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On the Hosoya Index of Some Families of Graph

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Abstract

We obtain the exact relations of the Hosoya index that is defined as the sum of the number of all the matching sets, on some classes of cycle-related graphs. Moreover, this index of three graph families, namely, chain triangular cactus, Dutch windmill graph, and Barbell graph is determined.

Keywords: Hosoya index, Helm graph, graph lotus, chain triangular cactus, Dutch windmill graph.

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1. Introduction

The Hosoya index $Z(G)$, first proposed by Haruo Hosoya. This index is correlated with certain properties of alkane isomers [6]. We consider G as a simple graph that contains the vertex set V and the edge set E. The cardinalities of V and E are called the order and size of graph G. A subset of the edge set in graph G is called the matching set if the end vertices of no two of its edges overlap. Assume that the number of the matching sets in G with cardinality k is given by $M_G(k)$. Then, the Hosoya index in graph G is given as follows

$$Z(G) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} M_G(k),$$

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in which n is the order of G and $\lfloor \frac{n}{2} \rfloor$ is the integer part of $\frac{n}{2}$. By definition, for any graph G, $M_G(0) = 1$ and $M_G(1) = m$. For $k > \lfloor \frac{n}{2} \rfloor$, $M_G(k) = 0$.

Many results concerning the Hosoya index of graphs have been obtained. Some nice results can be found in [2, 4, 9, 12, 14, 15, 17]. In this paper, the Hosoya index of some categories of cycle-related graphs such as helm graph, gear graph and graph lotus is determined. We also obtain exact formulas of three families of graphs, namely, chain triangular cactus, Dutch windmill graph and Barbell graph. Let vertex u be in graph G. The set $N_G(u)$ is the neighborhood of u that is defined as the set of the vertices $v \in V$ such that u and v are adjacent and $|N_G(u)| = d_u$ is the degree u in G. If $d_u = 1$ for the vertex u in graph G is called a leaf in the graph. Two graphs K and L are known isomorphic, denoted by $K \simeq L$, if there is a bijective correspondence between their vertices and edges [8].

The corona $K \circ L$, is the graph obtained from K and $|V(K)|$ copies of L, such that the ith vertex of K is adjacent to all vertices in the ith copy of L. Especially, the corona $G \circ K_1$, is obtained of G and adding a leaf to each vertex in G [5].

Throughout the paper, a cycle graph, a path graph and a complete graph with n vertices are denoted by K_n, C_n and P_n, respectively. A wheel W_n, where $n \geq 4$, is a graph obtained by adding a vertex to the cycle C_{n-1} and connecting it to all vertices of C_{n-1}. A star graph S_n is a graph that contains a central vertex x and n leaves attached to x.

2. Preliminaries

Now, we state some needed results that we use in the next section.

Lemma 2.1. [8] *Suppose that for a graph G, the vertex v in $V(G)$ and uv in $E(G)$, then*

(i) $Z(G) = Z(G - \{uv\}) + Z(G - \{u, v\})$,

(ii) $Z(G) = Z(G - \{v\}) + \sum_{u \in N_G(v)} Z(G - \{u, v\})$,

(iii) *If graph G contains the connected components $G_1, G_2, \ldots, G_t$, then $Z(G)$ is equal to $\prod_{i=1}^{t} Z(G_i)$.*

Lemma 2.2. [8]

(i) *The Hosoya index of the path of order $n > 0$ is equal to F_{n+1},*

(ii) *The Hosoya index of the cycle of order $n \geq 3$ is equal to $F_{n-1} + F_{n+1}$,*

in which F_n denotes the Fibonacci number and defines by the recursive formula $F_{n+1} = F_n + F_{n-1}$ and the conditions $F_0 = 0$, $F_1 = 1$.

The subdivision-related graph, denoted by $R(G)$, is a graph obtained by placing a cycle C_3 cycle instead of each edge in graph G [17].

Lemma 2.3. [17] *Suppose that $R(G)$ is the subdivision-related graph of a simple graph G, then $Z(R(G)) = \prod_{i=1}^{n}(d_i + 1)$ where d_i is the degree of i'th vertex in G for $i = 1\ldots, n$.*

Lemma 2.4. [16] *If $Z(K_n) = a_n$ where $n \geq 3$, then $a_n = a_{n-1} + (n-1)a_{n-2}$ with initial conditions $a_1 = 1$ and $a_2 = 2$.*

Lemma 2.5. [12] *For $n \geq 3$,*

(i) $Z(P_n \circ K_1) = \frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{2\sqrt{2}}$,

(ii) $Z(C_n \circ K_1) = \frac{(2+\sqrt{2})(1+\sqrt{2})^{n-1}-(2-\sqrt{2})(1-\sqrt{2})^{n-1}}{\sqrt{2}}$.

