Mathematics Interdisciplinary Research **6** (2021) 275 – 291

Original Scientific Paper

Integrals Involving Product of Polynomials and Daubechies Scale Functions

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Abstract

In this paper, we will introduce an algorithm for obtaining integrals of the form

$$\int_0^x t^m \varphi(t)dt, \quad m \in \mathbb{N} \cup \{0\},$$

where φ is the scaling functions of Daubechies wavelet. In order to obtain these integrals in dyadic points for x's, we have to solve a linear system. We will investigate, sparseness, well-conditioning and strictly diagonal dominant of matrices of these systems.

Keywords: Daubechies wavelets, scaling functions, dyadic points, diagonal dominant, well-condition.

2010 Mathematics Subject Classification: 65T60, 15A12

How to cite this article
A. Alipanah, M. Pendar and K. Sadeghi, Integrals involving product of polynomials and Daubechies scale functions, *Math. Interdisc. Res.* **6** (2021) 275–291.

1. Introduction

The orthogonal projection operator of function f onto V_j, the space of scaling function at resolution j of orthonormal wavelet, denoted by $\mathcal{P}_j f(x)$ and defined as

$$\mathcal{P}_j f(x) = \sum_k \langle \varphi_{j,k}, f\rangle \varphi_{j,k}(x).$$

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Academic Editor: Hikmet Koyunbakan
Received 16 July 2021, Accepted 5 November 2021
DOI: 10.22052/mir.2021.239849.1225

When we expand a function f via Daubechies scaling functions basis, we need to calculate integrals of the form $\langle \varphi_{j,k}, f \rangle = \int f(x)\varphi_{j,k}(x)dx$ where $\varphi_{j,k}(x) = 2^{\frac{j}{2}}\varphi(2^j x - k)$. If the function f is a polynomial or replacing it by Taylor expansion, these integrals can be formulated easily (see [6, 12]).

Let $\Phi_m(x)$, for all $x \in [0, L]$ be defined as

$$\Phi_m(x) = \int_0^x t^m \varphi(t)dt, \quad x > 0, \tag{1}$$

where m is a nonnegative integer, φ is the Daubechies scaling function of order N and $L = 2N - 1$. Integrals of the forms Equation (1) have many applications in solving integral equations [9], quantum mechanic [7], and wavelet-Galerkin method. Beylkin et al. in [2] studied some methods for these integrals. Sweldens and Piessens in [12] have used a variety of quadrature methods for evaluating these integrals. Some of these integrals in particular case of Daubechies wavelet have been obtained by Finek in [6] by formed a linear system of equations in terms of these integrals. Also Shann and Yen in thier work [10] formed a linear system of equations for calculation these integrals (Equation (1)) and they show that the matrices associated with these integrals have a uniform bound in their L_2-norm. In this paper, we will prove that the linear systems which obtaining from integrals in dyadic points for x's of Equation (1) are well-condition. Also, we will prove that for $2 \leq N \leq 17$ vanishing moment of Daubechies wavelets and under some conditions on m, for all $N \geq 18$ these matrices are strictly diagonally dominant.

This paper is organized as follows:
We first briefly introduce Daubechies wavelet and its properties. In Section 2, we will calculate integrals in Equation (1) for $m = 0$ at dyadic points for x. Also we will prove some properties of the matrix coefficients of these linear systems, such as boundedness and well-conditioning. In Section 3, we will calculate these integrals for any $m \geq 1$ at dyadic points of x. Finally, the strictly diagonally dominant property of matrix coefficients of these linear systems will be investigated.

2. Daubechies Orthonormal Wavelets

In this section, we shortly review some definitions and basic properties of Daubechies scaling and wavelets functions.

Definition 2.1. A multiresolution analysis of $L_2(\mathbb{R})$ is defined as a sequence of closed subspaces $V_j \subseteq L_2(\mathbb{R}), j \in \mathbb{Z}$ with the following properties.

i) $V_j \subseteq V_{j+1},\ j \in \mathbb{Z}$,

ii) $\bigcup_j V_j$ is dense in $L_2(\mathbb{R})$,

iii) $\bigcap_j V_j = \{0\}$,

iv) $f(\cdot) \in V_0$ if and only if $f(2^j \cdot) \in V_j$,

v) $f(2^j \cdot) \in V_j \Leftrightarrow f(2^j \cdot +1) \in V_j$.

vi) A function $\varphi \in V_0$ with a non-vanishing integral, exists so that the collection $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for V_0.

