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Hopf-Zero Bifurcation in Three-Cell Networks with Two Discrete Time Delays

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Abstract

In this paper, we study a delayed three-cell network which is introduced by coupled cell theory and neural network theory. We investigate this model with two different discrete delays. The aim is to obtain necessary conditions for the stability and the existence of Hopf-zero bifurcation in this model. Moreover, we find the normal form of this bifurcation by using linearization and the Multiple Time Scale method. Finally, the theoretical results are verified by numerical simulations.

Keywords: coupled cell theory, neural network, stability, Hopf-zero bifurcation, normal form

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1. Introduction

In the past decades, the complex network have attracted more and more attention from researchers, [8, 15, 25]. It is partially due to the fact that any large-scale and complicated system in real world can be modeled by a complex network, in which nodes can be seen as the elements of the system and edges can be considered as interactions between nodes, [2, 7, 14]. For example, the WWW, the internet,

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neural networks, social networks are all complex networks. To master the complicated nature of complex networks, more and more people began to investigate the dynamics of complex networks, such as robustness, pinning synchronization, [4, 10, 13]. The dynamical behavior (including stability, periodic oscillatory, and chaos) of continuous-time neural networks has received much attention due to their applications in optimization, signal processing, image processing, solving nonlinear algebraic equations, pattern recognition, associative memories, see [16, 17, 29] and references therein. It is well known that time delays in the information processing of neurons exist. As time delays always occur in the signal transmission, Marcus and Westervelt proposed an NN model with delay. Some dynamical behaviors such as periodic phenomenon, bifurcation, and chaos have been discussed in these systems. Moreover, it should be mentioned that only some researchers studied delayed neural network models with high codimension bifurcations because of complicated computations, see [1, 8, 12].

Most phenomena are complex networks that sometimes simulate as a differential equations system. For example, some neurological diseases, gene expression, homeostasis and other phenomena have been modeled and analyzed by using differential equation theory, [3, 11, 22]. Moreover, we know that interaction between elements of a phenomenon is not instantaneous. The delay in interactions is known as time delay which is constant or nonconstant. Also, it should be noted that the delay in any interaction may be different especially in networks with different input and output data/signals. Therefore, researchers considered time delay in realistic models of neural networks, [21, 23, 24].

In order to simplify the analysis of the dynamical behavior of differential equations near an equilibrium, normal form theory has been widely used. The basic idea of the method of normal forms is to employ successive coordinate transformations to systematically construct a simplified form of the original differential equations without changing the fundamental dynamical behavior of the system in the vicinity of the equilibrium, [5, 28].

Multiple Time Scale and center manifold are methods to compute normal forms of a differential equations system, [19, 26]. Nayfeh, [20] and Ding et al, [8] used both techniques in two different models with delay and showed that these techniques are equivalent. We would like to obtain a normal form of Hopf-zero bifurcation in a three-cell network with different delays in inputs and outputs, see Figure1. Delays in clockwise and counterclockwise directions are τ_1, τ_2, respectively.

Three-dimensional delay differential equations systems have attracted the attention of many researchers, see [6, 18, 27]. They are applicable in real models; Gopalsamy neural network model, Oregonator oscillator, Fitzhugh-Nagumo model, cell signaling networks, and so on. In this paper, we will investigate the following generalized Gopalsamy neural network model that is studied in [9, 18]:

$$\begin{cases} \dot{X}_1 = -X_1(t) + a_1 \tanh[X_3(t) - bX_3(t-\tau)], \\ \dot{X}_2 = -X_2(t) + a_2 \tanh[X_1(t) - bX_1(t-\tau)], \\ \dot{X}_3 = -X_3(t) + a_3 \tanh[X_2(t) - bX_2(t-\tau)], \end{cases}$$

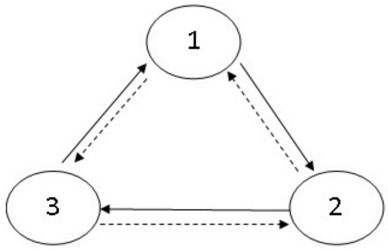

Figure 1: The architecture of a three-cell network with two different delays.

where a_i $(i = 1, 2, 3)$ corresponds to the range of the continuous variable X_i, $b > 0$ is the measure of the inhibitory influence of the past history, and $\tau > 0$ is a time delay.

In fact, we will consider the following system for our network (Figure 1)

$$\begin{cases} \dot{X}_1 = f(X_1) + g(X_2(t-\tau_2), X_3(t-\tau_1)), \\ \dot{X}_2 = f(X_2) + g(X_1(t-\tau_1), X_3(t-\tau_2)), \\ \dot{X}_3 = f(X_3) + g(X_1(t-\tau_2), X_2(t-\tau_1)). \end{cases} \tag{1}$$

This model with different delays is the first time to be considered for studying its dynamical behaviors. We will use Multiple Time Scale method to derive the normal form of Hopf-zero bifurcation of system (1). Furthermore, we will carry out other bifurcations occurrence possibilities in the model.

The rest of this paper is organized as follows. In Section 2, we investigate the existence of Hopf-zero bifurcation in three-cell networks. In Section 3, the normal form of Hopf-zero bifurcation in a three-cell network is obtained by using the method of Multiple Time Scales (MTS). In Section 4, simulations are given to illustrate the effectiveness of the proposed method. Finally, a brief discussion is given to conclude the work of this paper.

2. Existence of Hopf-Zero Bifurcation in a Three-Cell Network

In this section, we consider the general form of three-cell networks with delays τ_1 and τ_2, as mentioned in the previous section. System (1) and Figure 1 show our model and its network, where f and g are analytic functions in a neighborhood of zero. Our aim is to study the stability of (1). For this purpose, let $X = (0, 0, 0)$ be an equilibrium point of (1) and $f(0) = 0$, $g(0, 0) = 0$. Then the Taylor expansion of $f(X_i)$ around the equilibrium point is written as follows:

$$\begin{cases} f(X_1) = X_1 f'(0) + O(2), \\ f(X_2) = X_2 f'(0) + O(2), \\ f(X_3) = X_3 f'(0) + O(2). \end{cases}$$

Also, the Taylor expansion of g is

$$\begin{cases} g(X_2(t-\tau_2), X_3(t-\tau_1)) = g_{X_2}(0)X_2(t-\tau_2) + g_{X_3}(0)X_3(t-\tau_1) + O(2), \\ g(X_1(t-\tau_1), X_3(t-\tau_2)) = g_{X_1}(0)X_1(t-\tau_1) + g_{X_3}(0)X_3(t-\tau_2) + O(2), \\ g(X_1(t-\tau_2), X_2(t-\tau_1)) = g_{X_1}(0)X_1(t-\tau_2) + g_{X_2}(0)X_2(t-\tau_1) + O(2). \end{cases}$$

