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New Oscillation Results for a Nonlinear Generalization of Euler Differential Equation

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Abstract

In the present work the behavior of the solutions of a nonlinear generalization of Euler type equation will be considered. First, the equation will be transformed to an equivalent planar system. Then, the intersection of the solutions with the vertical isocline of the system will be proven. Some conditions will also be presented under which the positive semitrajectory of the system starting from a point on the characteristic curve does not tend to the origin. Finally, new sufficient conditions will be established ensuring oscillation of all solutions of this equation. Examples will also be provided to show the relevance of the main results.

Keywords: oscillation, Liénard system, Euler Equations

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1. Introduction

Originated from the monumental paper of Sturm ([18]), oscillation theory is now a very important branch of the theory of differential equations and dynamical systems which are related to the study of oscillatory phenomena in technology, natural, social and physical sciences. The issue of the theory is to investigate the properties of the solutions through the distribution analysis of zeros of the solutions of differential equations under consideration.

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In current manuscript the generalized nonlinear Euler equation

$$\big(t^2q'' + t(1+\phi(q))q'\big)\varrho(tq' + \Phi(q)) + \psi(q) = 0, \quad t > 0, \tag{1}$$

will be considered in which functions $\phi(q)$, $\varrho(q)$ and $\psi(q)$ satisfy the smoothness conditions and $\Phi(q) = \int_0^q \phi(\zeta)d\zeta$. Also, it is assumed that $\phi(\rho)$ and $\psi(\rho)$ satisfy the locally Lipschitz condition on $\mathbb{R}$ and

$$\varrho(\rho) > 0 \ \ and \ \ \rho\psi(\rho) > 0 \ \ if \ \ \rho \neq 0. \tag{2}$$

The oscillation of solutions of (1) will be discussed. If a nontrivial solution of (1) has arbitrarily large zeros, then it is called *oscillatory* and otherwise, the solution is called *nonoscillatory.*

Some explicit results for property (X^+) have been presented in [6] for the equivalent Liénard system of the following Euler type equation.

$$t^2\varrho(t\rho')\rho'' + t(\phi(\rho) + \varrho(t\rho'))\rho' + \psi(\rho) = 0, \quad t > 0.$$

Equation (1) can be reduced to

$$\big(q'' + q'\phi(q)\big)\varrho(q' + \Phi(q)) + \psi(q) = 0, \quad s \in \mathbb{R}, \tag{3}$$

by the change of variable $t = e^s$ where $' = \frac{d}{ds}$. The special cases of (3) have been widely studied for their interest for physical applications ([7] and [17]). Since Van der Pol's well known paper ([21]) many articles have been published about the existence, uniqueness and boundedness of the solutions, oscillation or multiplicity of periodic solutions.

Assuming $w = L(q' + \Phi(q))$, where $L(\rho) = \int_0^\rho \varrho(\zeta)d\zeta$, transfers equation (3) to

$$\begin{aligned} \dot{q} &= \Gamma(w) - \Phi(q) \\ \dot{w} &= -\psi(q), \end{aligned} \tag{4}$$

which is a generalization of the Liénard system in which $\Gamma(\rho) = L^{-1}(\rho)$. Note that $\Gamma(\rho)$ is strictly increasing since $\varrho(\rho) > 0$ and $L(\rho)$ is strictly increasing. Hereafter, s will be denoted by t again. The assumptions on the functions indicate that the unique critical point of (4) is origin. Many results about system (4) and its special cases can be found in the literatures ([1-24]).

To get the oscillatory results for (4), the intersection of all orbits with the vertical isocline $\Gamma(w) = \Phi(q)$ and negative w-axis will be first studied. Then, under quite general assumptions, some very sharp oscillation criteria for system (4) will be presented. The results are extension and improvements of the presented results in [2, 3, 4, 8, 9, 11, 12, 20, 22, 23].

Definition 1.1. Let $P(q_0, w_0)$ be an arbitrary point with $\Gamma(w_0) > \Phi(q_0)$ for $q_0 \geq 0$ (resp., $\Gamma(w_0) < \Phi(q_0)$ for $q_0 \leq 0$). If, passing through P, the positive semiorbit of (4) crosses the curve $\Gamma(w) = \Phi(q)$, then we say, in the right half-plane (RHP)(resp., in the left half-plane (LHP)), (4) has property (X^+).

To consider some results about this property see [1, 2, 3, 4, 8, 9, 10, 11, 19, 20, 22, 23]. Here, some of the previous results will be stated.

Hara, Yoneyama and Sugie ([11]) and Villari and Zanolin ([22]) presented some conditions for all positive semiorbits of system (4) with $\Gamma(w) = w$ to intersect the characteristic curve.

Aghajani and Moradifam [3] considered

$$\begin{aligned} \dot{q} &= \frac{1}{a(q)}[\Gamma(w) - \Phi(q)] \\ \dot{w} &= -a(q)\psi(q), \end{aligned} \tag{5}$$

and, under condition $\Gamma(w) \leq mw$ for $w < 0$, proved a theorem about the property (X^+) in RHP. Using the same approach, Gyllenberg and Yan ([8]) presented an extension theorem of the result in ([3]), which has already included the most of the previous results for system (4) with $\Gamma(q) = q$. Also, the authors in [2] considered system

$$\begin{aligned} \dot{q} &= \Gamma(w - \Phi(q)) \\ \dot{w} &= -\psi(q), \end{aligned} \tag{6}$$

and proved a theorem about the property (X^+) under weaker conditions. Recently, Yang, Kim and Lo in [24] used the same method and proved another theorem about the property (X^+) for system (4).