Definition 2.2. Let $N \in \mathbb{N}$ and $L = 2N-1$, the function $\varphi \in L_2(\mathbb{R})$ with support $[0, L]$ that satisfies multiresolution analysis is called Daubechies scaling function of order N. In the space V_j for all $x \in [0, L]$ Daubechies scaling function is defined as [4]:

$$\varphi_{j,k}(\cdot) := 2^{\frac{j}{2}}\varphi(2^j \cdot -k), \quad j, k \in \mathbb{Z}. \tag{2}$$

If $j = k = 0$ then $\varphi_{0,0}(x) = \varphi(x)$, and since $\int_0^L \varphi(x)dx \neq 0$, a normalization of Daubechies scaling function is as

$$\int_0^L \varphi(x)dx = 1, \tag{3}$$

and $\{\varphi(x-k) : k \in \mathbb{Z}\}$, is an orthonormal basis for V_0 , i.e,

$$\langle \varphi(x-k), \varphi(x-k')\rangle = \delta_{k,k'}, \quad k, k' \in \mathbb{Z}, \tag{4}$$

Also $\{2^{\frac{j}{2}}\varphi(2^j x - k) : k \in \mathbb{Z}\}$ generating an orthonormal basis for V_j, i.e,

$$V_j = \overline{Span\{\varphi_{j,k} : k \in \mathbb{Z}\}}. \tag{5}$$

Definition 2.3. Since for all $j \in \mathbb{Z}$ we have $V_j \subseteq V_{j+1}$ thus the spaces W_j which are orthogonal complements of V_j in V_{j+1}, spanning by set of functions as $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ that called Daubechies wavelet functions and they are defined as follows:

$$\psi_{j,k}(\cdot) := 2^{\frac{j}{2}}\psi(2^j \cdot -k), \quad j, k \in \mathbb{Z}. \tag{6}$$

By above definitions, we have

$$\begin{aligned}
&V_j \perp W_j, \quad V_j \cap W_j = \{0\}, \\
&W_j \cap W_i = \{0\}, \quad i \neq j, \\
&V_{j+1} = V_j \oplus W_j, \quad \forall j \in \mathbb{Z}, \\
&L_2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j,
\end{aligned}$$

where $\oplus$ is direct sum in $L_2(\mathbb{R})$ ([4, 6, 11]). There is orthonormality property between the Daubechies wavelet and scaling functions as follows:

$$\langle \varphi_{j,k}, \psi_{j',k'}\rangle = \delta_{j,j'}\delta_{k,k'}, \quad j, j', k, k' \in \mathbb{Z},$$

As we know $V_1 = V_0 \oplus W_0$ therefore, if $\varphi \in V_0 \subseteq V_1$, $\psi \in W_0 \subseteq V_1$ and from Equation (5) we conclude that there exist sequences h_k and g_k in $\ell^2\mathbb{R}$ such that satisfying the following equations:

$$\varphi(x) = \sum_{k=0}^{L} h_k \varphi(2x - k), \quad x \in [0, L], \tag{7}$$

$$\psi(x) = \sum_{k=0}^{L} g_k \varphi(2x - k), \quad x \in [0, L]. \tag{8}$$

Equations (7) and (8) are called dilation or the two-scale equations. Also h_k and g_k are called filter coefficients or scaling parameters and it has been shown that $g_k = (-1)^k h_{L-k}$. In fact, there are no explicit formulas for Daubechies wavelet and scaling functions. But by using the two-scale equations (7) and (8) we can calculate the Daubechies scaling and wavelet functions in integer and dyadic points. In Equations (7) and (8), φ, ψ and h_k all are unknowns and we can obtain these unknowns under some conditions which imposed. In order to determine filter coefficients Daubechies imposed some useful conditions as follows:

1. Scaling function must be able to exactly represent polynomials of order smaller than p.
2. $\int_0^L \varphi(x)dx = 1$.
3. Orthogonality of scaling function to its integer translates and orthogonality of scaling function to Daubechies wavelet:

$$\int_0^L \varphi(x-k)\varphi(x-l)dx = \delta_{k,l}, \quad k, l \in \mathbb{Z},$$

$$\int_0^L \varphi(x)\psi(x-l)dx = 0, \quad l \in \mathbb{Z}.$$

From above conditions we conclude the following equations:

$$\sum_{k=0}^{L} (-1)^k h_k k^m = 0, \quad m = 0, 1, 2, \ldots, N-1. \tag{9}$$

$$\sum_{k=0}^{L} h_k = 2, \tag{10}$$

$$\sum_{k=0}^{L} h_k h_{2m+k} = 2\delta_{0,m}, \quad m = 1-N, \ldots, N-1, \tag{11}$$

$$h_k = 0, \quad \forall k \notin \{0, 1, 2, \ldots, L\}, \tag{12}$$

Equations (9)-(12) are a nonlinear system of equations in which by solving them the filter coefficients $h_k,\ k = 0, 1, 2, \ldots, L$ can be obtained. In [10] the exact values of h_k for differen N are evaluated. For computation Daubechies scaling function $\varphi(x)$ in integer $x \in [0, L]$, we must solve a linear system of equations, that arising from two-scale equations in Equation (7) (see [8, 13]). Also the algorithm for computing the values of the scaling function $\varphi(x)$ in dyadic points $x = \frac{k}{2^j},\ j \geq 0,\ k = 0, 1, \ldots, 2^j L$ are given in [4, 10].