Then, the linearization of system (1) is

$$\begin{cases} \dot{X}_1 = X_1(t)f'(0) + g_{X_2}(0)X_2(t-\tau_2) + g_{X_3}(0)X_3(t-\tau_1), \\ \dot{X}_2 = X_2(t)f'(0) + g_{X_1}(0)X_1(t-\tau_1) + g_{X_3}(0)X_3(t-\tau_2), \\ \dot{X}_3 = X_3(t)f'(0) + g_{X_1}(0)X_1(t-\tau_2) + g_{X_2}(0)X_2(t-\tau_1). \end{cases}$$

In the other words,

$$\dot{X} = AX + BX_{\tau_1} + CX_{\tau_2}. \tag{2}$$

where

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, X_{\tau_1} = \begin{pmatrix} X_1(t-\tau_1) \\ X_2(t-\tau_1) \\ X_3(t-\tau_1) \end{pmatrix}, X_{\tau_2} = \begin{pmatrix} X_1(t-\tau_2) \\ X_2(t-\tau_2) \\ X_3(t-\tau_2) \end{pmatrix},$$

$$A = f'(0)I_3, B = \begin{pmatrix} 0 & 0 & g_{X_3}(0) \\ g_{X_1}(0) & 0 & 0 \\ 0 & g_{X_2}(0) & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & g_{X_2}(0) & 0 \\ 0 & 0 & g_{X_3}(0) \\ g_{X_1}(0) & 0 & 0 \end{pmatrix}. \tag{3}$$

For investigating the stability of the equilibrium point of system (1), it is sufficient to compute,

$$\det[A + Be^{-\lambda\tau_1} + Ce^{-\lambda\tau_2} - \lambda I] = 0. \tag{4}$$

Hence, the characteristic equation (4) is as follows

$$\begin{aligned} p(\lambda, \tau_1, \tau_2) &= (-\lambda^3 + 3\lambda^2 f'(0) - 3\lambda f'^2(0) + f'^3(0)) \\ &+ (\lambda - f'(0))e^{-\lambda(\tau_1+\tau_2)}[g_{X_2}g_{X_3} + g_{X_1}g_{X_2} + g_{X_1}g_{X_3}] \\ &+ g_{X_1}g_{X_2}g_{X_3}(e^{-3\lambda\tau_1} + e^{-3\lambda\tau_2}) = 0, \end{aligned}$$

such that $g_{X_i}{}'s$ are used to show $g_{X_i}(0)'s$, i=1,2,3. For simplifying, let $g_{X_2} = 0$, $g_{X_1}g_{X_3} \neq 0$ and $\tau_2 := \delta\tau_1, \quad \delta > 0$. Therefore

$$p(\lambda, \tau) = -\lambda^3 + 3\lambda^2 f'(0) - 3\lambda f'^2(0) + f'^3(0) + (\lambda - f'(0))(g_{X_1}g_{X_3}e^{-(\delta+1)\lambda\tau_1}) = 0. \tag{5}$$

Now, we suppose that $\lambda = i\omega$ $(\omega > 0)$ and we get the following equation

$$i\omega^3 - 3\omega^2 f'(0) - 3i\omega f'^2(0) + f'^3(0) + (i\omega g_{X_1} g_{X_3} - f'(0) g_{X_1} g_{X_3}) e^{-i(\delta+1)\omega\tau_1} = 0. \quad (6)$$

Equation (6) equals to the following system,

$$\begin{cases} \omega^3 - 3\omega f'^2(0) + \omega g_{X_1} g_{X_3} \bar{c}(\omega) - f'(0) g_{X_1} g_{X_3} \bar{s}(\omega) = 0, \\ -3\omega^2 f'(0) + f'^3(0) - \omega g_{X_1} g_{X_3} \bar{s}(\omega) - f'(0) g_{X_1} g_{X_3} \bar{c}(\omega) = 0, \end{cases} \quad (7)$$

where

$$\bar{c}(\omega) := \cos(-(\delta+1)\omega\tau_1),$$
$$\bar{s}(\omega) := \sin(-(\delta+1)\omega\tau_1).$$

Therefore

$$\tau_1^j = \frac{1}{(\delta+1)\omega}[\arcsin[\frac{2\omega f'(0)}{g_{X_1} g_{X_3}}] + 2\pi j], \quad j \in Z; \qquad \tau_1^j >= 0.$$

One can obtain the following equation by using system (7),

$$\overline{p}(\omega) = \omega^8 + a_2\omega^6 + a_4\omega^4 + a_6\omega^2 + a_8,$$

where

$$\begin{aligned} a_2 &= 4f'^2(0), \\ a_4 &= 6f'^4(0) - g_{X_1}^2 g_{X_3}^2, \\ a_6 &= 4f'^6(0) - 2f'^2(0) g_{X_1}^2 g_{X_3}^2, \\ a_8 &= f'^8(0) - f'^4(0) g_{X_1}^2 g_{X_3}^2. \end{aligned} \quad (8)$$

If $\Omega := \omega^2$, then

$$\overline{p}(\Omega) = \Omega^4 + \overline{a}_1\Omega^3 + \overline{a}_2\Omega^2 + \overline{a}_3\Omega + \overline{a}_4, \quad (9)$$

such that

$$\bar{a}_1 = a_2, \qquad \bar{a}_2 = a_4, \qquad \bar{a}_3 = a_6, \qquad \bar{a}_4 = a_8.$$

By using the Routh-Hurwitz criterion, we have the following theorem.

Theorem 2.1. *Equation* (9) *is stable if and only if*

$$\overline{a}_1 > 0, \qquad \overline{a}_3 > 0, \qquad \overline{a}_4 > 0, \qquad \overline{a}_1\overline{a}_2\overline{a}_3 > \overline{a}_3^2 + \overline{a}_1^2 a_4.$$

Proof. It is clear. □

By using Equations (8), if $f'(0) = 0$, then $\overline{a}_1 = \overline{a}_3 = \overline{a}_4 = 0$. Therefore, $\overline{p}(\Omega) = 0$ has roots $\Omega = 0$ and $\Omega = \pm g_{X_1} g_{X_3}$. We assume that $g_{X_1} g_{X_3} \neq 0$ thus Equation (9) has only one positive root, $\Omega = g_{X_1} g_{X_3}$ or $\Omega = -g_{X_1} g_{X_3}$.

According to the above discussion, we can state the following theorem.