However, the mentioned and previous results in [2, 3, 8, 11, 22, 24] are not applicable to system

$$\dot{q} = m\sinh^{-1}(w) + m\ln(|q| + 1), \quad m > 0 \qquad \textit{and} \qquad \dot{w} = -q. \tag{7}$$

Within the present work, these results will be generalized to deal with system of the form of (7) (see Example 2.7).

2. Property (X^+)

Here, the outcomes of [3, 8] and the presented results in [2, 11, 22] will be extended and improved and a theorem for system (4) consistent with the result of [2] will be presented. Also, some criteria will be given which are applicable to (7).

Now, consider the following theorem which is more exact than the theorem in [24] in the conditions on functions Γ and ϱ.

Let $\Psi(q) = \int_0^q \psi(\zeta)d\zeta$. Changing variables $\rho = \sqrt{2\Psi(q)}sgn(q)$, $\sigma = w$, $d\iota = \frac{\psi(q)sgn(q)}{\sqrt{2\Psi(q)}}dt$ and denoting ι by t again, system (4) can be transformed to

$$\begin{aligned} \dot{\rho} &= \Gamma(\sigma) - \Phi^*(\rho) \\ \dot{\sigma} &= -\rho, \end{aligned} \tag{8}$$

where $\Phi^*(\rho) = \Phi(\Psi^{-1}(\frac{1}{2}\rho^2 sgn(\rho)))$ and the inverse of $\Psi(q)sgn(q)$ is shown by $\Psi^{-1}(w)$. Hara and Yoneyama in [10] showed that (4) and (8) are equivalent when $\Gamma(q) = q$. Similarly, it can be proven that this is also true for this system. Consequently, system (8) (instead of (4)) will be considered to determine whether it has the property (X^+) or not. Let

$$S(\Gamma, \Phi^*, \mu, \varrho) = \limsup_{\rho \to +\infty} \left(\int_\mu^\rho \frac{\Phi^*(s)\varrho'(s) + 2\sqrt{\varrho'(s)}\sqrt{s\varrho(s)\Gamma'(\Gamma^{-1}(\Phi^*(s)))}}{\varrho^2(s)} ds + \frac{\Phi^*(\rho)}{\varrho(\rho)} \right). \tag{9}$$

Theorem 2.1. *Let Γ' be increasing on $(-\infty, 0)$. Assume that $\varrho(t)$ with $\varrho'(t) > 0$ for $t \geq \gamma > 0$ and $\varrho(\eta) \geq 0$ for some $\eta \geq \gamma$ exists such that for some $\mu > 0$*

$$S(\Gamma, \Phi^*, \mu, \varrho) = +\infty. \tag{10}$$

Then, (8) *has property* (X^+) *in RHP.*

Proof. Let $p(\rho_0, \sigma_0)$ with $\Gamma(\sigma_0) > \Phi^*(\rho_0)$ be a point that the positive semiorbit of (8), starting from p, does not cross $\Gamma(\sigma) = \Phi^*(\rho)$. Suppose also that $\rho = \rho(t)$ and $\sigma = \sigma(t)$ are the solution of (8) passing through p for $t \in [0, \omega_+)$. It will be shown that

$$\lim_{t \to \omega_+} \rho(t) = +\infty. \tag{11}$$

Assume that (11) does not satisfied. In this case, $\lim_{t\to\omega_+} \rho(t) = \rho^* < +\infty$. Let $\bar{op^*}$ be the characteristic curve connecting the origin to $p^* = (\rho^*, \Gamma^{-1}(\Phi^*(\rho^*))) \in \{(\rho, \Gamma^{-1}(\Phi^*(\rho))) : \ \rho \geq 0\}$. Then, starting from p, the positive semiorbit of (8) is surrounded by $\bar{op^*}$, $\sigma = \sigma_0$, $\rho = \rho^*$ and $\sigma-$axis. Therefore, $\lim_{t\to\omega_+}(\rho(t), \sigma(t))$ have to exist and is a critical point of (8). Hence, $\rho^* = 0$ from the uniqueness of equilibrium of (8). However, $\rho(t) > \rho_0$ for $t > 0$ and so $\rho^* > \rho_0 \geq 0$. This contradiction implies that (11) is satisfied and therefore ρ_0 can be large enough that $\sigma_0 < 0$. Because $\sigma(t) \leq \sigma_0 - \rho_0 t < 0$ as $t \to \omega_+$. Hence, $\rho(t) \geq \rho_0 > 0$ and $\sigma(t) \leq \sigma_0 < 0$.