3. Integrals Involving Daubechies Scaling Funtions

In this section, we will evaluate the integrals given in Equation (1) for $m = 0$. So by substituting Equation (7) in Equation (1) we have

$$\begin{aligned}\Phi_m(x) &= \int_0^x t^m \varphi(t)dt = \sum_{k=0}^{L} h_k \int_0^x t^m \varphi(2t-k)dt \\ &= \frac{1}{2^{m+1}} \sum_{k=0}^{L} h_k \sum_{i=0}^{m} \binom{m}{i} k^i \int_0^{2x-k} u^{m-i} \varphi(u)du.\end{aligned}$$

Therefore,

$$\Phi_m(x) = \frac{1}{2^{m+1}} \sum_{k=0}^{L} h_k \sum_{i=0}^{m} \binom{m}{i} k^i \Phi_{m-i}(2x-k). \tag{13}$$

Now if $m = 0$, we have:

$$\Phi_0(x) = \frac{1}{2} \sum_{k=0}^{L} h_k \Phi_0(2x-k), \quad x \in (0, L]. \tag{14}$$

We first obtain $\Phi_0(x)$ for integers $x = 1, 2, 3, \ldots, L$ by forming a linear system of equations that, elements of the coefficients matrix is composed of $h_k,\ \ 0 \leq k \leq L$. Then we using a recursivly formula to calculate the values of the function $\Phi_0(x)$ in dyadic points $x = 2^{-j}r$, for $\ j \geq 1$ and $\ 0 \leq r \leq 2^j L - 1$.

Here for simplicity we define [6]

$$I_n := \Phi_0(n) = \int_0^n \varphi(t)dt, \quad n = 1, 2, 3, \ldots, L. \tag{15}$$

So by this notation the Equation (14) can be written as:

$$I_n = \frac{1}{2} \sum_{k=0}^{L} h_k I_{2n-k}, \quad 1 \leq n \leq L. \tag{16}$$

According to the Equation (3) we know that,

$$I_L = \Phi_0(L) = \int_0^L \varphi(x)dx = 1.$$

From Equation (16) we have that

$$I_1 = \frac{1}{2}\sum_{k=0}^{L} h_k I_{2-k} = \frac{h_0}{2} I_2 + \frac{h_1}{2} I_1. \tag{17}$$

Then Equation (17) can be written as

$$\Big(1 - \frac{1}{2}h_1\Big) I_1 - \frac{1}{2}h_0 I_2 = 0. \tag{18}$$

Also similarly for $n = 2$ and from Equation (16) we have

$$I_2 = \frac{1}{2}\sum_{k=0}^{L} h_k I_{4-k} = \frac{h_0}{2} I_4 + \frac{h_1}{2} I_3 + \frac{h_2}{2} I_2 + \frac{h_3}{2} I_1,$$

therefore,

$$-\frac{1}{2}h_3 I_1 + \Big(1 - \frac{1}{2}h_2\Big) I_2 - \frac{1}{2}h_1 I_3 - \frac{1}{2}h_0 I_4 = 0. \tag{19}$$

By continuing this procedure for each $1 \le n \le N-1$ we have

$$\Big(1 - \frac{1}{2}h_n\Big) I_n - \frac{1}{2}\sum_{\substack{k=0\\ k\neq n}}^{2n-1} h_k I_{2n-k} = 0. \tag{20}$$

As we know $I_k = 1,\ k \ge L$, so from Equation (16) we conclude that

$$\Big(1 - \frac{1}{2}h_n\Big) I_n - \frac{1}{2}\sum_{\substack{k=2n-2N+2\\ k\neq n}}^{L} h_k I_{2n-k} = \frac{1}{2}\Big(\sum_{k=0}^{2n-2N+1} h_k\Big). \tag{21}$$

Now, Equations (18)-(21) can be written as a linear system as follows

$$\Big(\mathbf{I} - \frac{1}{2}\mathbf{A}\Big)\mathbf{x} = \mathbf{b}, \tag{22}$$

where, $\mathbf{I}$ is the identity matrix $(L-1)\times(L-1)$ and $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{x}$ are matrices as follows respectively:

$$\mathbf{A}_{(L-1)\times(L-1)} = \begin{pmatrix} h_1 & h_0 & 0 & 0 & \dots & 0 \\ h_3 & h_2 & h_1 & h_0 & \dots & 0 \\ h_5 & h_4 & h_3 & h_2 & \dots & 0 \\ \vdots & \vdots & \vdots\ddots & \vdots & \vdots & \vdots \\ h_{L-2} & h_{L-3} & \dots & h_{N-1} & \dots & h_0 \\ h_L & h_{L-1} & \dots & h_N & \dots & h_2 \\ 0 & 0 & h_L & h_{L-1} & \dots & h_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & h_{L-1} & h_{(L-2)} & h_{(L-3)} \\ 0 & 0 & \dots & 0 & h_L & h_{(L-1)} \end{pmatrix}, \tag{23}$$

$$\mathbf{b}_{(L-1)\times 1} = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h_0+h_1 \\ h_0+h_1+h_2+h_3 \\ \vdots \\ h_0+h_1+\ldots+h_{L-2} \end{pmatrix}, \tag{24}$$

$$\mathbf{x}_{(L-1)\times 1} = \begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_{L-1} \end{pmatrix}. \tag{25}$$