Theorem 2.2. *System* (1) *satisfies the following results,*

1. *If* $f'(0) = 0$ *or* $f'^2(0) = g_{X_1}g_{X_3}$*, then* $\lambda = 0$ *is the root of Equation* (5)*.*
2. *Equation* (9) *has a positive root or in the other words,* $\lambda = i\omega, (\tau = \tau_1^0)$ *is the root of Equation* (5)*, if*
$$\sim \overline{a}_1 > 0 \quad \vee \quad \sim \overline{a}_3 > 0 \quad \vee \quad \overline{a}_4 < 0 \quad \vee \quad \sim \overline{a}_1\overline{a}_2\overline{a}_3 > \overline{a}_3^2 + \overline{a}_1^2 a_4.$$
3. $\lambda = i\omega$ *is a simple root of the characteristic equation* (5)*, if*
$$\begin{cases} 3\omega^2 - 3f'(0) + g_{X_1}g_{X_3}\bar{c}(\omega) + (\delta+1)\omega\tau_1 g_{X_1}g_{X_3}\bar{s}(\omega) - (\delta+1)\tau_1 f'(0)g_{X_1}g_{X_3}\bar{c}(\omega) \neq 0, \\ -6\omega f'(0) + g_{X_1}g_{X_3}\bar{s}(\omega) + (\delta+1)\omega\tau_1 g_{X_1}g_{X_3}\bar{c}(\omega) - (\delta+1)\tau_1 f'(0)g_{X_1}g_{X_3}\bar{s}(\omega) \neq 0. \end{cases} \tag{10}$$
4. *If* $f'(0) = 0$*,* $\tau = \tau_1^0$ *and the condition* (10) *hold, then* $\lambda = 0$ *and* $\lambda = i\omega$ *are simple roots of the characteristic equation* (5)*.*

Proof.

1. We suppose that $\lambda = 0$ is a root of Equation (5). Then
$$f'^3(0) - f'(0)g_{X_1}g_{X_3} = 0,$$
which it implies $f'(0) = 0$ or $f'^2(0) = g_{X_1}g_{X_3}$.
2. Equation (5) has $\lambda = i\omega$, $\omega \neq 0$, as a root, if and only if Equation (9) has a positive root. This means that Equation (9) is not stable and has no zero root. Also, Equation (9) has not zero root if and only if $\overline{a}_4 \neq 0$. By using Routh-Hurwitz criterion, Equation (9) is not stable if and only if
$$\sim \overline{a}_1 > 0 \quad \vee \quad \sim \overline{a}_3 > 0 \quad \vee \quad \sim \overline{a}_4 > 0 \quad \vee \quad \sim \overline{a}_1\overline{a}_2\overline{a}_3 > \overline{a}_3^2 + \overline{a}_1^2 a_4.$$
Therefore, Equation (5) has $\lambda = i\omega$, $\omega \neq 0$, if and only if
$$\sim \overline{a}_1 > 0 \quad \vee \quad \sim \overline{a}_3 > 0 \quad \vee \quad \overline{a}_4 < 0 \quad \vee \quad \sim \overline{a}_1\overline{a}_2\overline{a}_3 > \overline{a}_3^2 + \overline{a}_1^2 a_4.$$
3. It is obvious, because $\lambda = i\omega$ is simple root of $p(\lambda, \tau) = 0$ if and only if $p'(\lambda = i\omega) \neq 0$.
4. By case 3, we know that $\lambda = i\omega$ is the simple root if condition (10) holds. Moreover, we know that $p'(0, \tau) = g_{X_1}g_{X_3}$where $g_{X_1}g_{X_3} \neq 0$. It is obvious that $\lambda = 0$ is also a simple root.

□

Theorem 2.2 helps us to identify different bifurcations occurrence possibilities. Therefore, we have the following corollary.

Corollary 2.3. *Consider the cases of Theorem* 2.2*, therefore*

1. *Saddle-Node bifurcation, pitchfork bifurcation, transcritical bifurcation or Takens-Bogdanov bifurcation can occur, if case* 1 *holds.*
2. *In case* 2*, it is possible to occur multiple Hopf bifurcation.*
3. *Simple Hopf bifurcation will happen if case* 3 *is established.*
4. *Hopf-zero bifurcation occurs if case* 4 *is established.*

3. Normal Form of Hopf-Zero Bifurcation in a Three-Cell Network

In this section, we will compute the normal form of Hopf-zero bifurcation in system (1) by using the MTS method. First, we set

$$a := f'(0), \qquad b := g_{X_1}(0), \qquad c := g_{X_3}(0), \qquad g_{X_2}(0) := 0. \tag{11}$$

According to case 4 of Theorem 2.2, we know that $\lambda = 0$ and $\lambda = i\omega$ are two simple eigenvalues of system (2), if the $f'(0) = 0$ and $\tau = \tau_1^0$. Therefore, we introduce the following parameters as bifurcation parameters

$$a_{bif} = a = 0, \qquad \tau_{bif} = \tau_1^0.$$

Now, we compute eigenvectors $p_1, \bar{p_1}, p_1^*, p_2, \bar{p_2}, p_2^*$, corresponding to A, A^* and eigenvalues $i\omega, -i\omega$ and zero, respectively

$$p_1 = \begin{pmatrix} i\omega \\ be^{-i\omega\tau_1^0} + i\omega e^{-i\omega(\tau_2-\tau_1^0)} \\ be^{-i\omega\tau_1^0} \end{pmatrix}, \bar{p}_1 = \begin{pmatrix} -i\omega \\ be^{i\omega\tau_1^0} - i\omega e^{i\omega(\tau_2-\tau_1^0)} \\ be^{i\omega\tau_1^0} \end{pmatrix}, p_1^* = \begin{pmatrix} \frac{i}{-2\omega} \\ 0 \\ \frac{e^{i\omega\tau_1^0}}{2b} \end{pmatrix},$$

$$p_2 = \bar{p}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, p_2^* = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}. \tag{12}$$

Note that with the help of inner product, we obtain p_1^*, p_2^* such that

$$\langle p_1^*, p_1 \rangle = 1, \qquad \langle p_2^*, p_2 \rangle = 1.$$

To obtain the normal form of system (1), we consider the system as follows with linear and nonlinear parts

$$\dot{X} = AX + BX_{\tau_1} + CX_{\tau_2} + F(X, X_{\tau_1}, X_{\tau_2}), \tag{13}$$

where A, B and C are defined at Equations (3) and $F(X, X_{\tau_1}, X_{\tau_2}) = O(2)$ such that there are second and third-order terms in Taylor expansion of F.

