iii*) $H^{(n)}, l_n(H) \prec H$.

Proof. By Definition and induction on n, the proof is clear. □

Theorem 3.12. *Assume* H *is a polygroup,* $K \leq H$, $K \prec H$ *and* $\alpha \in Aut(K)$. *Then* $\exists\ \alpha' \in Aut(H)$ *that* $\alpha'|_K = \alpha$.

Proof. Let $\alpha \in Aut(K)$. Then $\forall\ x \in H$, define

$$\alpha'(x) = \begin{cases} \alpha(x) & x \in K, \\ x & x \notin K. \end{cases}$$

Since $K \prec H$, we have α' is a well-defined map and clearly $\alpha' \in Aut(H)$. □

Example 3.13. Consider dihedral group D_8 and $K = \{1, r^2, s, r^2 s\}$. Thus $K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $Aut(K) \cong S_3$ and $K \not\prec D_8$. Now, $\alpha = (s, r^2 s) \in Aut(K)$ while $\alpha' \notin Aut(D_8)$.

Theorem 3.14. *Assume* H_1, H_2 *are hypergroups,* $K_1 \subseteq H_1$ *and* $K_2 \subseteq H_2$. *If* $K_1 \times K_2 \prec H_1 \times H_2$, *then* $K_1 \prec H_1$ *and* $K_2 \prec H_2$.

Proof. Let $\alpha_1 \in Aut(H_1)$ and $\alpha_2 \in Aut(H_2)$. Define $\alpha = (\alpha_1, \alpha_2)$ by $\alpha(x, y) = (\alpha_1(x), \alpha_2(y))$, then clearly $\alpha \in Aut(H_1 \times H_2)$. Since $\alpha(K_1 \times K_2) \subseteq K_1 \times K_2$, we get that $\alpha_1(K_1) \subseteq K_1$ and $\alpha_2(K_2) \subseteq K_2$. Thus $K_1 \prec H_1$ and $K_2 \prec H_2$. □

Example 3.15. Consider the groups $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Clearly, $\{\bar{0}\} \prec \mathbb{Z}_2$ and $\{\bar{1}\} \prec \mathbb{Z}_2$. Since for $\alpha = ((\bar{1}, \bar{0}), (\bar{1}, \bar{1})) \in Aut(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $\alpha(\{\bar{1}\} \times \{\bar{0}\}) = \{(\bar{1}, \bar{1})\} \not\subseteq \{\bar{1}\} \times \{\bar{0}\}$, we get $\{\bar{1}\} \times \{\bar{0}\} \not\prec \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence the converse of Theorem 3.14, is not necessarily true.

Lemma 3.16. *Assume* $f : H_1 \longrightarrow H_2$ *is an isomorphism of hypergroups,* $K \prec H_1$ *and* $K' \prec H_2$. *Then*

(*i*) $f(K) \prec H_2$,

(*ii*) $f^{-1}(K') \prec H_1$.

Proof. (i) Let $\alpha \in Aut(H_2)$. Then $(f^{-1}\alpha f)(K) \subseteq K$ and so $f(K) \prec H_2$.

(ii) Since f^{-1} is an isomorphism, so similar to the item (i), the proof is obtained. □

Assume H is a hypergroup, $K \subseteq H$ and $\alpha \in Aut(H)$. We define $\overline{\alpha}$ by $\overline{\alpha(x)} = \overline{\alpha}(\overline{x})$, where $x \in H$ and $\overline{x} = R^*_K(x)$.

Theorem 3.17. *Assume H is a hypergroup and $\alpha \in Aut(H)$. If $K \prec H$, then $\overline{\alpha} \in Aut(\frac{H}{R^*_K})$.*

Proof. Let $(x, y) \in R_K$ and $x \neq y$. Then $\exists\ (z_1, \ldots, z_n) \in H^n$, $u = \varrho(z_1, z_2, \ldots, z_n)$ and $\sigma \in S_n$ in a way $x \in u, y \in u_\sigma$ and $\forall\ 1 \leq i \leq n, z_i \in K$ implies that $\sigma(i) = i$. Since $\alpha \in Aut(H)$, we get $\exists\ u = \varrho(z_1, z_2, \ldots, z_n)$ and $\sigma \in S_n$ that $\alpha(x) \in \alpha(u), \alpha(y) \in \alpha(u_\sigma)$ and $\forall\ 1 \leq i \leq n, \alpha(z_i) \in K$ implies that $\sigma(i) = i$. Thus $(\alpha(x), \alpha(y)) \in R_K$ and $(x, y) \in R^*_K$ implies that $(\alpha(x), \alpha(y)) \in R^*_K$. □

Theorem 3.18. *Assume H is a hypergroup, $K \prec H$, $G = H/R^*_K$ and $\alpha, \theta \in Aut(H)$. Then*

(i) $\overline{\alpha \circ \theta} = \overline{\alpha} \circ \overline{\theta}$,

(ii) $\overline{\alpha}^{-1} = \overline{\alpha^{-1}}$,

(iii) $\overline{Aut(H)} = \{\overline{\alpha} \mid \alpha \in Aut(H)\} \leq Aut(G)$.

Proof. (i) Let $x \in H$. Define $\overline{\alpha}(R^*_K(x)) = R^*_K(\alpha(x))$, thus

$$\overline{\alpha \circ \theta}(\overline{x}) = \overline{\alpha \circ \theta(x)} = \overline{\alpha(\theta(x))} = \overline{\alpha}(\overline{\theta(x)}) = \overline{\alpha} \circ \overline{\theta}(\overline{x}),$$

where $\overline{x} = R^*_K(x)$. So the proof is obtained.

$(ii), (iii)$ are similar to the item (i). □

Theorem 3.19. *Assume H is a hypergroup, $K \prec H$ and $\pi : H \longrightarrow H/R^*_K$ is the canonical homomorphism. Then $w_K = \{x \in H \mid R^*_K(x) = 1_{H/R^*_K}\} \prec H$.*

Proof. Let $x \in H$. If $x \in w_K$, then $R^*_K(x) = 1_{H/R^*_K}$. Now

$$1_{H/R^*_K} = \overline{\alpha}(1_{H/R^*_K}) = \overline{\alpha}(R^*_K(x)) = R^*_K(\alpha(x)),$$

implies that $\alpha(x) \in w_K$ and so $w_K \prec H$. □

Corollary 3.20. *Suppose H is a hypergroup. Then $w_H \prec H$.*

3.1 Characteristic-Closure Set in Hypergroups

In this subsection, we apply the concept of characteristic subsets of hypergroups and introduce the concept of characteristic-closure set in hypergroups. In addition, the relationship between of characteristic-closure set in hypergroups and heart of hypergroups investigated.