Elements of the matrix $\mathbf{A}$ and vector $\mathbf{b} = (b_1, b_2, \ldots, b_{L-1})^T$ are obtained as follows

$$\mathbf{A}_{ij} = h_{2i-j}, \quad i, j = 1, 2, \ldots, L-1,$$

$$b_i = \left\{ \begin{array}{ll} 0, & 1 \le i \le N-1 \\ \\ \frac{1}{2}\sum\limits_{k=0}^{2i-2N+1} h_k\ , & N \le i \le L-1 \end{array} \right].$$

The necessary and sufficient condition for solving Equation (22) is that $\mathbf{I} - \frac{1}{2}\mathbf{A}$ be nonsingular matrix.

Theorem 3.1. [10] If $\mathbf{A}$ is the matrix given in Equation (23) and $\|\cdot\|_2$ is the Euclidean norm, then

$$\|\mathbf{A}\|_2 \le \sqrt{2}. \tag{26}$$

According to the theorem 3.1, since $\|\mathbf{A}\|_2 \le \sqrt{2}$, then

$$\|\mathbf{I} - \frac{1}{2}\mathbf{A}\|_2 \le 1 + \frac{\sqrt{2}}{2}. \tag{27}$$

Theorem 3.2. [3] If $\mathbf{B}$ is a square matrix such that $\|\mathbf{B}\|_2 < 1$, then the inverse of $\mathbf{I} - \mathbf{B}$ is calculated by the following series:

$$(\mathbf{I} - \mathbf{B})^{-1} = \sum_{m=0}^{\infty} \mathbf{B}^m.$$

According to the Theorem 3.1 , $\left\|\frac{1}{2}\mathbf{A}\right\|_2 < 1$ and therefore from Theorem 3.2 the invers of $\mathbf{I} - \frac{1}{2}\mathbf{A}$ is calculated by the series

$$\Big(\mathbf{I} - \frac{1}{2}\mathbf{A}\Big)^{-1} = \sum_{m=0}^{\infty}\Big(\frac{1}{2}\mathbf{A}\Big)^m.$$

Since $\left\|\frac{1}{2}\mathbf{A}\right\|_2 < 1$, from Theorem 3.2 the above series is convergence. So the Euclidean norm of $\Big(\mathbf{I} - \frac{1}{2}\mathbf{A}\Big)^{-1}$ is bounded, i.e,

$$\left\|\Big(\mathbf{I} - \frac{1}{2}\mathbf{A}\Big)^{-1}\right\|_2 \leq \frac{2}{2-\sqrt{2}}\ . \tag{28}$$

From Equations (27) and (28) we have that

$$\mathrm{cond}\Big(\mathbf{I} - \frac{1}{2}\mathbf{A}\Big) = \left\|\Big(\mathbf{I} - \frac{1}{2}\mathbf{A}\Big)^{-1}\right\|_2 \left\|\Big(\mathbf{I} - \frac{1}{2}\mathbf{A}\Big)\right\|_2 \leq \frac{2+\sqrt{2}}{2-\sqrt{2}}.$$

Therefore, the linear system of Equation (22) is well-conditioned. The condition number of $\mathbf{I} - \frac{1}{2}\mathbf{A}$ corresponding to the integration of Daubechies scaling functions with different N vanishing moments takes in Table 1. In this table, κ_2 indicative condition number of $\mathbf{I} - \frac{1}{2}\mathbf{A}$ by L^2-norm.

Table 1: Condition number of $\mathbf{I} - \frac{1}{2}\mathbf{A}$ in calculating integral of Daubechies scaling function on integer intervals for different N.

N	2	3	4	7	8	16	17	18
κ_2	2.21	3.18	3.48	4.18	4.17	4.52	4.51	4.53

The integral values of scaling function on integer intervals $[n-1, n]$ are obtained as follows:

$$\int_{n-1}^{n} \varphi(x)dx = I_n - I_{n-1}, \qquad 1 \leq n \leq L,\ I_0 = 0.$$