Now, for some $\delta \geq \eta$ define

$$A(t) = \int_\delta^{\varrho(\rho(t))} \frac{\Phi^*(\varrho^{-1}(s))}{s^2} ds + \frac{\Gamma(\sigma(t))}{\varrho(\rho(t))}.$$

Then,

$$\begin{aligned} \dot{A}(t) &= \frac{\dot{\rho}(t)\varrho'(\rho(t))\Phi^*(\rho(t))}{\varrho^2(\rho(t))} + \frac{\dot{\sigma}(t)\Gamma'(\sigma(t))\varrho(\rho(t)) - \dot{\rho}(t)\varrho'(\rho(t))\Gamma(\sigma(t))}{\varrho^2(\rho(t))} \\ &= \frac{-\rho(t)\Gamma'(\sigma(t))\varrho(\rho(t)) + \dot{\rho}(t)\varrho'(\rho(t))(\Phi^*(\rho(t)) - \Gamma(\sigma(t)))}{\varrho^2(\rho(t))} \\ &= \frac{-\rho(t)\Gamma'(\sigma(t))\varrho(\rho(t)) - (\dot{\rho}(t))^2\varrho'(\rho(t))}{\varrho^2(\rho(t))}. \end{aligned}$$

Since $\Phi^*(\rho(t)) < \Gamma(\sigma(t)) < 0$ for $t > 0$ and Γ' is increasing on $(-\infty, 0)$, the inequality $\Gamma'(\sigma(t)) \geq \Gamma'(\Gamma^{-1}(\Phi^*(\rho(t))))$ holds. Hence,

$$\begin{aligned} \dot{A}(t) \quad & \leq \frac{-\rho(t)\Gamma'(\Gamma^{-1}(\Phi^*(\rho(t))))\varrho(\rho(t)) - (\dot{\rho}(t))^2 \varrho'(\rho(t))}{\varrho^2(\rho(t))} \\ & \leq \frac{-2\dot{\rho}(t)\sqrt{\varrho'(\rho(t))}\sqrt{\rho(t)\Gamma'(\Gamma^{-1}(\Phi^*(\rho(t))))}}{\varrho^{\frac{3}{2}}(\rho(t))}. \end{aligned}$$

Thus,

$$\frac{d}{dt}\left(A(t) + \int_{\delta}^{\rho(t)} \frac{2\sqrt{\varrho'(s)}\sqrt{s\Gamma'(\Gamma^{-1}(\Phi^*(s)))}}{\varrho^{\frac{3}{2}}(s)} ds \right) \leq 0.$$

Therefore, for $t \geq 0$

$$\begin{aligned} & \int_{\delta}^{\varrho(\rho(t))} \frac{\Phi^*(\varrho^{-1}(s))}{s^2} ds + \frac{\Gamma(\sigma(t))}{\varrho(\rho(t))} + \int_{\delta}^{\rho(t)} \frac{2\sqrt{\varrho'(s)}\sqrt{s\Gamma'(\Gamma^{-1}(\Phi^*(s)))}}{\varrho^{\frac{3}{2}}(s)} ds \\ \leq & \int_{\delta}^{\varrho(\rho_0)} \frac{\Phi^*(\varrho^{-1}(s))}{s^2} ds + \frac{\Gamma(\sigma_0)}{\varrho(\rho_0)} + \int_{\delta}^{\rho_0} \frac{2\sqrt{\varrho'(s)}\sqrt{s\Gamma'(\Gamma^{-1}(\Phi^*(s)))}}{\varrho^{\frac{3}{2}}(s)} ds < +\infty. \end{aligned}$$

Since $\Gamma(\sigma(t)) > \Phi^*(\rho(t))$ and $\lim_{t \to \omega_+} \rho(t) = +\infty$,

$$\int_{\delta}^{\varrho(\rho(t))} \frac{\Phi^*(\varrho^{-1}(s))}{s^2} ds + \frac{\Phi^*(\rho(t))}{\varrho(\rho(t))} + \int_{\delta}^{\rho(t)} \frac{2\sqrt{\varrho'(s)}\sqrt{s\Gamma'(\Gamma^{-1}(\Phi^*(s)))}}{\varrho^{\frac{3}{2}}(s)} ds < +\infty.$$

Now, let $z = \varrho^{-1}(s)$ and $\mu = \max\{\delta, \varrho^{-1}(\delta)\}$. It can be concluded that

$$S(\Gamma, \Phi, \mu, \varrho) < +\infty.$$

The proof is complete with this contradiction. $\square$

Recalling the definition of $\Phi^*(\rho) = \Phi\big(\Psi^{-1}\big(\frac{1}{2}\rho^2\big)\big)$ for $\rho \geq 0$ and putting $q = \Psi^{-1}(\frac{1}{2}\rho^2)$, the following result for system (4) can be obtained.