3. Main Results

We compute the $Z(G)$ of some families of graphs. One of the well-known graphs related to cycles is a helm graph. The helm graph H_n is a graph obtained by connecting pendant edges to any vertex of the cycle in a wheel W_n with n vertices [7]. The Hosoya index of the helm graph is computed in Theorem 3.1.

Theorem 3.1. *For $n \geq 3$, the Hosoya index of the helm graph of H_n is equal to*

$$Z(H_n) = \frac{\left[2 + (n+1)(1+\sqrt{2})\right](1+\sqrt{2})^{n-2} - \left[2 + (n+1)(1-\sqrt{2})\right](1-\sqrt{2})^{n-2}}{2\sqrt{2}}.$$

Proof. Let H_n be the helm graph of order $2n - 1$ with the vertices set $\{v_1, v_2, \ldots, v_{n-1}\}$ on cycle C_{n-1} of wheel graph in H_n. We suppose that the vertex v_i is adjacent to the leaf v_i' for $1 \leq i \leq n-1$ and the central vertex of the graph H_n denoted by x. For $n \geq 3$, by applying Lemma 2.1, we have

$$Z(H_n) = Z(H_n - \{x\}) + \sum_{i=1}^{n-1} Z(H_n - \{x, v_i\}).$$

According to the structure of H_n, we get $H_n - \{x\} \simeq C_{n-1} \circ K_1$ and $H_n - \{x, v_i\} \simeq P_{n-2} \circ K_1$ for $i = 1, 2, \ldots, n-1$. Therefore using Lemma 2.5, we get

$$\begin{aligned} Z(H_n) &= Z(C_{n-1} \circ K_1) + (n-1)Z(P_{n-2} \circ K_1) \\ &= \frac{(4+2\sqrt{2})(1+\sqrt{2})^{n-2} - (4-2\sqrt{2})(1-\sqrt{2})^{n-2}}{2\sqrt{2}} \\ &+ \frac{(n-1)\left[(1+\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}\right]}{2\sqrt{2}} \\ &= \frac{\left[3+n+(n+1)\sqrt{2}\right](1+\sqrt{2})^{n-2} - \left[3+n-(n+1)\sqrt{2}\right](1-\sqrt{2})^{n-2}}{2\sqrt{2}}. \end{aligned}$$

After rearranging, the result completes. □

Another class of graph related to cycles is the gear graph G_n, which is created from the wheel W_n by putting a new vertex between all vertices that are adjacent on cycle C_{n-1} of the W_n [3]. That is, G_n is the obtained graph from W_n by replacing each edge on the cycle of W_n with a path P_2.

Theorem 3.2. *For $n \geq 3$, the Hosoya index of the gear graph G_n is obtained as following,*

$$Z(G_n) = 2F_{2n-3} + nF_{2(n-1)},$$

in which F_n denotes the Fibonacci number.

Proof. Assume that G_n is the gear graph with $2n-1$ vertices for $n \geq 3$. Let the vertices on the cycle C_{n-1} be labeled by v_i for $i = 1, \ldots, n-1$ in graph G_n and x is the central vertex of G_n such that $N_{G_n}(x) = \{v_1, v_2, \ldots, v_{n-1}\}$. Using Lemma 2.1 (ii) and (iii), we get

$$Z(G_n) = Z(G_n - \{x\}) + \sum_{i=1}^{n-1} Z(G_n - \{x, v_i\}).$$

According to the structure of G_n, clearly $G_n - \{x\} \simeq C_{2n-2}$ and $G_n - \{x, v_i\} \simeq P_{2n-3}$ for $i = 1, 2, \ldots, n-1$. Thus by applying Lemma 2.2, we can get

$$\begin{aligned} Z(G_n) &= Z(C_{2n-2}) + (n-1)Z(P_{2n-3}) \\ &= F_{2n-3} + F_{2n-1} + (n-1)F_{2(n-1)} \\ &= F_{2n-3} + F_{2n-3} + F_{2n-2} + (n-1)F_{2(n-1)}. \end{aligned}$$

So, the result holds. □

If any of two cycles in a connected graph has at most one vertex in common, this graph is known as a cactus graph. In other words, a cactus graph block can be either a cycle or an edge. In the cactus graph, if each block is a triangle, the cactus graph is called triangular cactus T_n of the length n [13].