To obtain the normal form of system (1), we use the MTS method. Let

$$X(t) = \varepsilon^{\frac{1}{2}} X_{(1)} + \varepsilon^{\frac{3}{2}} X_{(2)} + \cdots ,$$

where

$$X_{(j)} = (X_{1,(j)}(T_0, T_1, \ldots), X_{2,(j)}(T_0, T_1, \ldots), X_{3,(j)}(T_0, T_1, \ldots))^T,$$

for $j = 1, 2, 3, \ldots$ and

$$T_k := \varepsilon^k t, \qquad k = 0, 1, 2, \ldots.$$

Also, we have

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \cdots \\ &= D_0 + \varepsilon D_1 + \cdots , \end{aligned} \tag{14}$$

such that

$$D_i = \frac{\partial}{\partial T_i}, \qquad i = 0, 1, 2, \ldots. \tag{15}$$

Moreover, we have

$$a = 0 + \varepsilon a_\varepsilon, \qquad \tau_1 = \tau_1^0 + \varepsilon \tau_\varepsilon. \tag{16}$$

Therefore, we get

$$\begin{aligned} X_i(t - \tau_1) = {} & \varepsilon^{\frac{1}{2}} X_{i,(1)}(T_0 - \tau_1, T_1 - \varepsilon\tau_1, \ldots) \\ & + \varepsilon^{\frac{3}{2}} X_{i,(2)}(T_0 - \tau_1, T_1 - \varepsilon\tau_1, \ldots) + \ldots, \quad i = 0, 1, 2, \ldots \end{aligned} \tag{17}$$

in which

$$X_{i,(1)}(T_0 - \tau_1, T_1 - \varepsilon\tau_1, \ldots) = X_{i,(1)}(T_0 - \tau_1^0 - \varepsilon\tau_\varepsilon, T_1 - \varepsilon(\tau_1^0 - \varepsilon\tau_\varepsilon), \ldots). \tag{18}$$

This implies that

$$\begin{aligned} \varepsilon^{\frac{1}{2}} X_{i,(1)}(T_0 - \tau_1^0 - \varepsilon\tau_\varepsilon, T_1 - \varepsilon(\tau_1^0 - \varepsilon\tau_\varepsilon), \ldots) = {} & \varepsilon^{\frac{1}{2}} X_{i,(1)}(T_0 - \tau_1^0, T_1, T_2, \ldots) \\ & + \varepsilon^{\frac{3}{2}} (-\tau_\varepsilon D_0 X_{i,(1)}(T_0 - \tau_1^0, T_1, T_2, \ldots) \\ & - \tau_1^0 D_1 X_{i,(1)}(T_0 - \tau_1^0, T_1, T_2, \ldots)) \\ & + \ldots. \end{aligned} \tag{19}$$

Also

$$\begin{aligned} \frac{dX}{dt} &= DX \\ &= (D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots)(\varepsilon^{\frac{1}{2}} X_{(1)} + \varepsilon^{\frac{3}{2}} X_{(2)} + \ldots) \\ &= \varepsilon^{\frac{1}{2}} D_0 X_{(1)} + \varepsilon^{\frac{3}{2}} (D_0 X_{(2)} + D_1 X_{(1)} + \ldots). \end{aligned} \tag{20}$$

Now, we assume

$$\tau_2 = \tau_1^0, \tag{21}$$

then by substituting Equations (11), (16) and (17) in the right hand of Equation (13), the following equations obtain

$$\begin{aligned} AX &= aIX \\ &= \varepsilon a_\varepsilon I(\varepsilon^{\frac{1}{2}} X_{(1)} + \varepsilon^{\frac{3}{2}} X_{(2)} + \dots) \\ &= \varepsilon^{\frac{3}{2}} a_\varepsilon X_{(1)} + \varepsilon^{\frac{5}{2}} a_\varepsilon X_{(2)} + \dots, \end{aligned}$$

$$BX_{\tau_1} = B \begin{pmatrix} \varepsilon^{\frac{1}{2}} X_{1,(1)}(T_0 - \tau_1^0, T_1, T_2, \dots) + \dots \\ \varepsilon^{\frac{1}{2}} X_{2,(1)}(T_0 - \tau_1^0, T_1, T_2, \dots) + \dots \\ \varepsilon^{\frac{1}{2}} X_{3,(1)}(T_0 - \tau_1^0, T_1, T_2, \dots) + \dots \end{pmatrix},$$

$$CX_{\tau_2} = C \begin{pmatrix} \varepsilon^{\frac{1}{2}} X_{1,(1)}(T_0 - \tau_1^0, T_1, T_2, \dots) + \dots \\ \varepsilon^{\frac{1}{2}} X_{2,(1)}(T_0 - \tau_1^0, T_1, T_2, \dots) + \dots \\ \varepsilon^{\frac{1}{2}} X_{3,(1)}(T_0 - \tau_1^0, T_1, T_2, \dots) + \dots \end{pmatrix}.$$

By assumption Equation (21) and using Equations (14)-(20), we have

$$D_0 X_{(1)} = (B + C) X_{(1),\tau_1^0}, \tag{22}$$

$$\begin{aligned} D_0 X_{(2)} + D_1 X_{(1)} = {} & a_\varepsilon X_{(1)} + (B + C) X_{(2),\tau_1^0} \\ & - (B + C)(\tau_1^0 D_1 X_{(1),\tau_1^0} + \tau_\varepsilon D_0 X_{(1),\tau_1^0}) + F^{(\frac{3}{2})}(X, X_{\tau_1}, X_{\tau_2}), \end{aligned} \tag{23}$$

and

$$X_{(i),\tau_1^0} := X_{(i)}(T_0 - \tau_1^0, T_1, T_2, \dots).$$

It is clear that Equation (22) is a first-order differential equation with the solution as follows

$$X_{(1)}(T_0, T_1) = G_1(T_1) p_1 e^{i\omega T_0} + \overline{G}_1(T_1) \overline{p}_1 e^{-i\omega T_0} + G_2(T_1) p_2. \tag{24}$$

Equation (23) can be rewritten as follows

$$\begin{aligned} D_0 X_{(2)} - (B + C) X_{(2)}(T_0 - \tau_1^0, T_1, T_2, \dots) = {} & -D_1 X_{(1)} \\ & - B \tau_1^0 D_1 X_{(1)}(T_0 - \tau_1^0, T_1, T_2) \\ & - \tau_\varepsilon B D_0 X_{(1)}(T_0 - \tau_1^0, T_1, T_2, \dots) \\ & + a_\varepsilon X_{(1)} + F^{(\frac{3}{2})}(X, X_{\tau_1}, X_{\tau_2}). \end{aligned} \tag{25}$$

In the other words,

$$\begin{aligned} D_0 X_{(2)} - (B + C) X_{(2),\tau_1^0} = {} & -D_1 X_{(1)} - B(\tau_1^0 D_1 X_{(1),\tau_1^0} + \tau_\varepsilon D_0 X_{(1),\tau_1^0}) \\ & + a_\varepsilon X_{(1)} + F^{(\frac{3}{2})}(X, X_{\tau_1}, X_{\tau_2}). \end{aligned} \tag{26}$$