Theorem 2.2. *Let $\Gamma'(q)$ be increasing for $q \in (-\infty, 0)$ and $\Psi(+\infty) = +\infty$. Suppose also that $\varrho(t)$ with $\varrho'(t) > 0$ for $t \geq \gamma > 0$ and $\varrho(\eta) \geq 0$ for some $\eta \geq \gamma$ exists such that for some $\mu > 0$*

$$\begin{aligned} \limsup_{q \to +\infty} \Bigg(\int_{\mu}^{q} \Bigg(& \frac{\Phi(\zeta)\varrho'(\sqrt{2\Psi(\zeta)})\psi(\zeta)}{\varrho^2(\sqrt{2\Psi(\zeta)})\sqrt{2\Psi(\zeta)}} + \frac{2\sqrt{\Gamma'(\Gamma^{-1}(\Phi(\zeta)))}\sqrt{\varrho'(\sqrt{2\Psi(\zeta)})\psi(\zeta)}}{\varrho^{3/2}(\sqrt{2\Psi(\zeta)})\sqrt[4]{2\Psi(\zeta)}} \Bigg) d\zeta \\ & + \frac{\Phi(q)}{\varrho(\sqrt{2\Psi(q)})} \Bigg) = +\infty. \end{aligned} \tag{12}$$

Then, (4) has property (X^+) in RHP.

Two corollaries from Theorem 2.2 are as follows.

Corollary 2.3. *Let $\Gamma'(q)$ be increasing for $q \in (-\infty, 0)$ and $\Psi(+\infty) = +\infty$. Suppose also that $\varrho(t)$ with $\varrho'(t) > 0$ for $t \geq \gamma > 0$ and $\varrho(\eta) \geq 0$ for some $\eta \geq \gamma$ exists such that for some $\mu > 0$*

$$\liminf_{q \to +\infty} \frac{\Phi(q)}{\varrho(\sqrt{2\Psi(q)})} > -\infty \quad and$$

$$\limsup_{q \to +\infty} \left(\int_\mu^q \left(\frac{\Phi(\zeta)\varrho'(\sqrt{2\Psi(\zeta)})\psi(\zeta)}{\varrho^2(\sqrt{2\Psi(\zeta)})\sqrt{2\Psi(\zeta)}} + \frac{2\sqrt{\Gamma'(\Gamma^{-1}(\Phi(\zeta)))}\sqrt{\varrho'(\sqrt{2\Psi(\zeta)})\psi(\zeta)}}{\varrho^{3/2}(\sqrt{2\Psi(\zeta)})\sqrt[4]{2\Psi(\zeta)}} \right) d\zeta = +\infty.$$

Then, (4) *has property* (X^+) *in RHP.*

Corollary 2.4. *Let $\Gamma'(q)$ be increasing for $q \in (-\infty, 0)$ and $\Psi(+\infty) = +\infty$. Suppose also that $\varrho(t)$ with $\varrho'(t) > 0$ for $t \geq \gamma > 0$, $\varrho(\eta) \geq 0$ for some $\eta \geq \gamma$ and $\int_\mu^\infty \frac{1}{\varrho(s)} ds = +\infty$ for some $\mu > 0$ exists such that*

$$\liminf_{q \to +\infty} \frac{\Phi(q)}{\varrho(\sqrt{2\Psi(q)})} > -\infty \quad and$$

$$\liminf_{q \to +\infty} \frac{\Phi(q)\sqrt{\varrho'(\sqrt{2\Psi(q)})}}{\sqrt{\Gamma'(\Gamma^{-1}(\Phi(q)))}\sqrt[4]{2\Psi(q)}\sqrt{\varrho(\sqrt{2\Psi(q)})}} > -2.$$

Then, (4) *has property* (X^+) *in RHP.*

Now, the following theorem will be stated which has useful applications.

Theorem 2.5. *Let $\Gamma_1(q) \leq \Gamma_2(q)$ for $q \in (0, \infty)$. Suppose also that* (4) *has property* (X^+) *in RHP with Γ_2. Then, it has the property with Γ_1 too.*

Proof. Assume that $O_1^+(p)$ and $O_2^+(p)$ are the positive semiorbit of (4) with $\Gamma_1(q)$ and $\Gamma_2(q)$ respectively started from $p(q_0, w_0)$ which lie in $D = \{(q, w) : \; q \geq 0 \; and \; \Gamma_i(w) > \Phi^*(q), i = 1, 2\}$. Suppose also that $O_1^+(p)$ does not cross $\Gamma_1(w) = \Phi(q)$. The following relation can be obtained since $\Gamma_1(q) \leq \Gamma_2(q)$ for $q > 0$.

$$\left(\frac{\dot{w}}{\dot{q}}\right)_{\Gamma_1} = \frac{-\psi(q)}{\Gamma_1(w) - \Phi(q)} \leq \frac{-\psi(q)}{\Gamma_2(w) - \Phi(q)} = \left(\frac{\dot{w}}{\dot{q}}\right)_{\Gamma_2} \leq 0.$$

Therefore, $O_1^+(p)$ has less slope than $O_2^+(p)$. Thus, $O_2^+(p)$ always remains above $O_1^+(p)$. Therefore, $O_2^+(p)$ does not cross $\Gamma_2(w) = \Phi(q)$ which is a contradiction. □

The following theorem can be proven by the same way.

Theorem 2.6. *Let Φ_1 and Φ_2 be decreasing and $\Phi_1(q) \leq \Phi_2(q)$ for $q \in (0, \infty)$. Suppose also that* (4) *has property* (X^+) *in RHP with Φ_1. Then, it has the property with Φ_2 too.*

In the following, with two examples, it will be shown that how our results improve the previous results listed in the introduction.