Assume H is a hypergroup, $K_1, K_2 \prec H$. Then one can see that $K_1 \cap K_2 \prec H$, so if $\{K_i\}_{i\in I}$ is a set of characteristic subsets of H, then $\bigcap_{i\in I} K_i$ is a characteristic subset of H.

Definition 3.21. Suppose $A \subseteq H$. An intersection of all characteristic subsets of H which contains A is called characteristic–closure of A in H and it will be denoted by $C_c(A)$.

Consider $T_1(A) = A$ and for every $n \in \mathbb{N}$,

$$T_{n+1}(A) = \{x \in H \mid \exists\ \alpha \in Aut(H)\ s.t\ \alpha(x) \in T_n(A)\} \text{ and } T(A) = \bigcup_{n\geq 1} T_n(A).$$

Example 3.22. Let $H = \{3, 5, 7\}$. Consider the polygroup (H, ϱ) as follows:

ϱ	7	3	5
7	$\{7\}$	$\{3\}$	$\{5\}$
3	$\{3\}$	$\{7,5\}$	$\{3,5\}$
5	$\{5\}$	$\{3,5\}$	$\{7,3\}$

Routine computations show that $Aut(H) = \{id, \alpha = (3, 5)\}$. If $K_1 = \{3\}$ and $K_2 = \{7, 5\}$ then $\forall\ n \in \mathbb{N}$, we have $T_n(K_1) = \{3, 5\}$ and $T_n(K_2) = \{7, 3, 5\}$.

Lemma 3.23. *Assume H is a hypergroup, $n \in \mathbb{N}$ and $A, B \subseteq H$. Then*

(i) *if $A \subseteq B$, then $T_n(A) \subseteq T_n(B)$,*

(ii) *$T_n(A \cap B) \subseteq T_n(A) \cap T_n(B)$,*

(iii) *$T_n(A \cup B) \subseteq T_n(A) \cup T_n(B)$,*

(iv) *if A is a characteristic subsets of H, then $T_n(A) = A$.*

Proof. We just prove (iii), by induction on n and other items are similar to.

(iii) For $n = 1$, we have $T_1(A \cup B) = A \cup B = T_1(A) \cup T_1(B)$. Let $x \in T_{n+1}(A \cup B)$. Then $\exists\ \alpha \in Aut(H)$ in a way $\alpha(x) \in T_n(A) \cup T_n(B)$ and so $x \in T_{n+1}(A) \cup T_{n+1}(B)$. □

Example 3.24. Consider the hypergroup (H, ϱ) in Example 3.7. If $A = \{0, -4\}$ and $B = \{0, -5\}$, then $T(A) = T(B) = T, T(A \cap B) = \{0\} \neq T(A) \cap T(B)$.

Corollary 3.25. *Suppose H is a hypergroup and $n \in \mathbb{N}$. Then $T_n(\Gamma_n(H)) = \Gamma_n(H)$ and $T_n(L_n(H)) = L_n(H)$.*

Theorem 3.26. *Assume H is a hypergroup and $A \subseteq H$. Then*

(*i*) $C_c(A) = T(A)$,

(*ii*) $C_c(A) = \bigcup_{a \in A} C_c(\{a\})$.

Proof. (*i*) Let $\alpha \in Aut(H)$ and $x \in T(A)$. Then, $\exists\ n \in \mathbb{N}$ that $x \in T_n(A)$. So we have $(\alpha^{-1} \circ \alpha)(x) = x \in T_n(A)$ and by definition, $\alpha(x) \in T_{n+1}(A)$. Thus $T(A) \prec H$.

If $A \subseteq B$ and $B \prec H$. Then $T_1(A) = A \subseteq B$. Suppose that $T_n(A) \subseteq B$. For every $x \in T_{n+1}(A)\ \exists\ \alpha \in Aut(H)$, *s.t* $\alpha(x) \in T_n(A) \subseteq B$, hence hypotheses of induction and $B \prec H$ imply $(\alpha^{-1} \circ \alpha)(x) \in B$. Thus we get $x \in B$.

(*ii*) By induction on n, we have $T_n(A) \subseteq \bigcup_{a \in A} T_n(\{a\})$. Hence $C_c(A) = \bigcup_{a \in A} C_c(\{a\})$. □

Corollary 3.27. *Suppose (H, ϱ, e) is a polygroup. Then*

(*i*) $C_c(\emptyset) = \emptyset$,

(*ii*) $C_c(\{e\}) = \{e\}$,

(*iii*) $C_c(H \setminus \{e\}) = H \setminus \{e\}$,

(*iv*) $C_c(w_H) = w_H$.

From now on, $\forall\ x \in H$ and $n \in \mathbb{N}$, we denote $T_n(\{x\})$ by $T_n(x)$ and $(C_c(\{x\}))T(\{x\})$ by $(C_c(x))T(x)$.

Lemma 3.28. *Assume H is a hypergroup and $n \in \mathbb{N}$. Then*

(*i*) $\forall\ x \in H$, *we have* $T_n(T_2(x)) = T_{n+1}(x)$,

(*ii*) $\forall\ x, y \in H$, $x \in T_n(y) \ \Leftrightarrow\ y \in T_n(x)$.