Theorem 3.3. *For the chain triangular cactus T_n, the Hosoya index is given by the following*

$$Z(T_n) = 4 \times 3^{n-2}.$$

Proof. According to the definition, it is clear that $T_n \simeq R(P_n)$. Therefore, using Lemma 2.3

$$Z(T_n) = Z(R(P_n)) = 4 \times 3^{n-2}.$$

□

The graph lotus inside a circle LC_n is formed from the cycle C_n with the vertices set $\{u_1, \ldots, u_n\}$ and a star graph S_n with central vertex x and the end vertices $\{v_1, \ldots, v_n\}$ by connecting each v_i to u_i and u_{i+1} (mod n) (see Figure 1(a)) [10].

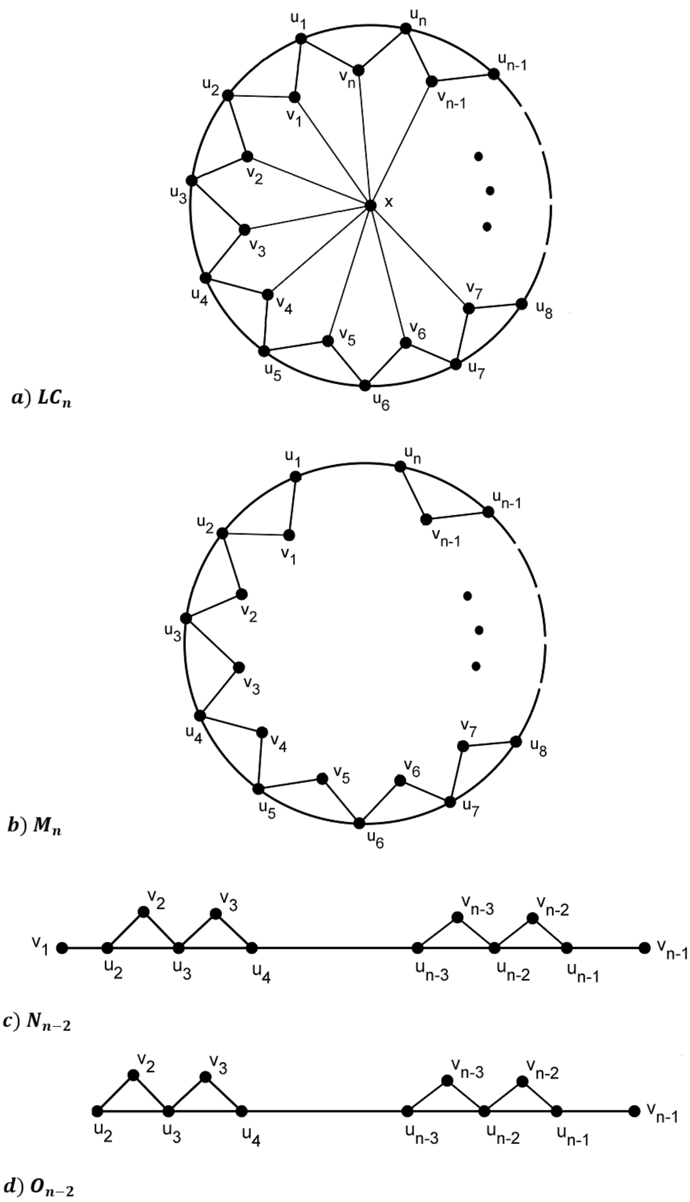

Figure 1: (a) The graph lotus inside a circle LC_n, (b, c and d) the used graphs in Theorem 3.4.

Theorem 3.4. *For $n \geq 3$, the Hosoya index of LC_n is obtained of following recursive formula,*

$$Z(LC_n) = 3^{n-2}(9 + 4n) + nZ(N_{n-2}),$$

such that for $n \geq 4$,

$$Z(N_{n-2}) = 4 \times 3^{n-4} + 2Z(N_{n-3}) - Z(N_{n-4}),$$

with initial conditions $Z(N_0) = 1$ and $Z(N_1) = 3$ and graph N_{n-2} is shown in Figure 1.

Proof. Let LC_n be a graph lotus inside a circle of order $2n+1$. Using Lemma 2.1 and Figure 1(a),

$$Z(LC_n) = Z(LC_n - \{x\}) + \sum_{i=1}^{n} Z(LC_n - \{x, v_i\}),$$

in which $LC_n - \{x\} \simeq R(C_n)$ and $LC_n - \{x, v_i\} \simeq M_n$ for $i = 1, 2, \ldots, n$ such that the graph M_n is shown in Figure 1(b). Therefore, by Lemma 2.3

$$Z(LC_n) = Z(R(C_n)) + nZ(M_n) = 3^n + nZ(M_n). \tag{1}$$

For computing $Z(M_n)$, we consider the edge u_1u_n. Using Lemma 2.1(i) and Theorem 3.3, we have