Note that

$$D_1X_{(1)} = \frac{\partial G_1}{\partial T_1}p_1e^{i\omega T_0} + \frac{\partial \overline{G}_1}{\partial T_1}\overline{p}_1e^{-i\omega T_0} + \frac{\partial G_2}{\partial T_1}p_2, \tag{27}$$

$$D_1X_{(1),\tau_1^0} = \frac{\partial G_1}{\partial T_1}p_1e^{i\omega(T_0-\tau_1^0)} + \frac{\partial \overline{G}_1}{\partial T_1}\overline{p}_1e^{-i\omega(T_0-\tau_1^0)} + \frac{\partial G_2}{\partial T_1}p_2, \tag{28}$$

$$D_0X_{(1),\tau_1^0} = G_1p_1i\omega e^{i\omega(T_0-\tau_1^0)} - i\omega\overline{G}_1\overline{p}_1e^{-i\omega(T_0-\tau_1^0)}, \tag{29}$$

$$a_\varepsilon X_{(1)} = a_\varepsilon G_1p_1e^{i\omega T_0} + a_\varepsilon G_2p_2 + a_\varepsilon\overline{G}_1\overline{p}_1e^{-i\omega T_0}, \tag{30}$$

and

$$F^{(\frac{3}{2})}(X, X_{\tau_1}, X_{\tau_2}) = \begin{pmatrix} f_{11} \\ f_{21} \\ f_{31} \end{pmatrix}, \tag{31}$$

where

$$\begin{aligned} f_{11} = \frac{f^{(3)}(0)}{6}X_{1,(1)}^3 &+ \frac{1}{6}g_{X_2X_2X_2}(0)X_{2,(1),\tau_1^0}^3 + \frac{1}{2}g_{X_2X_2X_3}(0)X_{2,(1),\tau_1^0}^2X_{3,(1),\tau_1^0} \\ &+\frac{1}{6}g_{X_3X_3X_3}(0)X_{3,(1),\tau_1^0}^3 + \frac{1}{2}g_{X_2X_3X_3}(0)X_{2,(1),\tau_1^0}X_{3,(1),\tau_1^0}^2, \end{aligned}$$

$$\begin{aligned} f_{21} = \frac{f^{(3)}(0)}{6}X_{2,(1)}^3 &+ \frac{1}{6}g_{X_1X_1X_1}(0)X_{1,(1),\tau_1^0}^3 + \frac{1}{2}g_{X_1X_1X_3}(0)X_{1,(1),\tau_1^0}^2X_{3,(1),\tau_1^0} \\ &+\frac{1}{6}g_{X_3X_3X_3}(0)X_{3,(1),\tau_1^0}^3 + \frac{1}{2}g_{X_1X_3X_3}(0)X_{1,(1),\tau_1^0}X_{3,(1),\tau_1^0}^2, \end{aligned}$$

$$\begin{aligned} f_{31} = \frac{f^{(3)}(0)}{6}X_{3,(1)}^3 &+ \frac{1}{6}g_{X_1X_1X_1}(0)X_{1,(1),\tau_1^0}^3 + \frac{1}{2}g_{X_1X_1X_2}(0)X_{1,(1),\tau_1^0}^2X_{2,(1),\tau_1^0} \\ &+\frac{1}{6}g_{X_2X_2X_2}(0)X_{2,(1),\tau_1^0}^3 + \frac{1}{2}g_{X_1X_2X_2}(0)X_{1,(1),\tau_1^0}X_{2,(1),\tau_1^0}^2. \end{aligned}$$

From (12), (21) and (24), we conclude

$$X_{(1)}(T_0, T_1) = \begin{pmatrix} G_1p_{11}e^{i\omega T_0} + \overline{G}_1\overline{p}_{11}e^{-i\omega T_0} + 0 \\ G_1p_{12}e^{i\omega T_0} + \overline{G}_1\overline{p}_{12}e^{-i\omega T_0} + G_2 \\ G_1p_{13}e^{i\omega T_0} + \overline{G}_1\overline{p}_{13}e^{-i\omega T_0} + 0 \end{pmatrix} \tag{32}$$

and

$$X_{(1)}(T_0 - \tau_1^0, T_1) = \begin{pmatrix} G_1p_{11}e^{-i\omega\tau_1^0}e^{i\omega T_0} + \overline{G}_1\overline{p}_{11}e^{i\omega\tau_1^0}e^{-i\omega T_0} + 0 \\ G_1p_{12}e^{-i\omega\tau_1^0}e^{i\omega T_0} + \overline{G}_1\overline{p}_{12}e^{i\omega\tau_1^0}e^{-i\omega T_0} + G_2 \\ G_1p_{13}e^{-i\omega\tau_1^0}e^{i\omega T_0} + \overline{G}_1\overline{p}_{13}e^{i\omega\tau_1^0}e^{-i\omega T_0} + 0 \end{pmatrix}. \tag{33}$$

By substituting Equations (26)-(33) into Equation (25), we have

$$D_0X_{(2)} - (B + C)X_{(2),\tau_1^0} = \xi_1e^{i\omega T_0} + \overline{\xi}_1e^{-i\omega T_0} + \xi_2 + NST, \tag{34}$$

such that NST shows the terms that do not produce secular terms, and

$$\xi_1 = \begin{pmatrix} \xi_{11} \\ \xi_{12} \\ \xi_{13} \end{pmatrix},$$

where

$$\begin{aligned}
\xi_{11} &= \frac{1}{2} f^{(6)}(0) G_1^2 \overline{G}_1 p_{11}^2 \overline{p}_{11} + \frac{1}{2} g_{X_2X_2X_2}(0) e^{-i\omega\tau_1^0} (G_1 G_2^2 p_{12} + G_1^2 \overline{G}_1 p_{12}^2 \overline{p}_{12}) \\
&+ \frac{1}{2} g_{X_2X_2X_3}(0) e^{-i\omega\tau_1^0} (2\overline{G}_1 \overline{p}_{12} G_1^2 p_{13} p_{12} + G_2^2 G_1 p_{13} + G_1^2 \overline{G}_1 p_{12}^2 \overline{p}_{13}) \\
&+ \frac{1}{2} g_{X_2X_3X_3}(0) e^{-i\omega\tau_1^0} (2\overline{G}_1 G_1^2 p_{12} p_{13} \overline{p}_{13} + \overline{G}_1 G_1^2 \overline{p}_{12} p_{13}^2) \\
&- \frac{\partial G_1}{\partial T_1} p_{11} - \frac{\partial G_1}{\partial T_1} g_{X_3} \tau_1^0 p_{13} - g_{X_3} \tau_\varepsilon p_{13} G_1 i\omega + a_\varepsilon p_{11} G_1 \\
&+ \frac{1}{2} g_{X_3X_3X_3}(0) e^{-i\omega\tau_1^0} (G_1^2 \overline{G}_1 p_{13}^2 \overline{p}_{13}),
\end{aligned}$$