After obtaining integrals of scaling function on integer intervals by length one, we use the following recursive equation for calculating integrals on dyadic intervals for all $j \in \mathbb{N}$ and $0 \leq k \leq L \times 2^j - 1$ [6]:

$$\begin{aligned} I_{[k2^{-j},(k+1)2^{-j}]} :&= \int_{k2^{-j}}^{(k+1)2^{-j}} \varphi(x)dx \\ &= \sum_{l=0}^{L} h_l \int_{k2^{-j}}^{(k+1)2^{-j}} \varphi(2x-l)dx \\ &= \frac{1}{2}\sum_{l=0}^{L} h_l \int_{k2^{(-j+1)}-l}^{(k+1)2^{(-j+1)}-l} \varphi(t)dt, \\ & j = 1, 2, \ldots (k = 0, 1, \ldots, L \times 2^j - 1). \end{aligned}$$

Example 3.3. We calculating the integral of Daubechies scaling function with $N = 2$ on integer and dyadic intervals in $[0, 3]$. As we know

$$I_1 := \int_0^1 \varphi(x)dx, \quad I_2 := \int_0^2 \varphi(x)dx, \quad I_3 := \int_0^3 \varphi(x)dx = 1.$$

So the linear system of equations given in Equation (22) is as follows:

$$\begin{pmatrix} 1 - \frac{1}{2}h_1 & -\frac{1}{2}h_0 \\ -\frac{1}{2}h_3 & 1 - \frac{1}{2}h_2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ h_0 + h_1 \end{pmatrix}. \tag{29}$$

By substituting values of h_k as :

$$h_0 = \frac{1+\sqrt{3}}{4}, \quad h_1 = \frac{3+\sqrt{3}}{4},$$
$$h_2 = \frac{3-\sqrt{3}}{4}, \quad h_3 = \frac{1-\sqrt{3}}{4}.$$

By solving the linear system (29) we obtain

$$I_1 = 0.849679, \qquad I_2 = 1.016346.$$

Tables 2 and 3 shows the results integration of Daubechies scaling function on integer and dyadic intevals for different N.

Table 2: Integrals of Daubechies scaling function I_n, on $[0, n]$ for different N.

n	$N = 2$	$N = 3$	$N = 4$	$N = 5$
1	0.84967936	0.60074159	0.37993441	0.22061401
2	1.01634604	1.09671145	1.15338184	1.11667365
3	1	0.98548673	0.95442792	0.95626923
4		0.99965909	1.00636553	1.01457715
5		1	1.00050094	0.99819794
6			0.99999616	0.99976213
7			1	0.99999787
8				0.99999999
9				1

4. Integrals Involving Daubechies Scaling Funtions and Polynomials

In this Section, by using evaluations of previous section we obtaining the function $\Phi_m(x)$ for any $m \geq 1$ and for integer and dyadic points of $x \in (0, L]$.

Table 3: The values of integrals, $I_{[\frac{k}{2},\frac{k+1}{2}]}$, for Daubechies scaling function with $N = 2, 3$ on diadic intervals $[\frac{k}{2}, \frac{k+1}{2}]$.

k	$N = 2$	$N = 3$
0	2.901709×10^{-1}	1.413146×10^{-1}
1	5.595084×10^{-1}	4.594269×10^{-1}
2	2.276709×10^{-1}	4.521666×10^{-1}
3	-6.100423×10^{-2}	4.380321×10^{-2}
4	-1.784180×10^{-2}	-1.116453×10^{-1}
5	1.495766×10^{-3}	4.205812×10^{-4}
6		1.783161×10^{-2}
7		-3.659253×10^{-3}
8		3.324193×10^{-4}
9		8.491666×10^{-6}

Now by separating the first sentence of Equation (13) for $i = 0$ we have

$$\Phi_m(x) - \frac{1}{2^{m+1}} \sum_{k=0}^{L} h_k \Phi_m(2x-k) = \frac{1}{2^{m+1}} \sum_{k=0}^{L} h_k \sum_{i=1}^{m} \binom{m}{i} k^i \Phi_{m-i}(2x-k). \quad (30)$$

As we know Equation (30) is a recursive relation for evaluation of $\Phi_m(x)$. For $x = L$, Equation (30) or $\Phi_m(L)$ can be calculated recursively by the following equation

$$\Phi_m(L) = \frac{1}{2\,(2^m - 1)} \sum_{i=1}^{m} \binom{m}{i} \mu_i \Phi_{m-i}(L), \quad \Phi_0(L) = 1, \quad (31)$$

where, $\mu_i = \sum_{k=0}^{L} h_k k^i$. So for any $x \geq L$ we have:

$$\Phi_m(x) = \Phi_m(L) = \int_0^L t^m \varphi(t) dt.$$

In both of Equations (30) and (31) for obtaining in m needs to evaluate $\Phi_k(x)$, for $k = 0, 1, 2, \ldots, m-1$.