Proof. (*i*) Clearly, $T_1(T_2(x)) = T_2(x)$. Now, by induction, if $T_{n-1}(T_2(x)) = T_n(x)$, Then

$$\begin{aligned} T_n(T_2(x)) &= \{x \mid \exists\ \alpha \in Aut(H) : \alpha(x) \in T_{n-1}(T_2(x))\} \\ &= \{x \mid \exists\ \alpha \in Aut(H) : \alpha(x) \in T_n(x)\} = T_{n+1}(x). \end{aligned}$$

(*ii*) Clearly, $x \in T_1(y) \Leftrightarrow y \in T_1(x)$. Let $\forall x, y \in H$,

$$x \in T_{n-1}(y) \Leftrightarrow y \in T_{n-1}(x).$$

If $x \in T_n(y)$, then $\exists\ \alpha \in Aut(H)$ *s.t* $\alpha(x) \in T_{n-1}(y)$. Using hypotheses of induction, we conclude that $y \in T_{n-1}(\alpha(x))$. In addition, $\alpha^{-1}(\alpha(x)) = x \in T_1(x)$ implies that $\alpha(x) \in T_2(x)$. Hence

$$y \in T_{n-1}(\alpha(x)) \subseteq T_{n-1}(T_2(x)) = T_n(x).$$

□

Lemma 3.29. *Assume H is a hypergroup. Then $S = \{(x,y) \in H \times H \mid x \in T(y)\}$, is an equivalence relation on H.*

Proof. Clearly $\{x\} \subseteq T(x)$, so S is a reflexive relation and by Lemma 3.28, S is a symmetric relation. Now, suppose that $(x,y) \in S$ and $(y,z) \in S$. If $z \in B \prec H$, then $y \in C_c(y) \subseteq B$ and consequently $x \in C_c(y) \subseteq B$. It follows that $x \in \bigcap_{z \in B \prec H} B = C_c(z)$ and so $x \in C_c(z) = T(z)$. Thus $(x,z) \in S$ and so S is an equivalence relation on H. □

Suppose H is a hypergroup and $R = \{(x,y) \mid \exists\ \alpha \in Aut(H) \text{ s.t } \ \alpha(x) = y\}$. Obviously, R is an equivalence relation on H.

Theorem 3.30. *Suppose H is a hypergroup. Then*

(i) $S = R$,

(ii) for $x \in H$, $R(x) \prec H$.

Proof. (i) Let $x, y \in H$. Then

$$\begin{aligned}(x,y) \in S \quad &\Rightarrow \quad x \in T(y) \Rightarrow\ \exists\ n \in \mathbb{N} \text{ s.t } \ x \in T_n(y)\\ &\Rightarrow \quad \exists\ \alpha_1 \in Aut(H) \text{ s.t } \ \alpha_1(x) \in T_{n-1}(y)\\ &\Rightarrow \quad \exists\ \alpha_2 \in Aut(H) \text{ s.t } \ \alpha_2(\alpha_1(x)) \in T_{n-2}(y)\end{aligned}$$

Thus by induction,

$$\exists\ \alpha_{n-1} \in Aut(H) \text{ s.t } \alpha_{n-1}(\alpha_{n-2}(\cdots(\alpha_2(\alpha_1(x)))\cdots)) \in T_1(y) = \{y\},$$

and so $(x,y) \in R$.

Conversely, $(x,y) \in R$ implies that

$$\exists\ \alpha \in Aut(H) \text{ s.t } \ \alpha(x) = y \in T_1(y) \Rightarrow x \in T_2(y) \Rightarrow (x,y) \in S.$$

(ii) By definition, the proof is obtained. □

Theorem 3.31. *Assume H is a hypergroup and $A \subseteq H$. Then*

$$C_c(A) = \bigcup_{\alpha \in Aut(H)} \alpha(A).$$

Proof. Let $x \in H$. Then

$$\begin{aligned}x \in C_c(A) &\Leftrightarrow \exists\ a \in A \text{ s.t } \ x \in C_c(a) = T(a)\\ &\Leftrightarrow\ \exists\ a \in A \text{ and } \ \exists\ \alpha \in Aut(H) \text{ s.t } \ x = \alpha(a) \Leftrightarrow x \in \bigcup_{\alpha \in Aut(H)} \alpha(A).\end{aligned}$$

□

Theorem 3.32. *Suppose H is a hypergroup and $K \prec H$. Then*

(i) *if $x \in w_K$, then $C_c(x) \subseteq w_K$,*

(ii) *if $x \in w_H$, then $C_c(x) \subseteq w_H$.*

Proof. $(i),(ii)$ Since $\{x\} \subseteq w_K$, by Lemma 3.23, we have $C_c(x) \subseteq w_K$. □

Example 3.33. Consider the hypergroup H in Example 3.8. Facility $w_H = \{1,-1\}$, while $C_c(-1) = \{-1\}$. So In Theorem 3.32, necessarily, $C_c(x) \neq w_K$.

Corollary 3.34. *Let $\emptyset \neq A \subseteq H$ and $x \in H$. Then*

(i) $C_c(x) = R(x)$,

(ii) $C_c(A) = \bigcup\limits_{a \in A} R(a)$.

Proof. $(i),(ii)$ Let $x \in H$. Then $x \in C_c(A)$ if and only if $\exists\ \alpha \in Aut(H)\ s.t\ x \in \alpha(A)$ if and only if $\exists\ \alpha \in Aut(H)$ and $a \in A$ in a way $x = \alpha(a)$ if and only if $\exists\ a \in A\ s.t\ x \in R(a)$ if and only if $x \in \bigcup\limits_{a \in A} R(a)$. □

4. Auto–Engel Polygroups

In this section, we introduce the concept of autocommutators via the automorphisms of hypergroups and define the notation of auto–Engel polygroups with respect to the hearts of hypergroups. Moreover, we consider some conditions to construct of auto–Engel polygroups(auto–Engel groups) via the general fundamental relation and automorphisms of hypergroups.

Assume H is a hypergroup, $x \in H$ and $\alpha \in Aut(H)$. Define $[x,\alpha] = \{h \in H \mid x \in \varrho(h,\alpha(x))\}$ and will call an autocommutator of x and α. Inductively, $\forall\ \alpha_1,\alpha_2,\ldots,\alpha_n \in Aut(H), [x,\alpha_1,\alpha_2,\ldots,\alpha_n] = \big[[x,\alpha_1,\alpha_2,\ldots,\alpha_{n-1}],\alpha_n\big]$ is an autocommutator of $x,\alpha_1,\alpha_2,\ldots,\alpha_n$ of weight $n+1$, where $\forall\ X \subseteq H$ we have $[X,\alpha] = \bigcup_{x\in X}[x,\alpha]$.