$$\begin{aligned}
\xi_{12} &= \frac{1}{2} f^{(3)}(0) (G_1 G_2^2 p_{12} + \overline{G}_1 G_1^2 p_{12}^2 \overline{p}_{12}) + \frac{1}{2} g_{X_1X_1X_1}(0) e^{-i\omega\tau_1^0} (G_1^2 \overline{G}_1 p_{11}^2 \overline{p}_{11}) \\
&+ \frac{1}{2} g_{X_1X_1X_3}(0) e^{-i\omega\tau_1^0} (\overline{G}_1 G_1^2 p_{11}^2 \overline{p}_{13} + 2G_1^2 \overline{G}_1 p_{11} \overline{p}_{11} p_{13}) \\
&+ \frac{1}{2} g_{X_1X_3X_3}(0) e^{-i\omega\tau_1^0} (2G_1^2 \overline{G}_1 p_{11} \overline{p}_{13} p_{13} + G_1^2 \overline{G}_1 p_{13}^2 \overline{p}_{11}) \\
&- \frac{\partial G_1}{\partial T_1} p_{12} - \frac{\partial G_1}{\partial T_1} g_{X_1} \tau_1^0 p_{11} - g_{X_1} \tau_\varepsilon p_{11} G_1 i\omega + a_\varepsilon p_{12} G_1 \\
&+ \frac{1}{2} g_{X_3X_3X_3}(0) e^{-i\omega\tau_1^0} G_1^2 \overline{G}_1 p_{13}^2 \overline{p}_{13},
\end{aligned}$$

$$\begin{aligned}
\xi_{13} &= \frac{1}{2} f^{(3)}(0) (\overline{G}_1 G_1^2 p_{13}^2 \overline{p}_{13}) + \frac{1}{2} g_{X_1X_1X_1}(0) e^{-i\omega\tau_1^0} (G_1^2 \overline{G}_1 p_{11}^2 \overline{p}_{11}) \\
&+ \frac{1}{2} g_{X_1X_1X_2}(0) e^{-i\omega\tau_1^0} (\overline{G}_1 G_1^2 p_{11}^2 \overline{p}_{12} + 2G_1^2 \overline{G}_1 p_{11} p_{12} \overline{p}_{11}) \\
&+ \frac{1}{2} g_{X_1X_2X_2}(0) e^{-i\omega\tau_1^0} (G_2^2 G_1 p_{11} + 2G_1^2 \overline{G}_1 \overline{p}_{12} p_{12} p_{11} + G_1^2 p_{12}^2 \overline{G}_1 \overline{p}_{11}) \\
&- \frac{\partial G_1}{\partial T_1} p_{13} - \frac{\partial G_1}{\partial T_1} g_{X_2} \tau_1^0 p_{12} - g_{X_2} \tau_\varepsilon p_{12} G_1 i\omega + a_\varepsilon p_{13} G_1 \\
&+ \frac{1}{2} g_{X_2X_2X_2}(0) e^{-i\omega\tau_1^0} (G_1 G_2^2 p_{12} + G_1^2 \overline{G}_1 p_{12}^2 \overline{p}_{12}).
\end{aligned}$$

Also,

$$\xi_2 = \begin{pmatrix} \xi_{21} \\ \xi_{22} \\ \xi_{23} \end{pmatrix},$$

at which

$$\xi_{21} =_{X_2X_2X_2}(0)(\frac{1}{6}G_2^3 + G_1\overline{G}_1G_2p_{12}\overline{p}_{12}) + g_{X_2X_2X_3}(0)G_1\overline{G}_1G_2(p_{12}\overline{p}_{13} + p_{13}\overline{p}_{12}) + g_{X_2X_3X_3}(0)(G_1\overline{G}_1G_2p_{13}\overline{p}_{13}) - \frac{\partial G_2}{\partial T_1}p_{21} - \frac{\partial G_2}{\partial T_1}\tau_1^0 g_{X_3}p_{23} + a_\varepsilon p_{21}G_2,$$

$$\xi_{22} = f^{(3)}(0)(\frac{1}{6}G_2^3 + G_1\overline{G}_1G_2p_{12}\overline{p}_{12}) - \frac{\partial G_2}{\partial T_1}p_{22} - \frac{\partial G_2}{\partial T_1}\tau_1^0 g_{X_1}p_{21} + a_\varepsilon p_{22}G_2,$$

$$\xi_{23} = g_{X_1X_1X_2}(0)(G_1\overline{G}_1G_2p_{11}\overline{p}_{11}) + g_{X_1X_2X_2}(0)G_1\overline{G}_1G_2(\overline{p}_{11}p_{12} + p_{11}\overline{p}_{12}) + g_{X_2X_2X_2}(0)(\frac{1}{6}G_2^3 + G_1\overline{G}_1G_2p_{12}\overline{p}_{12}) - \frac{\partial G_2}{\partial T_1}p_{23} - \frac{\partial G_2}{\partial T_1}\tau_1^0 g_{X_2}p_{22} + a_\varepsilon p_{23}G_2.$$

It is clear that Equation (34) is a linear non-homogeneous differential equation. We know that the non-homogeneous differential equation has a solution if and only if a solvability condition is satisfied. Then, it suffices $\langle p_j^*, \xi_j\rangle = 0, j = 1, 2$, so $\frac{\partial G_1}{\partial T_1}$ and $\frac{\partial G_2}{\partial T_2}$ are solved. It is not difficult to rewrite ξ_{11} and ξ_{13} as follows

$$\xi_{11} = \zeta_1G_1 + \zeta_2\overline{G}_1G_1^2 + \zeta_3G_1G_2^2 + \frac{\partial G_1}{\partial T_1}(-p_{11} - g_{X_3}\tau_1^0 p_{13}),$$
$$\xi_{13} = \zeta_4G_1 + \zeta_5\overline{G}_1G_1^2 + \zeta_6G_1G_2^2 + \frac{\partial G_1}{\partial T_1}(-p_{13} - g_{X_2}\tau_1^0 p_{12}).$$

By using the solvability condition, we get

$$\frac{\partial G_1}{\partial T_1} = \frac{1}{d_1}\{\eta_1G_1 + \eta_2\overline{G}_1G_1^2 + \eta_3G_1G_2^2\},$$

such that

$$d_1 = -1 + \frac{i}{2\omega}g_{X_1}g_{X_3}\tau_1^0 e^{-i\omega\tau_1^0},$$
$$\eta_i = \frac{-i}{2\omega}\zeta_i + \frac{e^{-i\omega\tau_1^0}}{2g_{X_1}}\zeta_{i+3}; \quad i = 1, 2, 3.$$

In a similar way to obtain $\frac{\partial G_2}{\partial T_1}$, we rewrite ξ_{21}, ξ_{22} and ξ_{23}

$$\xi_{21} = \varrho_1G_2^3 + \varrho_2G_2 + \varrho_3\overline{G}_1G_1G_2 + \varrho_4\frac{\partial G_2}{\partial T_1},$$
$$\xi_{22} = \varrho_5G_2^3 + \varrho_6G_2 + \varrho_7\overline{G}_1G_1G_2 + \varrho_8\frac{\partial G_2}{\partial T_1},$$
$$\xi_{23} = \varrho_9G_2^3 + \varrho_{10}G_2 + \varrho_{11}\overline{G}_1G_1G_2 + \varrho_{12}\frac{\partial G_2}{\partial T_1}.$$