By substituting $x = 1, 2, 3, \ldots, L-1$ in Equation (30) for any Daubechies scaling function with N vanishing moment and for fixed $m \geq 1$ generated a linear system of equations as follows:

$$(\mathbf{I} - \frac{1}{2^{m+1}} \mathbf{A})\mathbf{x} = \mathbf{b}, \quad (32)$$

where $\mathbf{A}$ is the well known matrix $(L-1) \times (L-1)$ in Equation (26) and

$$\mathbf{A}_{ij} = h_{2i-j}, \quad i, j = 1, 2, \ldots, L-1.$$

Also $\mathbf{b}$ is the right hand side known vector that its elements are as follows:

$$\mathbf{b}_i = \frac{1}{2^{m+1}} \sum_{k=0}^{L} h_k \sum_{r=1}^{m} \binom{m}{r} k^r \Phi_{m-r}(2i-k), \quad 1 \leq i \leq N-1,$$

and for any $N \leq i \leq L-1$ we have

$$\mathbf{b}_i = \frac{1}{2^{m+1}} \sum_{k=0}^{L} h_k \sum_{r=1}^{m} \binom{m}{r} k^r \Phi_{m-r}(2i-k) + \Big(\sum_{r=0}^{2i-2N+1} h_r \Big) \Phi_0(L).$$

The unknown vector is $\mathbf{x} = \Big(\Phi_m(1), \Phi_m(2), \ldots, \Phi_m(L-1)\Big)^T$. From the Theorem 3.1, we have $\|\mathbf{A}\|_2 \leq \sqrt{2}$, therefore for any $m \geq 0$, $\|\frac{1}{2^{m+1}}\mathbf{A}\|_2 < 1$, and using the theorem 3.2 the matrix $\mathbf{I} - \frac{1}{2^{m+1}}\mathbf{A}$ is nonsingular and its inverse can be computed by the following series

$$\Big(\mathbf{I} - \frac{1}{2^{m+1}}\mathbf{A}\Big)^{-1} = \sum_{n=0}^{\infty} \Big(\frac{1}{2^{m+1}}\mathbf{A}\Big)^n.$$

Thus, the solution of the linear system (32) for all integer $x \in (0, L]$ and for given $m \geq 1$ is obtained. Now by using Equation (30) we can obtain $\Phi_m(x)$ for $x = \frac{r}{2^J}$, where $r = 1, 3, 5, \ldots, 2^J L - 1$. So,

$$\Phi_m\Big(\frac{r}{2^J}\Big) = \frac{1}{2^{m+1}} \sum_{k=0}^{L} h_k \Phi_{m,k}\Big(\frac{r}{2^{J-1}}\Big), \tag{33}$$

where

$$\Phi_{m,k}(x) = \sum_{i=0}^{m} \binom{m}{i} k^i \Phi_{m-i}(x-k).$$

Example 4.1. For an example we use Equations (31) and (32) to calculating $\Phi_m(x)$ for Daubechies scaling function with $N = 3$ and for different m in integer points of $[0, 5]$ and for dyadic points with $J = 1$. The linear system of equations with (32) is as follows

$$(\mathbf{I} - \frac{1}{2^{m+1}}\mathbf{A})\mathbf{x} = \mathbf{b}, \tag{34}$$

where

$$\mathbf{A}_{4\times 4} = \begin{pmatrix} h_1 & h_0 & 0 & 0 \\ h_3 & h_2 & h_1 & h_0 \\ h_5 & h_4 & h_3 & h_2 \\ 0 & 0 & h_5 & h_4 \end{pmatrix}, \quad \mathbf{x} = \Big(\Phi_m(1), \Phi_m(2), \Phi_m(3), \Phi_m(4)\Big)^T,$$

Table 4: The values of $\Phi_m(x)$ for $m = 0, 1, 2, 3$ in integer points of $(0, 5]$ for Daubechies scaling function with $N = 3$.

x	$\Phi_0(x)$	$\Phi_1(x)$	$\Phi_2(x)$	$\Phi_3(x)$
1	0.60074156	0.40662888	0.30755930	0.24746347
2	1.09671144	1.01386038	1.05421749	1.16181930
3	0.98548672	0.77363651	0.53674234	0.05290363
4	0.99965908	0.81603480	0.66267182	0.42355380
5	1	0.81740116	0.66814466	0.44546004

$$\mathbf{b}_{4\times 1} = \frac{1}{2^{m+1}} \begin{pmatrix} \sum_{k=0}^{5}\sum_{r=1}^{m} \binom{m}{r} h_k k^r \Phi_{m-r}(2-k) \\ \sum_{k=0}^{5}\sum_{r=1}^{m} \binom{m}{r} h_k k^r \Phi_{m-r}(4-k) \\ \sum_{k=0}^{5}\sum_{r=1}^{m} \binom{m}{r} h_k k^r \Phi_{m-r}(6-k) + \sum_{k=0}^{1} h_k \\ \sum_{k=0}^{5}\sum_{r=1}^{m} \binom{m}{r} h_k k^r \Phi_{m-r}(8-k) + \sum_{k=0}^{3} h_k \end{pmatrix}.$$