Example 2.7. Consider the system

$$\begin{aligned}\dot{q} &= m\sinh^{-1}(w) + m\ln(|q|+1) \quad \textit{with} \;\; m>0\\ \dot{w} &= -q.\end{aligned}$$

Here

$$\Gamma(w) = m\sinh^{-1}(w), \quad \Phi(q) = -m\ln(|q|+1) \quad \textit{and} \quad \psi(q) = q.$$

Note that, there is no $l>0$ such that $\Gamma(w)\le lw$ for $w<0$ with $|w|$ large enough. Therefore, the results of [2, 3, 8] and previous results in [11, 22] are not applicable to this system. Also, since $\Gamma'(w)=\frac{m}{\sqrt{w^2+1}}>0$ and Γ' is increasing just for $w<0$ (not for every w), the result of [24] is not applicable to this system. Now, let $\varrho(q)=q$. Then,

$$\begin{aligned}&\liminf_{q\to+\infty}\frac{\Phi(q)}{\varrho(\sqrt{2\Psi(q)})} = \liminf_{q\to+\infty}\frac{-m\ln(q+1)}{q} = 0 > -\infty \quad \textit{and}\\ &\liminf_{q\to+\infty}\frac{\Phi(q)\sqrt{\varrho'(\sqrt{2\Psi(q)})}}{\sqrt{\Gamma'(\Gamma^{-1}(\Phi(q)))}\sqrt[4]{2\Psi(q)}\sqrt{\varrho(\sqrt{2\Psi(q)})}}\\ &\qquad = \liminf_{q\to+\infty}\frac{-\sqrt{m}\ln(q+1)}{\sqrt{2q}} = 0 > -2.\end{aligned}$$

Hence, this system has property (X^+) in RHP by Corollary 2.4.

Example 2.8. Consider the system

$$\begin{aligned}\dot{q} &= \gamma\tan^{-1}(w) + \eta w + \delta q\cos^2 q\ln(|q|+1)\\ \dot{w} &= -q,\end{aligned}$$

where $\gamma,\eta,\delta>0$. Here

$$\begin{aligned}\Gamma(w) &= \gamma\tan^{-1}(w) + \eta w \quad \textit{with} \quad \gamma,\eta>0,\\ \Phi(q) &= -\delta q\cos^2 q\ln(|q|+1) \quad \textit{with} \quad \delta>0 \quad \textit{and} \quad \psi(q)=q.\end{aligned}$$

Notice that for every $m>0$,

$$\begin{aligned}&\limsup_{q\to+\infty}\left(\int_\mu^q\left(\frac{\Phi(\zeta)\psi(\zeta)}{(2\Psi(\zeta))^{\frac{3}{2}}} + \frac{\sqrt{m}\psi(\zeta)}{\Psi(\zeta)}\right)d\zeta + \frac{\Phi(q)}{\sqrt{2\Psi(q)}}\right)\\ &= \limsup_{q\to+\infty}\left(\int_\mu^q\left(\frac{-\delta\cos^2\zeta\ln(\zeta+1)+2\sqrt{m}}{\zeta}\right)d\zeta - \delta\cos^2 q\ln(q+1)\right) \ne +\infty.\end{aligned}$$

Therefore, the results of [2, 3, 8] are not applicable to this system. Also, since $\Gamma'(w) = \frac{\gamma}{1+w^2} + \eta > \eta$ and Γ' is increasing just for $w < 0$ (not for every w), the results of [24] is not applicable to this system. Now, choosing $\varrho(q) = \frac{q}{q+1}$, the following relation can be obtained.

$$\begin{aligned}
&\limsup_{q\to+\infty}\Bigg(\int_\mu^q\Bigg(\frac{\Phi(\zeta)\varrho'(\sqrt{2\Psi(\zeta)})\psi(\zeta)}{\varrho^2(\sqrt{2\Psi(\zeta)})\sqrt{2\Psi(\zeta)}}+\frac{2\sqrt{\Gamma'(\Gamma^{-1}(\Phi(\zeta)))}\sqrt{\varrho'(\sqrt{2\Psi(\zeta)})\psi(\zeta)}}{\varrho^{3/2}(\sqrt{2\Psi(\zeta)})\sqrt[4]{2\Psi(\zeta)}}\Bigg)d\zeta\\
&\qquad+\frac{\Phi(q)}{\varrho(\sqrt{2\Psi(q)})}\Bigg)\\
&\geq \limsup_{q\to+\infty}\Bigg(\int_\mu^q\Bigg(\frac{-\delta\cos^2 s\ln(s+1)+2\sqrt{\gamma}\sqrt{s+1}}{s}\Bigg)ds-\delta\cos^2 q\ln(q+1)\Bigg)\\
&\geq \limsup_{q=2k\pi+\frac{\pi}{2}\to+\infty}\Bigg(\int_\mu^q\Bigg(\frac{-\delta\ln(s+1)+2\sqrt{\gamma}\sqrt{s+1}}{s}\Bigg)ds\Bigg)=+\infty.
\end{aligned}$$

Hence, this system has property (X^+) in RHP by Theorem 2.2.

In this part, a theorem and some corollaries about the property of (X^+) in LHP will be presented.