Notice that $p_2^T = (0,1,0)$, so we have

$$\frac{\partial G_2}{\partial T_1} = \frac{1}{d_2}\{\gamma_1 G_2^3 + \gamma_2 G_2 + \gamma_3 \overline{G}_1 G_1 G_2\},$$

such that

$$\begin{aligned} d_2 &= -1 - \tau_1^0 g_{X_2}, \\ \gamma_i &= -\varrho_i + \varrho_{i+4} - \varrho_{i+8}; \quad i = 1,2,3,4. \end{aligned}$$

Now, let $G_1 = re^{i\theta}$ and $G_2 = z$. One obtains the following normal form in cylindrical coordinates

$$\begin{cases} \dfrac{dr}{dt} = (\frac{\eta_1}{d_1})^{Re} r + (\frac{\eta_2}{d_1})^{Re} r^3 + (\frac{\eta_3}{d_1})^{Re} r z^2, \\ \dfrac{d\theta}{dt} = (\frac{\eta_1}{d_1})^{Im} + (\frac{\eta_2}{d_1})^{Im} r^2 + (\frac{\eta_3}{d_1})^{Im} z^2, \\ \dfrac{dz}{dt} = \frac{1}{d_2}(\gamma_1 z^3 + \gamma_2 z + \gamma_3 r^2 z). \end{cases} \tag{35}$$

where $(\frac{\eta_i}{d_1})^{Re} = Re[\frac{\eta_i}{d_1}], (\frac{\eta_i}{d_1})^{Im} = Im[\frac{\eta_i}{d_1}], i = 1,2,3.$

Consider the first two equations of the above normal form

$$\begin{cases} \dfrac{dr}{dt} = (\frac{\eta_1}{d_1})^{Re} r + (\frac{\eta_2}{d_1})^{Re} r^3 + (\frac{\eta_3}{d_1})^{Re} r z^2, \\ \dfrac{dz}{dt} = \frac{1}{d_2}(\gamma_1 z^3 + \gamma_2 z + \gamma_3 r^2 z). \end{cases} \tag{36}$$

Thus one can obtain equilibria if

$$\frac{dr}{dt} = \frac{dz}{dt} = 0.$$

Therefore, system (36) has the following equilibria

$$\begin{aligned} E_0 &= (0,0), \\ E_1 &= (\sqrt{-(\frac{\eta_1}{\eta_2})^{Re}}, 0) \qquad if \qquad (\frac{\eta_1}{\eta_2})^{Re} < 0, \\ E_2^{\pm} &= (0, \pm\sqrt{-\frac{\gamma_2}{\gamma_1}}) \qquad if \qquad \frac{\gamma_2}{\gamma_1} < 0, \\ E_3^{\pm} &= (\sqrt{\frac{\eta_3^{Re}\gamma_2 - \eta_1^{Re}\gamma_1}{\eta_2^{Re}\gamma_1 - \eta_3^{Re}\gamma_3}}, \pm\sqrt{\frac{\eta_2^{Re}\gamma_2 - \eta_1^{Re}\gamma_3}{\eta_3^{Re}\gamma_3 - \eta_2^{Re}\gamma_1}}) \\ & \quad if \qquad \frac{\eta_3^{Re}\gamma_2 - \eta_1^{Re}\gamma_1}{\eta_2^{Re}\gamma_1 - \eta_3^{Re}\gamma_3} > 0 \qquad and \qquad \frac{\eta_2^{Re}\gamma_2 - \eta_1^{Re}\gamma_3}{\eta_3^{Re}\gamma_3 - \eta_2^{Re}\gamma_1} > 0. \end{aligned}$$

Therefore, we have three critical lines as follows

$$\begin{aligned} L_1 &: \eta_1^R = 0, \\ L_2 &: \gamma_2 = 0, \\ L_3 &: \eta_3^{Re}\gamma_2 - \eta_1^{Re}\gamma_1 = 0 \quad and \quad \eta_2^{Re}\gamma_2 - \eta_1^{Re}\gamma_3. \end{aligned}$$

Now, one can state the following theorem.

Theorem 3.1. *If $a = a_{bif}$ and $\tau = \tau_{bif}$, then the local asymptotic behavior of solutions of the original system* (1) *is determined by behavior of the solutions on center manifold* (35). *Thus, equilibria on the z-axis of system* (35) *remain equilibria and system* (35) *has periodic orbits with period $\frac{2\pi}{\omega}$ for equilibria away from the z-axis.*

4. Numerical Simulation

In this section, numerical simulations are presented to verify our analytical results. Consider the following system

$$\begin{cases} \dot{X}_1(t) = -a^* X_1(t) + a_1 \tanh(X_1(t)) + c\tanh(X_3(t-\tau_1)), \\ \dot{X}_2(t) = -a^* X_2(t) + a_1 \tanh(X_2(t)) + b\tanh(X_1(t-\tau_1)) + c\tanh(X_3(t-\tau_2)), \\ \dot{X}_3(t) = -a^* X_3(t) + a_1 \tanh(X_3(t)) + b\tanh(X_1(t-\tau_2)). \end{cases} \tag{37}$$

The parameter values used in the simulations are given in Table 1. The first, we find the Hopf-zero bifurcation parameter for system (37); $(a_{bif}, \tau_{bif}) = (0, \frac{\pi}{\sqrt{2}})$, then we simulate the solutions of this system, see Figure 2.

It should be noted that its characteristic equation has a zero root and a pair of purely imaginary eigenvalues. Then by using Matlab and Mathematica software, we study the dynamical behaviours of the system when $a \neq a_{bif}$ or $\tau \neq \tau_{bif}$, see Figures 3-5. Similar results can be obtained for the same value of $a = a_1 - a^*$ where $\tau_1 = \pi$ and $\tau_1 = \frac{\pi}{2}$.

Table 1: Parameter values.

Example	a_1	a^*	b	c	τ_1	τ_2	Inital value	Figure
1	-1	-1	2	-1	$\frac{\pi}{\sqrt{2}}$	$\frac{\pi}{\sqrt{2}}$	$(0.1, 0.1, 0.2)$	Figure 2
2	-1	0.275	2	-1	$\frac{\pi}{\sqrt{2}}$	$\frac{\pi}{\sqrt{2}}$	$(0.1, 0.1, 0.2)$	Figure 3
3	-1	0.35	2	-1	$\frac{\pi}{\sqrt{2}}$	$\frac{\pi}{\sqrt{2}}$	$(0.1, 0.1, 0.2)$	Figure 4
4	-1	-1.5	2	-1	$\frac{\pi}{\sqrt{2}}$	$\frac{\pi}{\sqrt{2}}$	$(0.1, 0.1, 0.2)$	Figure 5

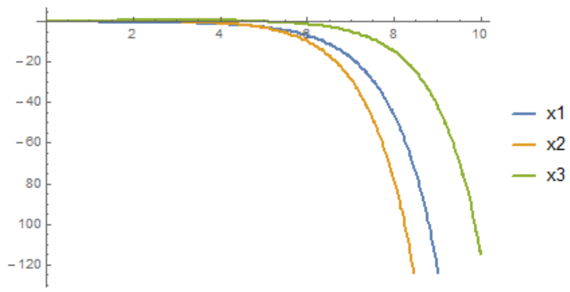

Figure 2: Example 1 (Existence of unstable equilibrium point): Simulated solutions of system 37 at bifurcation parameters $a_{bif} = 0$ and $\tau_{bif} = \frac{\pi}{\sqrt{2}}$.