The solution of the linear system in Equation (34) for different m is listed in Table 4 and the condition number of matrix $\mathbf{I} - \frac{1}{2^{m+1}}\mathbf{A}$ for several different m and N are listed in Table 5 and the values of the function $\Phi_m(x)$ in dyadic points for different m and $J = 1$ is shown in Table 6. Also from 5, we see that the condition number of the matrix $\mathbf{I} - \frac{1}{2^{m+1}}\mathbf{A}$ woth respect to the Euclidean vector norm, decreases with increasing m for any N. In Table 5 notation κ_m^N indicates the condition number of $\mathbf{I} - \frac{1}{2^{m+1}}\mathbf{A}$ for Daubechies scaling function correspond to the wavelet with N vanishing moment. As we will prove in Theorm 3.2, as m tends to infinity, κ_m^N tends to one (see Figure 1).

Table 5: The condition number of $\mathbf{I} - \frac{1}{2^{m+1}}\mathbf{A}$ for several m and integer points of the interval $[0, L]$ for Daubechies scaling function with several N.

m	κ_m^2	κ_m^3	κ_m^4	κ_m^5	κ_m^8	κ_m^{16}
0	2.21	3.17	3.48	3.91	4.17	4.52
1	1.36	1.64	1.74	1.84	1.89	1.95
2	1.15	1.26	1.30	1.34	1.36	1.38
3	1.07	1.12	1.14	1.15	1.16	1.17
4	1.03	1.05	1.07	1.07	1.07	1.08
2	1.01	1.02	1.03	1.03	1.03	1.04
6	1.007	1.014	1.016	1.017	1.019	1.020

Table 6: *The values of* $\Phi_m(x)$ *for* $m = 0, 1, 2, 3$ *in dyadic points of* $[0, 5]$ *for* $j = 1$ *and Daubechies scaling function with* $N = 3$.

x	$\Phi_0(x)$	$\Phi_1(x)$	$\Phi_2(x)$	$\Phi_3(x)$
$\frac{1}{2}$	0.14131460	0.04782638	0.01808707	0.00727646
$\frac{3}{2}$	1.052908	0.95455990	0.98018841	1.08388038
$\frac{5}{2}$	0.98506614	0.77005185	0.519996944	−0.01275429
$\frac{7}{2}$	1.00331834	0.82906010	0.70901844	0.58839001
$\frac{9}{2}$	0.99999150	0.81736292	0.66797242	0.44468447

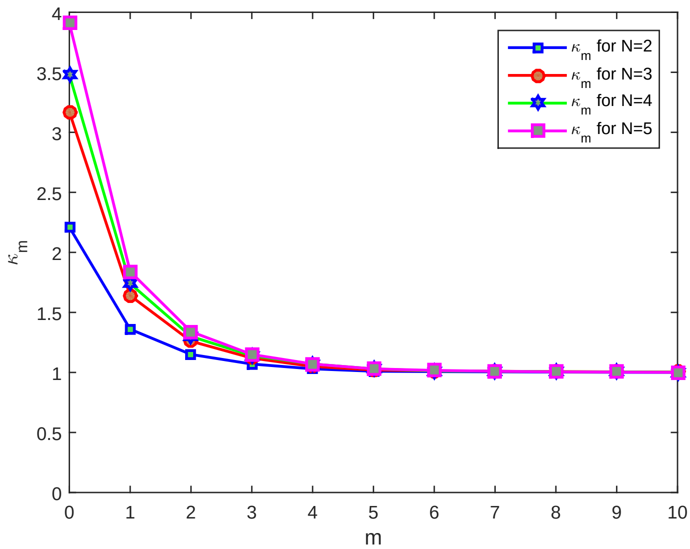

Figure 1: The condition number of $\mathbf{I} - \frac{1}{2^{m+1}}\mathbf{A}$ for different m and for Daubechies scaling functions with different N.

Theorem 4.2. If $\mathbf{I} - \frac{1}{2^{m+1}}\mathbf{A}$ be the coefficient matrix of linear system (32) for Daubechies scaling function with N vanishing moment and κ_m be its condition number for $m \geq 0$, then

$$\lim_{m \longrightarrow \infty} \kappa_m = 1.$$

Proof. Let κ_m be the condition number of a matrix $\mathbf{I} - \frac{1}{2^{m+1}}\mathbf{A}$, i.e.,

$$\kappa_m = \left\| \mathbf{I} - \frac{1}{2^{m+1}}\mathbf{A} \right\|_2 \left\| \left(\mathbf{I} - \frac{1}{2^{m+1}}\mathbf{A} \right)^{-1} \right\|_2,$$

and from Theorem 3.1 the norm of $\mathbf{A}$ independent of N is bounded, and

$\|\mathbf{A}\|_2 \leq \sqrt{2}$, thus

$$\lim_{m \longrightarrow \infty} \left\| \mathbf{I} - \frac{1}{2^{m+1}} \mathbf{A} \right\|_2 \leq \lim_{m \longrightarrow \infty} \left(\|\mathbf{I}\|_2 + \frac{1}{2^{m+1}} \|\mathbf{A}\|_2 \right)$$
$$\leq \left(1 + \lim_{m \longrightarrow \infty} \frac{1}{2^{m+1}} \sqrt{2} \right) = 1.$$