Example 4.1. Let $H = \{1,2,3,4,5,6,7\}$. Then $(H,\varrho,1,\vartheta)$ is a non–commutative polygroup as follows:

ϱ	1	2	3	4	5	6	7
1	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$
2	$\{2\}$	$\{1,2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$
3	$\{3\}$	$\{3\}$	$\{1,2\}$	$\{7\}$	$\{6\}$	$\{5\}$	$\{4\}$
4	$\{4\}$	$\{4\}$	$\{6\}$	$\{1,2\}$	$\{7\}$	$\{3\}$	$\{5\}$
5	$\{5\}$	$\{5\}$	$\{7\}$	$\{6\}$	$\{1,2\}$	$\{4\}$	$\{3\}$
6	$\{6\}$	$\{6\}$	$\{4\}$	$\{5\}$	$\{3\}$	$\{7\}$	$\{1,2\}$
7	$\{7\}$	$\{7\}$	$\{5\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{6\}$

.

If $\alpha_1 = (3,4,5)$, $\alpha_2 = (3,5,4)$ and $\alpha_3 = (6,7)$, then clearly $\forall\, x \in \{2,6,7\}, [x,\alpha_1] = \{1,2\}$ and $\forall\, x \in \{3,4,5\}, [x,\alpha_1] = \{7\}$. Also computations show that $\forall\, x \neq 1, [x,\alpha_1,\alpha_2] = \{1,2\}$, $\forall\, x \in \{2,6,7\}, [x,\alpha_1,\alpha_3] = \{1,2\}$, $\forall\, x \in \{3,4,5\}$, $[x,\alpha_1,\alpha_3] = \{6\}$ and $\forall\, x \neq 1, [x,\alpha_1,\alpha_2,\alpha_3] = \{1,2\}$.

Theorem 4.2. *Let (H,e) be a polygroup, $x \in H$, $K \prec H$ and $\alpha \in Aut(H)$. Then*

(i) $[x,id] = [e,x] = \varrho(x,\vartheta(x))$ *and* $[e,\alpha] = e$,

(ii) $[x,\alpha] = \varrho(x,\alpha(\vartheta(x)))$,

(iii) $R^*_K([x,\alpha]) = [R^*_K(x),\overline{\alpha}]$,

(iv) $\vartheta([x,\alpha]) = [\alpha(x),\alpha^{-1}]$,

Proof. We prove only (iv). Let $x \in H$. Then we have

$$\vartheta([x,\alpha]) = \vartheta(\varrho(x,\alpha(\vartheta(x))) = \varrho(\alpha(x),\vartheta(x)) = [\alpha(x),\alpha^{-1}].$$

□

Suppose H is a hypergroup. Define $A_{0,\alpha}(H) = H$, for every $n \in \mathbb{N}^*$, $A_{n+1,\alpha}(H) = \{h \in [x,\alpha] \mid x \in A_{n,\alpha}(H)\}$ and $A_n(H) = \bigcup_{\alpha\in Aut(H)} A_{n,\alpha}(H)$.

Example 4.3. Consider the hypergroup (H,ϱ) in Example 4.1. If $\alpha = (6,7)$, then $A_{0,\alpha}(H) = H$ and $\forall\, n \geq 1$ we have $A_{n,\alpha}(H) = \{1,2,6,7\}$. Also $\alpha = (3,4,5)$, $\forall\, n \geq 2$ implies that $A_{n,\alpha}(H) = \{1,2\}$, $A_n(H) = \{1,2,6,7\}$ and $\forall\, n \geq 1, A_{n,id} = \{1,2\}$.

Theorem 4.4. *Assume H is a hypergroup, $n \in \mathbb{N}^*$ and $\alpha \in Aut(H)$. Then*

(i) $A_{n+1,\alpha}(H) \subseteq A_{n,\alpha}(H)$ *and* $A_{n+1}(H) \subseteq A_n(H)$,

(ii) $A_{n,\alpha}(H) = \{h \in [x,_n \alpha] \mid x \in H\}$,

(iii) $A_{n,id}(H) \subseteq w_H$,

(iv) $A_n(H) = \bigcup\limits_{\alpha\in Aut(H)} \bigcup\limits_{x\in H} [x,_n \alpha]$.

Proof. We prove by induction on n.

(i) Let $h \in A_{n+2,\alpha}(H)$. Then $\exists\, x \in A_{n+1,\alpha}(H)$ *s.t* $h \in [x,\alpha]$. Using the hypotheses of induction, $x \in A_{n,\alpha}(H)$. It follows that $h \in [x,\alpha]$ and so $h \in A_{n+1,\alpha}(H)$. Thus by definition, we get $A_{n+1}(H) \subseteq A_n(H)$.

(ii) $A_{n+1,\alpha}(H) = \{h \in [x,\alpha] \mid x \in A_{n,\alpha}(H)\}$. By induction, we have

$$\begin{aligned} A_{n+1,\alpha}(H) = \bigcup_{y\in A_{n,\alpha}(H)} [y,\alpha] = \bigcup_{y\in \bigcup\limits_{x\in H}[x,_n \alpha]} [y,\alpha] &= \bigcup_{x\in H} [[x,_n \alpha],\alpha] \\ &= \bigcup_{x\in H} [x,_{n+1} \alpha]. \end{aligned}$$

Other items are obtained in a similar way by induction on n. □

Definition 4.5. Suppose H is a polygroup and $\alpha \in Aut(H)$. Then H is called:

(i) an α-auto–Engel polygroup, if $\forall\ x \in H\ \exists\ n \in \mathbb{N}$ in a way $[x,_n \alpha] \subseteq w_H$($\alpha$ is fixed),

(ii) an (n,α)-auto–Engel polygroup, if $\forall\ x \in H$, $[x,_n \alpha] \subseteq w_H$($\alpha$ and n are fixed),

(iii) an n-auto–Engel polygroup, if $\forall\ x \in H$ and $\forall\ \alpha \in Aut(H)$, $[x,_n \alpha] \subseteq w_H$($n$ is fixed),

(iv) an auto–Engel polygroup, if $\forall\ x \in H$ and $\forall\ \alpha \in Aut(H)\ \exists\ n \in \mathbb{N}\ s.t\ [x,_n \alpha] \subseteq w_H$,

Example 4.6. Let $H = \{1, -2, -6\}$. Then $(H, \varrho, 1, \vartheta)$ is a polygroup as follows:

ϱ	1	-2	-6
1	$\{1\}$	$\{-2\}$	$\{-6\}$
-2	$\{-2\}$	$\{1,-6\}$	$\{-2,-6\}$
-6	$\{-6\}$	$\{-2,-6\}$	$\{1,-2\}$

.