Theorem 2.9. *Let $\Gamma'(q)$ be decreasing for $q \in (0,\infty)$ and $\Psi(-\infty) = +\infty$. Suppose also that $\varrho(t)$ with $\varrho'(t) > 0$ for $t \leq \gamma < 0$ and $\varrho(\eta) \leq 0$ for some $\eta \leq \gamma$ exists such that for some $\mu < 0$*

$$\begin{aligned}
&\liminf_{q\to-\infty}\Bigg(\int_q^\mu\Bigg(-\frac{\Phi(\zeta)\varrho'(-\sqrt{2\Psi(\zeta)})\psi(\zeta)}{\varrho^2(-\sqrt{2\Psi(\zeta)})\sqrt{2\Psi(\zeta)}}\\
&\qquad+\frac{2\sqrt{\Gamma'(\Gamma^{-1}(\Phi(\zeta))\varrho'(-\sqrt{2\Psi(\zeta)})\psi(\zeta)}}{(-\varrho(-\sqrt{2\Psi(\zeta)}))^{3/2}\sqrt[4]{2\Psi(\zeta)}}\Bigg)d\zeta-\frac{\Phi(q)}{\varrho(-\sqrt{2\Psi(q)})}\Bigg)=-\infty.
\end{aligned}\tag{13}$$

Then, (4) has property (X^+) in LHP.

Corollary 2.10. *Let Γ' be decreasing on $(0,\infty)$ and $\Psi(-\infty) = +\infty$. Suppose also that $\varrho(t)$ with $\varrho'(t) > 0$ for $t \leq \gamma < 0$ and $\varrho(\eta) \leq 0$ for some $\eta \leq \gamma$ exists such that for some $\mu < 0$*

$$\begin{aligned}
&\limsup_{q\to-\infty}\frac{-\Phi(q)}{\varrho(-\sqrt{2\Psi(q)})}<+\infty \quad \textit{and}\\
&\liminf_{q\to-\infty}\Bigg(\int_q^\mu\Bigg(-\frac{\Phi(\zeta)\varrho'(-\sqrt{2\Psi(\zeta)})\psi(\zeta)}{\varrho^2(-\sqrt{2\Psi(\zeta)})\sqrt{2\Psi(\zeta)}}\\
&\qquad+\frac{2\sqrt{\Gamma'(\Gamma^{-1}(\Phi(\zeta))\varrho'(-\sqrt{2\Psi(\zeta)})\psi(\zeta)}}{(-\varrho(-\sqrt{2\Psi(\zeta)}))^{3/2}\sqrt[4]{2\Psi(\zeta)}}\Bigg)d\zeta=-\infty.
\end{aligned}$$

Then, (4) has property (X^+) in LHP.

Corollary 2.11. *Let Γ' be decreasing on $(0,\infty)$ and $\Psi(-\infty) = +\infty$. Suppose also that $\varrho(t)$ with $\varrho'(t) > 0$ for $t \le \gamma < 0$, $\varrho(\eta) \le 0$ for some $\eta \le \gamma$ and $\int_{-\infty}^{\mu} \frac{1}{\varrho(s)} ds = -\infty$ for some $\mu < 0$ exists such that*

$$\limsup_{q \to -\infty} \frac{-\Phi(q)}{\varrho(-\sqrt{2\Psi(q)})} < +\infty \quad \text{and}$$

$$\limsup_{q \to -\infty} \frac{\Phi(q)\sqrt{\varrho'(-\sqrt{2\Psi(q)})}}{\sqrt{-\varrho(-\sqrt{2\Psi(q)})\Gamma'(\Gamma^{-1}(\Phi(q))}\sqrt[4]{2\Psi(q)}} < 2.$$

Then, (4) has property (X^+) in LHP.

3. Property (Z^+) and Oscillation

In the sequel, some oscillation criteria for system (4) will be found. First, some conditions will be presented under which, starting from $P(q_0, w_0)$ with $\Gamma(w_0) = \Phi(q_0)$ and $q_0 \ge 0$ (resp., $q_0 \le 0$), the positive semiorbit of (4) does not approach to the origin through the first (resp., third) quadrant but intersects the negative (resp., positive) w-axis.

Definition 3.1. System (4) has property (Z_1^+) (resp., (Z_3^+)) if there exists a point $P(q_0, w_0)$ with $q_0 \ge 0$ (resp., $q_0 \le 0$) on the curve $\Gamma(w_0) = \Phi(q_0)$ such that, starting at P, the positive semitrajectory of (4) tends to the origin through only the first (resp., third) quadrant.

To consider results about the property (Z_1^+) see [1, 4, 19]. Now, consider the following theorem and corollary about the property (Z_1^+) which are proven in [4].

Theorem 3.2. ([4]) *Let $\Phi > 0$ on $(0, \varepsilon)$ for some $\varepsilon > 0$. Suppose also that for any $\varrho \in [0,1]$ constant $\delta_\varrho > 0$ exists such that*

$$\liminf_{q \to 0^+} \left(\frac{\int_0^q \frac{\psi(\zeta)}{\Phi(\zeta)} d\zeta}{(1 - \varrho + \delta_\varrho)\Gamma^{-1}((\varrho + \delta_\varrho)\Phi(q))} \right) > 1. \tag{14}$$

Then, (4) does not have property (Z_1^+).