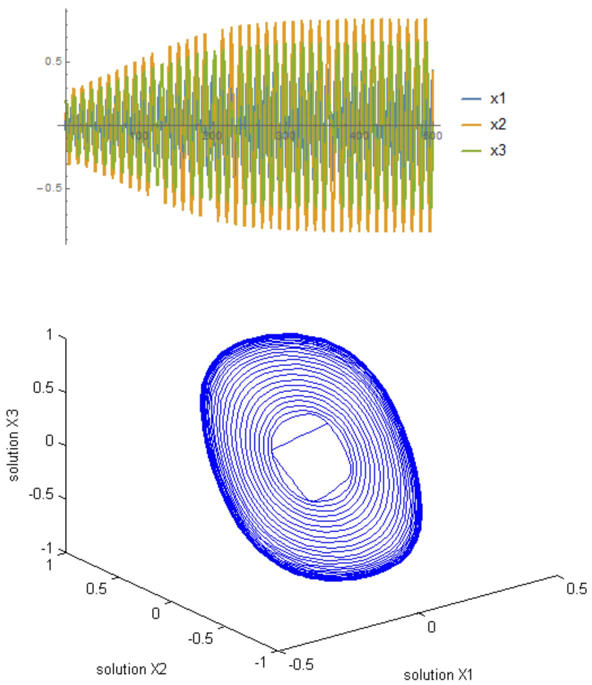

Figure 3: Example 2 (Existence of a periodic solution): Simulated solutions of system 37 for $(a, \tau) = (-1.275, \frac{\pi}{\sqrt{2}})$ where $a < a_{bif}$ and $\tau = \tau_{bif}$; left) shows the time history, and right) shows the phase space.

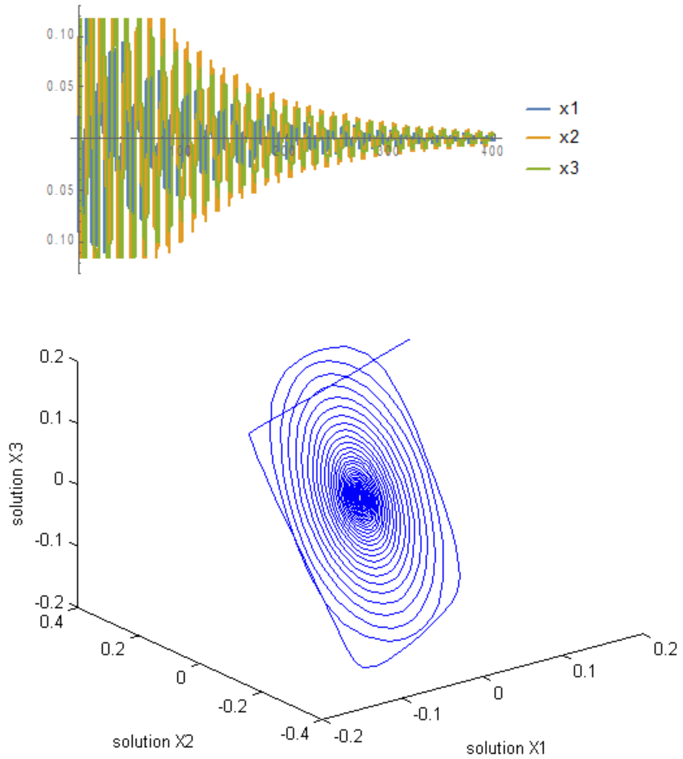

Figure 4: Example 3 (Existence of a stable equilibrium point): Simulated solutions of system 37 for $(a, \tau) = (-1.35, \frac{\pi}{\sqrt{2}})$ where $a << a_{bif}$ and $\tau = \tau_{bif}$; left) shows the time history, and right) shows the phase space.

5. Conclusion

In this paper, a three-cell network with two different delays is considered. Necessary conditions are obtained to occur some bifurcations, such as saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation, simple Hopf bifurcation, and Hopf-zero bifurcation. Also, the Multiple Time Scale method is used to obtain the normal form of Hopf-zero bifurcation. Finally, the presented numerical simulations have demonstrated the correctness of the theoretical analysis. Our work is a future study of the coupled cell networks with different delays which will be useful in the research of the complex phenomena.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

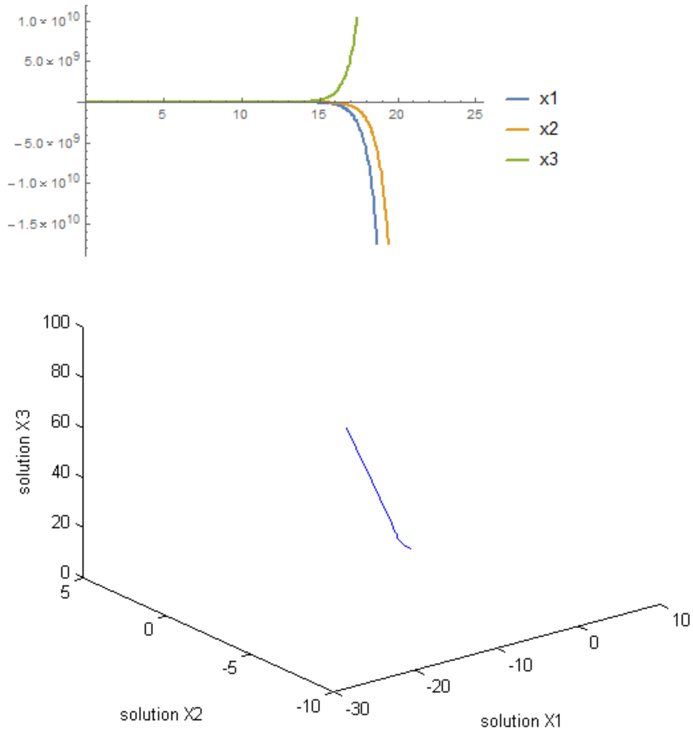

Figure 5: Example 4 (Existence of a unstable equilibrium point): Simulated solutions of system 37 for $(a, \tau) = (0.5, \frac{\pi}{\sqrt{2}})$ where $a > a_{bif}$ and $\tau = \tau_{bif}$; left) shows the time history, and right) shows the phase space.

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