Clearly $Aut(H) = \{id, \alpha = (-2\ -6)\}$ and $\overline{Aut(H)} = Aut(H/\beta^*)$. In addition, for every $n \in \mathbb{N}, x \in H$ and $\alpha \in Aut(H)$ we have $[x,_n \alpha] \subseteq w_H$. It concludes that H is an auto-Engel polygroup.

Example 4.7. Let $H = \mathbb{Z}_6 \cup \{\sqrt{2}\}$. Then $(H, +, \overline{0})$ is a polygroup as follows:

$+$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\{\sqrt{2}\}$
$\overline{0}$	$\{\overline{0}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$	$\{\overline{3}\}$	$\{\overline{4}\}$	$\{\overline{5}\}$	$\{\sqrt{2}\}$
$\overline{1}$	$\{\overline{1}\}$	$\{\overline{2}\}$	$\{\overline{3}\}$	$\{\overline{4}\}$	$\{\overline{5}\}$	$\{\overline{0},\sqrt{2}\}$	$\{\overline{1}\}$
$\overline{2}$	$\{\overline{2}\}$	$\{\overline{3}\}$	$\{\overline{4}\}$	$\{\overline{5}\}$	$\{\overline{0},\sqrt{2}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$
$\overline{3}$	$\{\overline{3}\}$	$\{\overline{4}\}$	$\{\overline{5}\}$	$\{\overline{0},\sqrt{2}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$	$\{\overline{3}\}$
$\overline{4}$	$\{\overline{4}\}$	$\{\overline{5}\}$	$\{\overline{0},\sqrt{2}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$	$\{\overline{3}\}$	$\{\overline{4}\}$
$\overline{5}$	$\{\overline{5}\}$	$\{\overline{0},\sqrt{2}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$	$\{\overline{3}\}$	$\{\overline{4}\}$	$\{\overline{5}\}$
$\sqrt{2}$	$\{\sqrt{2}\}$	$\{\overline{1}\}$	$\{\overline{2}\}$	$\{\overline{3}\}$	$\{\overline{4}\}$	$\{\overline{5}\}$	$\{\overline{0}\}$

Since $\forall\ n \geq 1, \alpha \neq id, A_{n,\alpha}(H) = \{\overline{0}, \sqrt{2}, \overline{2}, \overline{4}\}$ and $w_H = \{\overline{0}, \sqrt{2}\}$, we get that H is not an auto-Engel polygroup.

Corollary 4.8. *Assume $n \in \mathbb{N}$, H is a polygroup and $\alpha \in Aut(H)$. Then the following hold:*

(i) *Every polygroup is an id–auto–Engel polygroup (id is the identity automorphism).*

(ii) *Every n–auto–Engel polygroup is an (n,α)–auto–Engel polygroup.*

(iii) *Every n–auto–Engel polygroup is an auto–Engel polygroup.*

Theorem 4.9. *Suppose H is a polygroup. Then*

(*i*) *H is an (n, α)- auto–Engel polygroup if and only if $A_{n,\alpha}(H) \subseteq w_H$,*

(*ii*) *H is an n-auto–Engel polygroup if and only if $\forall\ \alpha \in Aut(H)$, $A_{n,\alpha}(H) \subseteq w_H$.*

Proof. By Theorem 4.4, the proof is obtained. □

Suppose H is a hypergroup. Define $K_0(H) = H$ and for every $n \in \mathbb{N}^*$, $K_{n+1}(H) = \{h \in [x, \alpha] \mid x \in K_n(H),\ \alpha \in Aut(H)\}$. H is called an autonilpotent polygroup of class at most n, if $K_n(H) \subseteq w_H$ [2].

Theorem 4.10. *Assume H is a hypergroup and $n \in \mathbb{N}^*$. Then $A_n(H) \subseteq K_n(H)$.*

Proof. By induction on n, for $n = 0$, we have $A_{0,\alpha}(H) = H \subseteq K_0(H)$. Let $h \in A_{n+1,\alpha}(H)$, then $\exists\ x \in A_{n,\alpha}(H)$ *s.t* $h \in [x, \alpha]$. Applying of hypotheses of induction we get that $x \in K_n(H)$ and so $h \in K_{n+1}(H)$. □

Corollary 4.11. *Assume H is a polygroup, then*

(*i*) *If H be an autonilpotent polygroup then H is an auto-Engel polygroup.*

(*ii*) *If H be an n-auto–Engel polygroup then H is an $(n+1)-$auto-Engel polygroup.*

(*iii*) *H is an n-auto–Engel polygroup if and only if $A_n(H) \subseteq w_H$.*

Proof. By Theorems 4.4 and 4.9, it is clear. □

Example 4.12. Consider the hypergroup (H, ϱ) in Example 4.1. Computations show that $w_H = \{1, 2\}$ and $\forall\ n \in \mathbb{N}$, we have $K_n(H) = \{1, 2, 6, 7\}$. If $\alpha_1 = (3, 4, 5)$ and $\alpha_2 = (6, 7)$, then $\forall\ 2 \leq n \in \mathbb{N}, x \in H$ we have $[x,_n \alpha_1] = \{1, 2\}$ and $\forall\ 1 \leq n \in \mathbb{N}$, we have $[6,_{2n} \alpha_2] = \{6\}$ and $[6,_{2n-1} \alpha_2] = \{7\}$. Thus $\forall\ n \geq 2, H$ is an (n, α_1)-auto–Engel polygroup, while it is not an (n, α_2)-auto–Engel polygroup and it is not an auto-nilpotent polygroup.