Corollary 3.3. ([4]) *Let $\Phi > 0$ on $(0, \varepsilon)$ for some $\varepsilon > 0$. Suppose also that $\xi \in (1, 2]$ exists such that*

$$\liminf_{q \to 0^+} \left(\frac{\int_0^q \frac{\psi(\zeta)}{\Phi(\zeta)} d\zeta}{\Gamma^{-1}(\xi\Phi(q))} \right) > 1. \tag{15}$$

Then, (4) does not have property (Z_1^+).

Using Theorem 3.2, the following lemma will be proven about the asymptotic behavior of solutions of (4) which is needed in the main theorem.

Lemma 3.4. *Assume that one of the following conditions hold.*

(i) $\Phi < 0$ *on* $(0, \varepsilon)$ *for some* $\varepsilon > 0$ *or* $\Phi(q)$*, clustering at* $q = 0$*, has infinite number of positive zeroes.*

(ii) *Conditions of Theorem* 3.2 *hold if* $\Phi > 0$ *on* $(0, \varepsilon)$ *for some* $\varepsilon > 0$.

Then, starting from P*, the positive semiorbit of* (4) *intersects the negative* w*-axis for each point* $P(p, \Gamma^{-1}(\Phi(p)))$ *with* $p > 0$.

Proof. Let $p > 0$ and $P(p, \Gamma^{-1}(\Phi(p)))$ be a point that, starting from P, the positive semiorbit of (4) does not cross the negative w-axis. Suppose also that $(q(t), w(t))$ be the solution of (4) on $[t_0, \infty)$ with $(q(t_0), w(t_0)) = P$. Then, starting from P, the positive semiorbit of (4) corresponds to the solution $(q(t), w(t))$. Taking into account the vector field of (4), it can be verified that

$$0 < q(t) \leq q(t_0) \quad for \quad t \geq t_0.$$

On the other hand, if condition **(i)** holds, then $w(t) \to -\infty \quad as \quad t \to +\infty$ and in the case of condition **(ii)**, by (14) (or (15)), this system fails to have property (Z_1^+). Therefore, $w(t) \nrightarrow 0$ as $t \to +\infty$ and so $\lim_{t\to+\infty} w(t) = -\infty$. Thus, in the both cases:

$$\lim_{t\to+\infty} w(t) = -\infty.$$

Therefore, from $\Gamma(-\infty) = -\infty$ and the first equation of (4), $\lim_{t\to+\infty} \dot{q}(t) = -\infty$. Hence, $t_1 > t_0$ exists such that $\dot{q}(t) \leq -1$ for $t \geq t_1$. Thus,

$$-q(t_1) < q(t) - q(t_1) \leq t_1 - t \to -\infty \quad as \quad t \to \infty.$$

The proof is complete with this contradiction. □

Similarly, changing of variables $(q, w) \longrightarrow (-q, -w)$, Theorem 3.2 and Corollary 3.3 can be formulated for the property (Z_3^+) as follows.

Theorem 3.5. *Let* $\Phi < 0$ *on* $(-\varepsilon, 0)$ *for some* $\varepsilon > 0$*. Suppose also that for any* $\varrho \in [0, 1]$ *a constant* $\delta_\varrho > 0$ *exists such that*

$$\liminf_{q\to 0^-} \left(\frac{\int_0^q \frac{\psi(\zeta)}{\Phi(\zeta)} d\zeta}{(1 - \varrho + \delta_\varrho)\Gamma^{-1}((\varrho + \delta_\varrho)\Phi(q))} \right) > 1. \tag{16}$$

Then, (4) *does not have property* (Z_3^+).

Corollary 3.6. *Let $\Phi < 0$ on $(-\varepsilon, 0)$ for some $\varepsilon > 0$. Suppose also that $\xi \in (1, 2]$ exist such that*

$$\liminf_{q\to 0^-} \left(\frac{\int_0^q \frac{\psi(\zeta)}{\Phi(\zeta)} d\zeta}{\Gamma^{-1}(\xi\Phi(q))} \right) > 1. \tag{17}$$

Then, (4) *does not have property* (Z_3^+).

By the same method as adopted for the proof of Lemma 3.4, the following lemma can be proven.

Lemma 3.7. *Assume that one of the following conditions hold.*

(i) $\Phi > 0$ *on* $(-\varepsilon, 0)$ *for some* $\varepsilon > 0$ *or* $\Phi(q)$*, clustering at* $q = 0$*, has infinite number of negative zeroes.*

(ii) *Conditions of Theorem* 3.5 *hold if* $\Phi < 0$ *on* $(-\varepsilon, 0)$ *for some* $\varepsilon > 0$.

Then, starting from P, the positive semiorbit of (4) *intersects the positive w-axis for each point $P(-p, \Gamma^{-1}(\Phi(-p)))$ with $p > 0$.*

Now, applying Theorems 2.2, 2.9 and Lemmas 3.4, 3.7, we state our main result.