We also have

$$\lim_{m \to \infty} \left\| \left(\mathbf{I} - \frac{1}{2^{m+1}} \mathbf{A} \right)^{-1} \right\|_2 = \lim_{m \to \infty} \left\| \sum_{n=0}^{\infty} \left(\frac{1}{2^{m+1}} \mathbf{A} \right)^n \right\|_2$$
$$\leq \lim_{m \to \infty} \sum_{n=0}^{\infty} \left(\frac{\|\mathbf{A}\|}{2^{m+1}} \right)^n$$
$$= \lim_{m \to \infty} \frac{2^{m+1}}{2^{m+1} - \|\mathbf{A}\|} = 1.$$

And therefore

$$\lim_{m \longrightarrow \infty} \kappa_m = \lim_{m \longrightarrow \infty} \left\| \mathbf{I} - \frac{1}{2^{m+1}} \mathbf{A} \right\|_2 \times \lim_{m \longrightarrow \infty} \left\| \left(\mathbf{I} - \frac{1}{2^{m+1}} \mathbf{A} \right)^{-1} \right\|_2 = 1.$$

□

Another properties of the coefficient matrix $\mathbf{I} - \frac{1}{2^{m+1}} \mathbf{A}$, is the strictly diagonally dominant property. The results of the numerical calculations show that for any $m \geq 1$ and Daubechies scaling functions with $2 \leq N \leq 17$ the matrix $\mathbf{I} - \frac{1}{2^{m+1}} \mathbf{A}$ is strictly diagonally dominant. But for $N \geq 18$ this property is not established. However, for $N \geq 18$ we have the following theorm.

Theorem 4.3. If $\mathbf{I} - \frac{1}{2^{m+1}} \mathbf{A}$ be the coefficients matrix of linear system (32) for Daubechies scaling function with N vanishing moment, then for any $m \geq 0$ and $N > 2$ which satisfies

$$m \geq \frac{\ln(L-1)}{\ln(2)} - \frac{1}{2}, \quad L = 2N - 1, \tag{35}$$

this matrix is strictly diagonally dominant.

Proof. Since the matrix elements of $\mathbf{A}$ in Equation (34) have elements h_k in every elements, and in all rows of $\mathbf{A}$ elements are distinct, also the maximum number of elements in each row is $L-1$ terms of $h_k s$. If we show the elements of $\left(\mathbf{I} - \frac{1}{2^{m+1}} \mathbf{A} \right)_{ij}$ with c_{ij}, this matrix is strictly diagonally dominant if

$$|c_{ii}| > \sum_{i \neq j} |c_{ij}|, \quad i = 1, 2, \ldots, L-1.$$

On the other hand, diagonal elements in i-th row is equal to $c_{ii} = 1 - \frac{1}{2^{m+1}} h_i$, and all of non-diagonal elements c_{ij} are equal to $c_{ij} = -\frac{1}{2^{m+1}} h_{2i-j}$. So we show that

$$\Big|1 - \frac{1}{2^{m+1}} h_i\Big| > \sum_{i \neq j} \Big| - \frac{1}{2^{m+1}} h_{2i-j}\Big|, \quad i = 1, 2, \ldots, L-1. \tag{36}$$

Also according to the Equation (11), we know that $|h_k| \leq \sqrt{2}$. So we have

$$\max \Big(\sum_{i \neq j} \Big| - \frac{1}{2^{m+1}} h_{2i-j}\Big| \Big) \leq \frac{L-2}{2^{m+1}} \sqrt{2}, \tag{37}$$

and

$$\Big|1 - \frac{1}{2^{m+1}} h_i\Big| \geq \min \Big|1 - \frac{1}{2^{m+1}} h_i\Big| = 1 - \frac{\sqrt{2}}{2^{m+1}}, \tag{38}$$

In order to the Equation (36) be hold, from Equations (37)-(38) we must have that

$$1 - \frac{\sqrt{2}}{2^{m+1}} \geq \frac{L-2}{2^{m+1}} \sqrt{2}, \tag{39}$$

and finally from Equation (39) we obtain that

$$m \geq \frac{\ln(L-1)}{\ln(2)} - \frac{1}{2}.$$

□

5. Conclusion

We investigated matrices obtained integrals involving Daubechies scaling functions and plynomials. Evaluations and calculations shows that these matrices are bounded by L_2-norm [10].

We proved that the systems of equations associated with these matrices are well-conditioned. Also we proved that in general for $2 \leq N \leq 17$ vanishing moment of Daubechies wavelets and under some conditions for all $N \geq 18$ on m, these matrices are strictly diagonally dominant.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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