Example 4.13. (*i*) Consider the polygroup H in Example 4.7. Since $\forall\ n \geq 1, \alpha \neq id, K_n(H) = \{\overline{0}, \sqrt{2}, \overline{2}, \overline{4}\}$ and $w_H = \{\overline{0}, \sqrt{2}\}$, we get that H is not an autonilpotent polygroup, while it is a nilpotent polygroup and an Engel polygroup.

(*ii*) Let $G = D_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b, \}$. Routine computations show that

$$Aut(G) = \{\alpha_{i,j} : G \to G \mid \alpha_{i,j}(a) = a^i, \alpha_{i,j}(b) = a^j b, \text{ s.t } i \in \{1, 3\}, j \in \{0, 1, 2, 3\}\}.$$

Hence we obtain that $K_1(G) = \langle a \rangle, K_2(G) = \langle a^2 \rangle$ and $K_3(G) = \{e\}$.

So $\forall\ n \geq 3$, we have $K_n(G) = \{e\}$. It follows that G is an autonilpotent group and by Corollary 4.11, G is an n-auto-Engel polygroup.

Example 4.14. Let $G = S_3 = \{e, a, a^2, b, ab, a^2b\}$. Routinely,

$$Aut(G) = \{\alpha_{i,j} : G \to G \mid \alpha_{i,j}(a) = a^i, \alpha_{i,j}(b) = a^j b, \text{ s.t } i \in \{1,2\}, j \in \{0,1,2\}\}.$$

Then we have

$$A_{1,\alpha_{1,j}}(G) = \{e, a^{-j}\}, A_{2,\alpha_{1,j}}(G) = \{e\},$$

and $\forall\ n \geq 1$, $A_{n,\alpha_{2,j}}(G) = \langle a \rangle$. So G is not an auto-Engel polygroup but it is an $(n, \alpha_{1,j})$-auto-Engel polygroup.

Theorem 4.15. *Assume H is a hypergroup, $G = H/R^*_K$ and $K \prec H$. If $Aut(G) \subseteq \overline{Aut(H)}$, then*

(i) $K_n(G) = \{\overline{h} \mid h \in K_n(H)\}$,

(ii) if $K = H$, then H is an autonilpotent polygroup if and only if G is an autonilpotent group.

Proof. (i) We prove by induction on n. Let $\overline{a} \in K_{n+1}(G)$. Then $\exists\ \alpha \in Aut(G)$ and $\overline{x} \in K_n(G)$ *s.t* $\overline{a} = [\overline{x}, \alpha]$. Thus $\exists\ \alpha_0 \in Aut(H)$ that $\overline{\alpha_0} = \alpha$. Using induction hypotheses, $\exists\ t \in K_n(H)$ *s.t* $\overline{x} = \overline{t}$. If $b \in [t, \alpha_0]$, then $b \in K_{n+1}(H)$ and $\overline{b} = [\overline{x}, \overline{\alpha_0}] = [\overline{x}, \alpha] = \overline{a}$. The converse is clear.

(ii) Assume H is an autonilpotent polygroup. Then $\exists\ n \in \mathbb{N}$ in a way $K_n(H) \subseteq w_H$. It follows that $K_n(G) \subseteq \{e\}$. The converse is similarly. □

Theorem 4.16. *Suppose H is a hypergroup, $G = H/R^*_K, \alpha \in Aut(H)$ and $K \prec H$. Then*

(i) $\overline{A_{n,\alpha}(H)} = \{\overline{h} \mid h \in A_{n,\alpha}(H)\} = A_{n,\overline{\alpha}}(G)$,

(ii) If $K = H$, then H is an (n, α)–auto–Engel polygroup, if and only if G is an $(n, \overline{\alpha})$–auto–Engel group.

Proof. It is similar to proof of Theorem 4.15. □

Proposition 4.17. *Assume H_1, H_2 are hypergroups, $\alpha_1 \in Aut(H_1), \alpha_2 \in Aut(H_2)$ and $(\alpha_1, \alpha_2) = \alpha \in Aut(H_1 \times H_2)$ by $\alpha(x, y) = (\alpha_1(x), \alpha_2(y))$. Then*

(i) $K_n(H_1) \times K_n(H_2) \subseteq K_n(H_1 \times H_2)$,

(ii) $A_{n,\alpha_1}(H_1) \times A_{n,\alpha_2}(H_2) \subseteq A_{n,\alpha}(H_1 \times H_2)$,

(iii) if $H_1 \times H_2$ is an (n, α)-auto-Engel polygroup, then H_1 is an (n, α_1)-auto-Engel polygroup and H_2 is an (n, α_2)-auto-Engel polygroup.

(iv) if $H_1 \times H_2$ is an n-auto-Engel polygroup, then H_1 and H_2 are n-auto-Engel polygroups.

Proof. (i) We prove by induction. Let $(h_1, h_2) \in K_{n+1}(H_1) \times K_{n+1}(H_2)$. Then $\exists\ x_1 \in K_n(H_1), x_2 \in K_n(H_2), \alpha_1 \in Aut(H_1)$ and $\alpha_2 \in Aut(H_2)$ $s.t$ $h_1 \in [x_1, \alpha_1]$ and $h_2 \in [x_2, \alpha_2]$. Define $\alpha = (\alpha_1, \alpha_2)$ by $\alpha(x, y) = (\alpha_1(x), \alpha_2(y))$. Clearly $\alpha \in Aut(H_1 \times H_2)$ and so by induction hypotheses, $(x_1, x_2) \in K_n(H_1) \times K_n(H_2) \subseteq K_n(H_1 \times H_2)$. So $(h_1, h_2) \in [(x_1, x_2), (\alpha_1, \alpha_2)] \subseteq K_{n+1}(H_1 \times H_2)$.

In a similar way the items (ii) and (iii) hold.