$$\begin{aligned} Z(M_n) &= Z(M_n - \{u_1u_n\}) + Z(M_n - \{u_1, u_n\}) \\ &= Z(T_n) + Z(N_{n-2}) \\ &= 4 \times 3^{n-2} + Z(N_{n-2}), \end{aligned}$$

where the graph N_{n-2} is shown in Figure 1(c). By substituting for $Z(M_n)$ in the Equation 1, we get

$$Z(LC_n) = 3^n + 4 \times 3^{n-2} \times n + nZ(N_{n-2}).$$

Now, we need to obtain $Z(N_{n-2})$ for $n \geq 4$. The reduction process can be applied to N_{n-2}. Therefore, with deleting the edge v_1u_2 we get

$$\begin{aligned} Z(N_{n-2}) &= Z(N_{n-2} - \{v_1u_2\}) + Z(N_{n-2} - \{v_1, u_2\}) \\ &= Z(O_{n-2}) + Z(N_{n-3}), \end{aligned} \tag{2}$$

and for $n \geq 4$,

$$\begin{aligned} Z(O_{n-2}) &= Z(R(P_{n-2})) + Z(O_{n-3}) \\ &= 4 \times 3^{n-4} + Z(O_{n-3}). \end{aligned}$$

Substituting for $Z(O_{n-2})$ in Equation (2) yields

$$Z(N_{n-2}) = 4 \times 3^{n-4} + Z(O_{n-3}) + Z(N_{n-3}),$$

Then, by substituting for $Z(O_{n-3})$ using Equation 2, we get

$$Z(N_{n-2}) = 4 \times 3^{n-4} + 2Z(N_{n-3}) - Z(N_{n-4}).$$

According to the structure of graph N_{n-2} in Figure 1(c), graphs N_0 and N_1 are obtained from deleting n and $n-1$ vertices on cycle C_n in graph M_n. Consequently, $N_0 \simeq (n-1)K_1$ and N_1 consists of path P_3 and $n-2$ isolated vertices. Therefore, $Z(N_0) = 1$ and $Z(N_1) = Z(P_3) = 3$. □

The Dutch windmill graph $D_n^{(m)}$ is a created graph by m cycles C_n, with a common vertex. The graph $D_n^{(m)}$ is also known as the generalized friendship graph $F_{m,n}$ which contains $(n-1)m+1$ vertices. If $n = 3$, then $F_{m,3}$ is called a friendship graph [11].

Theorem 3.5. *The Hosoya index of $D_n^{(m)}$ is equal to*

$$Z(D_n^{(m)}) = \big(F_n\big)^{m-1}\Big(F_n + 2mF_{n-1}\Big),$$

in which F_n denotes the Fibonacci number.

Proof. Let x be the common vertex of m cycles of the graph $D_n^{(m)}$. Using Lemma 2.1,

$$Z(D_n^{(m)}) = Z(D_n^{(m)} - \{x\}) + \sum_{v_i \in N_{D_n^{(m)}}(x)} Z(D_n^{(m)} - \{x, v_i\}).$$

According to the structure of $D_n^{(m)}$, vertex x is adjacent to two vertices of each of the cycles in the graph $D_n^{(m)}$. Thus, it is easy to have

$$\begin{aligned} Z(D_n^{(m)}) &= Z(P_{n-1})^m + 2m\Big(Z(P_{n-1})^{m-1} \times Z(P_{n-2})\Big) \\ &= Z(P_{n-1})^{m-1}\Big(Z(P_{n-1}) + 2mZ(P_{n-2})\Big). \end{aligned}$$

Using Lemma 2.2(i), the result holds. □

Corollary 3.6. *For the friendship graph $F_{m,3}$, the index of Hosoya is equal to*

$$Z(F_{m,3}) = 2^m(m+1).$$

The n-Barbell graph, denoted by $B(2, n)$, is a graph obtained by joining two complete graphs K_n by a bridge [1].

Theorem 3.7. *For $n \geq 2$,*

$$Z(B(2,n)) = a_n^2 + a_{n-1}^2,$$

such that for $n \geq 3$,

$$a_n = a_{n-1} + (n-1)a_{n-2},$$

with initial conditions $a_1 = 1$ and $a_2 = 2$.

Proof. By considering the bridge edge uv and using Lemmas 2.1 and 2.4, we get

$$\begin{aligned} Z(B(2,n)) &= Z(B(2,n) - uv) + Z(B(2,n) - \{u,v\}) \\ &= Z(K_n)^2 + Z(K_{n-1})^2 \\ &= a_n^2 + a_{n-1}^2. \end{aligned}$$

Since $a_n = a_{n-1} + (n-1)a_{n-2}$ then, the result holds. □

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