Theorem 3.8. *Let Γ' be increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$ and $\Psi(\pm\infty) = +\infty$. Suppose also that* (12) *and* (13) *hold for some $\varrho(t)$ and $\varrho_1(t)$ satisfying the conditions of Theorems* 2.2 *and* 2.9*, respectively, and the conditions of Lemmas* 3.4 *and* 3.7 *hold. Then, all nontrivial solutions of* (4) *are oscillatory.*

Example 3.9. Consider the following system

$$\begin{aligned} \dot{q} &= \varsigma \tanh(w) + \delta w + \gamma q - \eta q \cos^2(q) \\ \dot{w} &= -q, \end{aligned} \tag{18}$$

where $\varsigma, \delta, \eta > 0$, $0 < \gamma < 2\sqrt{\delta}$ and $\delta + \varsigma > (\gamma - \eta)^2$. Here

$$\begin{aligned} &\Gamma(w) = \varsigma \tanh(w) + \delta w \quad \textit{with} \quad \varsigma, \delta > 0, \\ &\Phi(q) = -\gamma q + \eta q \cos^2(q) \quad \textit{with} \quad 0 < \gamma < 2\sqrt{\delta},\ \eta > 0 \quad \textit{and} \quad \psi(q) = q. \end{aligned}$$

Notice that $\Gamma'(w) = \frac{\varsigma}{\cosh^2(w)} + \delta > \delta$. Now, take $\varrho(q) = q$ and therefore,

$$\begin{aligned} &\liminf_{|q|\to\infty} \frac{\Phi(q)}{\varrho(\sqrt{2\Psi(q)})} = \liminf_{|q|\to\infty} \frac{-\gamma q + \eta q \cos^2(q)}{q} = -\gamma > -\infty \quad \textit{and} \\ &\liminf_{q\to+\infty} \frac{\Phi(q)\sqrt{\varrho'(\sqrt{2\Psi(q)})}}{\sqrt{\Gamma'(\Gamma^{-1}(\Phi(q)))}\sqrt[4]{2\Psi(q)}\sqrt{\varrho(\sqrt{2\Psi(q)})}} \geq \liminf_{q\to+\infty} \frac{-\gamma + \eta \cos^2 q}{\sqrt{\delta}} \\ &\quad = \frac{-\gamma}{\sqrt{\delta}} > -2. \end{aligned}$$

Therefore, applying Corollary 2.4, the system has property (X^+) in RHP. On the other hand,

$$\limsup_{q\to-\infty}\frac{\Phi(q)\sqrt{\varrho'(-\sqrt{2\Psi(q)})}}{\sqrt{-\varrho(-\sqrt{2\Psi(q)})\Gamma'(\Gamma^{-1}(\Phi(q))}\sqrt[4]{2\Psi(q)}}\le\limsup_{q\to-\infty}\frac{\gamma-\eta\cos^2 q}{\sqrt{\delta}}=\frac{\gamma}{\sqrt{\delta}}<2.$$

Therefore, applying Corollary 2.11, this system has property (X^+) in LHP. Now, if $\gamma\ge\eta$, then $\Phi\le 0$ on $(0,\infty)$ and $\Phi\ge 0$ on $(-\infty,0)$. If $\gamma<\eta$, then $\varepsilon>0$ exists such that $\Phi>0$ on $(0,\varepsilon)$ and $\Phi<0$ on $(-\varepsilon,0)$. Therefore,

$$\lim_{|q|\to0}\left(\frac{\int_0^q\frac{\psi(\zeta)}{\Phi(\zeta)}d\zeta}{\Gamma^{-1}(\xi\Phi(q))}\right)=\lim_{|q|\to0}\frac{\Gamma'(\Gamma^{-1}(\xi\Phi(q)))}{(\gamma-\eta\cos^2 q)\xi(\gamma-\eta\cos^2 q+\eta q\sin 2q)}=\frac{\delta+\varsigma}{(\gamma-\eta)^2\xi}.$$

Thus, since $\delta+\varsigma>(\gamma-\eta)^2$, $\xi\in(1,2]$ exists satisfying (15) and (17). Therefore, the conditions of Lemmas 3.4 and 3.7 hold and, all nontrivial solutions of this system are oscillatory by Theorem 3.8. The phase portrait of the solutions of this system is plotted for parameter values $\gamma=\varsigma=1$, $\eta=2$ and $\delta=4$ in Fig. 1.

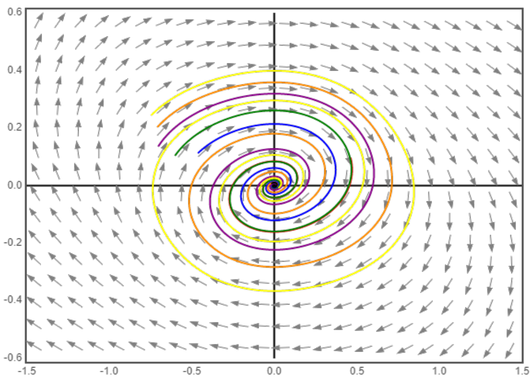

Figure 1: Portrait of system (18).

Remark 1. All obtained results for system (4) can be formulated to the same results for Euler Equation (1).

4. Conclusion

In this work the oscillatory behavior of solutions of the nonlinear generalization of Euler equation has been considered. Some new conditions have been established ensuring oscillation of all solutions of the equation. Examples have also been provided to illustrate the relevance of the main results.

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