(v) By item (ii) it is clear. □

Example 4.18. Consider the groups $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Clearly $K_1(\mathbb{Z}_2) = \{\overline{0}\}$, $K_1(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{[x, \alpha] \mid x \in \mathbb{Z}_2 \times \mathbb{Z}_2, \alpha \in Aut(\mathbb{Z}_2 \times \mathbb{Z}_2)\}$. Now, for $\alpha = ((\overline{1}, \overline{0}), (\overline{1}, \overline{1}))$ and for all x, we have $[x, \alpha] \subseteq K_1(\mathbb{Z}_2 \times \mathbb{Z}_2))$. Thus $\{(\overline{0}, \overline{0}), (\overline{0}, \overline{1})\} \subseteq K_1(\mathbb{Z}_2 \times \mathbb{Z}_2)$, while $\{(\overline{0}, \overline{0}), (\overline{0}, \overline{1})\} \nsubseteq K_1(\mathbb{Z}_2) \times K_1(\mathbb{Z}_2) = \{e\}$. Hence the converse of Proposition 4.17, is not necessarily true.

Corollary 4.19. *Assume H is an (auto–Engel)n–auto–Engel polygroup, $K \leq H$ and $K \prec H$, then K is an (auto–Engel)n–auto–Engel polygroup.*

Proof. By Theorem 3.12, the proof is obtained. □

Corollary 4.20. *If H is a polygroup and $|Aut(H)| = 1$, then H is an auto–Engel polygroup.*

Proof. Since $|Aut(H)| = 1$, by Theorem 4.4, for all $n \in \mathbb{N}$, we get $A_n(H) \subseteq \omega_H$. Thus H is an auto–Engel polygroup. □

Theorem 4.21. *Assume H is a polygroup, $K = H$, $G = H/R^*_K$ and $\alpha \in Aut(H)$. If $Aut(G) \subseteq \overline{Aut(H)}$, then*

(i) *H is an n–auto-Engel polygroup if and only if G is an n–auto-Engel group,*

(ii) *H is an auto-Engel polygroup if and only if G is an auto-Engel group,*

(iii) *H is an (n, α)–auto-Engel polygroup if and only if G is an $(n, \overline{\alpha})$–auto-Engel group,*

(iv) *H is an autonilpotent polygroup if and only if G is an autonilpotent group.*

Proof. We prove just (i), the other items are similar to (i). Suppose H is an n–auto-Engel polygroup and $\alpha \in Aut(G)$. Since $Aut(G) \subseteq \overline{Aut(H)}$, we get $\alpha \in \overline{Aut(H)}$ and so $\exists\ \beta \in Aut(H)$ $s.t$ $\overline{\beta} = \alpha$. Now, by Theorem 4.16, we have $A_{n,\alpha}(G) = \overline{A_{n,\beta}(H)} = \{1\}$, because of $A_{n,\beta}(H) \subseteq \omega_H$. The converse is obtained in a similar way. □

In Example 4.12, using definition we show that H is not an auto-Engel polygroup. Now, apply the Theorem 4.21, and show that H is not an auto-Engel polygroup.

Example 4.22. Consider the polygroup in Example 4.12. If $K = H$, then routinely $G = H/R^*_K \cong S_3$ and so by Example 4.14, H is not an auto-Engel polygroup. In addition, for every $j \in \{0, 1, 2\}$, G is an $(n, \alpha_{1,j})$-auto-Engel group and so H is an (n, α_j)-auto Engel polygroup where $\alpha_0 = id, \alpha_1 = (3, 4, 5), \alpha_2 = (3, 5, 4), \overline{\alpha_j} = \alpha_{1,j}$.

If $K = \emptyset$, then $Aut(H/R^*_K) = \{id\}$ and $\forall\ n \in \mathbb{N}$ we have $A_n(H/R^*_K) = \{1\}$. It follows that H/R^*_K is an auto-Engel group. It follows that Theorem 4.21, in general is not true.

Definition 4.23. Assume H is a polygroup, $n \in \mathbb{N}^*$ and $\alpha \in Aut(H)$. Define $Z_{0,\alpha}(H) = w_H$, $Z_{n+1,\alpha}(H) = \{x \mid [x, \alpha] \subseteq Z_{n,\alpha}(H)\}$ and

$$Z_n(H) = \bigcap_{\alpha \in Aut(H)} Z_{n,\alpha}(H).$$

Example 4.24. Consider $G = S_3$. By Example 4.14, $\forall\ n \geq 2$, we get that $Z_{n,\alpha_{1,j}}(G) = G$, $Z_{n,\alpha_{2,0}}(G) = \{e, b\}$, $Z_{n,\alpha_{2,1}}(G) = \{e, a^2b\}$ and $Z_{n,\alpha_{2,2}}(G) = \{e, ab\}$.

Example 4.25. Consider the dihedral polygroup $G = D_8$. By Example 4.13, $\forall\ n \geq 2$, we obtain $Z_{n,\alpha_{1,j}}(G) = Z_{n+1,\alpha_{3,j}}(G) = G$.

Theorem 4.26. *Assume H is a polygroup, $x \in H$, $\alpha \in Aut(H)$ and $n \in \mathbb{N}^*$. Then*

(i) $Z_{n,\alpha}(H) \subseteq Z_{n+1,\alpha}(H)$ *and so* $w_H \subseteq Z_{n,\alpha}(H)$,

(ii) $Z_{n,\alpha}(H)$ *is a complete part of* H,

(iii) $[x, id] \subseteq Z_{n,\alpha}(H)$,

(iv) *if* $|Aut(H)| = 1$, *then* $Z_{n+1,\alpha}(H) = H$.

Proof. (i) By Corollary 3.20, we have $\alpha(w_H) \subseteq w_H$. Hence we get that $Z_{0,\alpha}(H) \subseteq Z_{1,\alpha}(H)$ and so by induction the proof is obtained.

(ii) By item (i), $w_H \subseteq Z_{n,\alpha}(H)$ implies that $C(Z_{n,\alpha}(H)) = Z_{n,\alpha}\varrho((H), w_H) = Z_{n,\alpha}(H)$. Thus $Z_{n,\alpha}(H)$ is a complete part of H.

(iii) It is obtained from the item (i).

(iv) It is clear. □

Theorem 4.27. *Suppose H is a polygroup, $n \in \mathbb{N}$ and $\alpha \in Aut(H)$. Then $A_{n,\alpha}(H) \subseteq w_H$ if and only if $Z_{n,\alpha}(H) = H$.*

Proof. Let $Z_{n,\alpha}(H) = H$. Then by induction on i, we have $A_{i,\alpha}(H) \subseteq Z_{n-i,\alpha}(H)$. Now for $i = n$ we obtain that $A_{n,\alpha}(H) \subseteq Z_{0,\alpha}(H) = w_H$.

Conversely, if $A_{n,\alpha}(H) \subseteq w_H$, then by induction we conclude $A_{n-i,\alpha}(H) \subseteq Z_{i,\alpha}(H)$. Letting $i = n$ implies that $H = A_{0,\alpha}(H) \subseteq Z_{n,\alpha}(H) \subseteq H$. □

Corollary 4.28. *Assume H is a polygroup, $n \in \mathbb{N}$ and $\alpha \in Aut(H)$. Then*

(*i*) *H is an (n,α)-auto-Engel polygroup if and only if $Z_{n,\alpha}(H) = H$;*

(*ii*) *H is an n-auto-Engel polygroup if and only if $Z_n(H) = H$.*

Example 4.29. Consider the polygroup (H, ϱ) in Example 4.6. Thus $\forall\, n \in \mathbb{N}, \alpha \in Aut(H)$ we have $A_{n,\alpha}(H) \subseteq w_H$. Hence $Z_{n,\alpha}(H) = H$ and so $Z_n(H) = H$.

Theorem 4.30. *Assume H is an n-auto-Engel polygroup and $H \neq w_H$. Then, $\forall\, \alpha \in Aut(H)$ we have $Z_{1,\alpha}(H) \neq w_H$.*

Proof. If $\exists\, \alpha \in Aut(H)$ $s.t$ $Z_{1,\alpha}(H) = w_H$, then

$$Z_{2,\alpha}(H) = \{x \mid [x,\alpha] \subseteq Z_{1,\alpha}(H) = w_H\} = Z_{1,\alpha}(H) = w_H,$$

so $H = Z_{n,\alpha}(H) = w_H$, which is a contradiction. □

Theorem 4.31. *Suppose G is a group and $Aut(G) = Inn(G)$. Then G is an n-auto-Engel group if and only if G is an n-Engel group.*

Proof. Let $\alpha \in Aut(G)$,then $\exists\, y \in G$ $s.t$ $[x,\alpha] = [x,y]$, for every $x \in G$.Thus by induction on n, we have $[x,_n \alpha] = [x,_n y]$, $\forall\, x \in G$. □

Theorem 4.32. *Assume (G, e) is an n–auto–Engel group. Then G is an n–Engel group*

Proof. Let $x, y \in G$ and $\varphi_y \in Inn(G)$. Then $[x,y] = [x,\varphi_y]$ and by induction on n we get that $[x,_n y] = [x,_n \varphi_y]$. Since G is an n–auto–Engel group, for every $\alpha \in Aut(G)$ we have $A_{n,\alpha}(G) = \{e\}$. It follows that $\forall\, x, y \in G, [x,_n y] = 1$. □

Theorem 4.33. *Assume H is a polygroup and $n \in \mathbb{N}$. H is an n–Engel polygroup if and only if H/β^* is an n–Engel group.*

Proof. By induction on n, we have $\beta^*([x,_n y]) = [\beta^*(x),_n \beta^*(y)]$, hence the proof is obtained. □

Corollary 4.34. *Assume $n \in \mathbb{N}$, H is a n–auto–Engel polygroup, $G = H/\beta^*$ and $Aut(G) \subseteq Aut(H)$. Then H is an n–Engel polygroup.*

Proof. Using Theorem 4.21, G is an n–auto–Engel group and by Theorem 4.32, G is an n–Engel group. Now Theorem 4.33, implies that H is an n–Engel polygroup. □

Corollary 4.35. *Assume H is a polygroup, $n \in \mathbb{N}$ and $\alpha \in Aut(H)$. Then we have the tree diagram HT in Figure 1.*

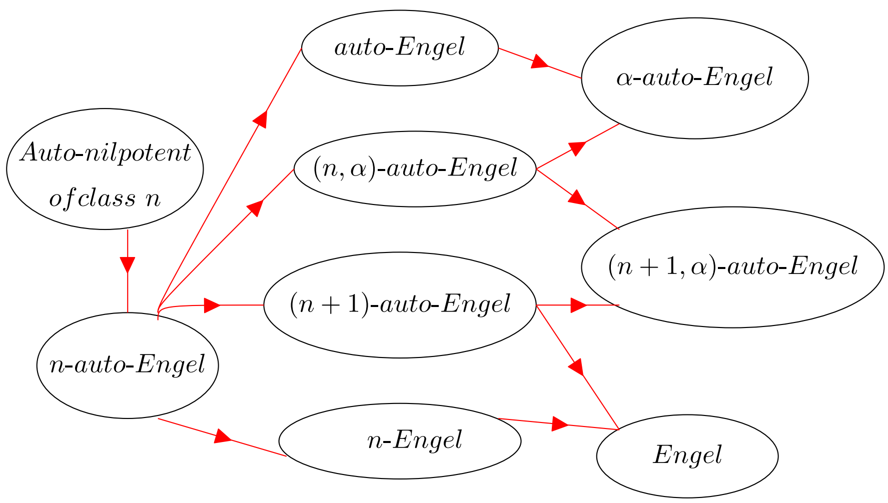

Figure 1: Tree diagram HT of polygroup H.

5. Conclusion

The current paper introduced a fundamental relation on hypergroups in such a way that under some conditions is a generalization of β^* and γ^*. Also

(i) The concept of characteristic subset of hypergroups is introduced and is shown that the heart of every hypergroup is a characteristic subset of hypergroup.

(ii) By using the concept of automorphisms and fundamental relation on hypergroups, we obtain some characteristic subset of hypergroup.

(iii) The notation of characteristic-cluster subset of hypergroups is defined and the relation between of characteristic-cluster subset and the heart of hypergroups is investigated.

(iv) With respect to the concept of characteristic-cluster subset and the hearts of hypergroups the concept of auto–Engel polygroups is defined.

(v) Through the concept of complete parts and fundamental relations , the relation between of some class of auto–Engel polygroups and Engel polygroups are considered and is shown in a Hass diagram.

We try to that these results are helpful in next studies in Engel–groups. In our next papers, we try to obtain more results regarding Engel–polygroups, groups, and their applications.

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