Mathematics Interdisciplinary Research **6** (2021) 257 – 273

Original Scientific Paper

Hyperideals of (Finite) General Hyperrings

Reza Ameri ⋆, *Mohammad Hamidi and Hoda Mohammadi*

Abstract

A general hyperring is an algebraic hypercompositional system $(R, +, \cdot)$ with two hyperoperations "+" and " · ", such that for all $x, y \in R$, $x + y$ and $x \cdot y$ are non-empty subsets of R, and R satisfies the axioms similar to a ring. We introduce and study hyperideals of a general hyperring. In this regards, we construct a connection between classical rings and general hyperrings, specifically, we extend a ring to a general hyperring in nontrivial way. Moreover, a way to construct a general hyperring from set are given. Also, we concentrate on an important class of general hyperrings, which is called Krasner hyperrings, and discuss on their hyperideals. Finally, the set of all hyperideals of a finite general (resp. Krasner) hyperring are considered and its hyperideals are investigated.

Keywords: general hyperring, hyperideals, Krasner hyperring

2010 Mathematics Subject Classification: 20N20.

How to cite this article
R. Ameri, M. Hamidi and H. Mohammadi, Hyperideals of (finite) general hyperrings, *Math. Interdisc. Res.* **6** (2021) 257 – 273.

1. Introduction

The hyperstructure theory, as an extension of classical structures, introduced by F. Marty in 1934 [10]. In a hyperalgebraic system (or a hyperstructure) a hyperproduct (or a hyperoperation) of elements is a (nonvoid) set, and so any algebraic system is an special case of a hyperalgebraic system. F. Marty extended

⋆Corresponding author (E-mail: rameri@ut.ac.ir)
Academic Editor: Mohammad Ali Iranmanesh
Received 19 January 2020, Accepted 28 December 2021
DOI:10.22052/mir.2021.240436.1269

the concept of a group to a hypergroup, and other researchers presented the hyperalgebraic concepts such as (Krasner) hyperrings, hypermodules, hyperfields, polygroups, multirings and etc. (for more details see [7], [8]). Nowadays hyperstructures are applied in several branches of sciences, such as artificial intelligence, chemistry and (hyper) complex network [8]. Hyperring is a hypercompositional structure as generalizing of a ring, that the sum and the product of two elements is not an element but a subset. The concept of Krasner hyperring was introduced by Krasner [9], who used it as a tool on the approximation of valued fields or the second type of a hyperring as multiplicative hyperring (the multiplication is a hyperoperation, while the addition is an operation was introduced by R. Rota in 1982 [12]. Further materials regarding hyperrings and mutirings are available in the literature too [1, 2, 3, 6]. Hyperideals are one of the important tools in hyperrings and some researchers have investigated in these concepts, see [4, 11].

This paper studies the general hyperrings as a generalization of Krasner hyperrings as well as classical rings. In this regards we give a manner to extend a finite ring to a (nontrivial) general hyperring. Indeed, we show how one can extend the class of (finite) rings to a class of general hyperrings. We will proceed to construct a general hyperring from an ideal nonempty set. Finally, we introduce some class of hyperideals of a special general hyperrings as well as the class of Krasner hyperrings and investigate their properties.

2. Preliminaries

In this section, we review some concepts from hyperstructures, which we need to development our paper [1, 5]. Let $n \in \mathbb{N}^*, H \neq \emptyset$ and $\mathcal{P}^*(H) = \{G \mid \emptyset \neq G \subseteq H\}$. A map $\varrho : H^n \to \mathcal{P}^*(H)$ is an n-ary hyperoperation with *arity* n, ($n = 0$, *nullary hyperoperation*) and $(H, \{\varrho_i\}_{i\in I})$ is a *hyperalgebra* ($|I| = 1$, *hypergroupoid*) of type $\varphi : I \to \mathbb{N}^*$. For two hyperalgebras $(H, \{\varrho_i\}_{i\in I})$ and $(H', \{\varrho'_i\}_{i\in I'})$, if $I = I'$, say *similar* hyperalgebras. Let H be a hyperalgebra, a $\emptyset \neq S \subseteq H$ is said to be a *subhyperalgebra* of H if for any $(a_1, \ldots, a_{n_i}) \in S^{n_i}$, $\varrho_i(a_1, \ldots, a_{n_i}) \subseteq S$. For similar hyperalgebras $(H, \{\varrho_i\}_{i\in I}), (H', \{\varrho'_i\}_{i\in I})$, a map $h : H \to H'$ is called a *homomorphism*, if for any $i \in I$ and any $(a_1, \ldots, a_{n_i}) \in H^{n_i}$, we have $h(\varrho_i((a_1, \ldots, a_{n_i})) \subseteq {\varrho'}_i(h(a_1), \ldots, h(a_{n_i}))$ and a *good homomorphism* if for any $i \in I$, and any $(a_1, \ldots, a_{n_i}) \in H^{n_i}$, $h(\varrho_i((a_1, \ldots, a_{n_i})) = {\varrho'}_i(h(a_1), \ldots, h(a_{n_i}))$. A *hypergroupoid* (H, ϱ), where ϱ is an associative is said to be a *semihypergroup*, and a semihypergroup (H, ϱ) is called a *hypergroup*, if for any $x \in H, \varrho(x, H) = \varrho(H, x) = H$. A hypergroup (H, ϱ) is a *canonical* hypergroup, if always

(i) $\varrho(x, y) = \varrho(y, x)$,

(ii) there is a unique element $e \in H$, that for any $x \in H$, $\varrho(e, x) = \varrho(x, e) = \{x\}$ (*neutral element*),

(iii) $x \in \varrho(y, z)$ concludes that $y \in \varrho(x, \vartheta(z))$ and $z \in \varrho(\vartheta(y), x)$, where ϑ is a unitary operation on $H\big($for any $x \in H$, there is a unique element $\vartheta(x) \in H$

i.e $e \in \big(\varrho(x, \vartheta(x)) \cap (\varrho(\vartheta(x), x)), \vartheta(e) = e, \vartheta(\vartheta(x)) = x\big)$ and is denoted by $(H, \varrho, e, \vartheta)$ or $(H, +, 0, -)$.

A *Krasner hyperring* is an algebraic hypercompositional $(R, +, \cdot)$, where

(i) $(R, +)$ is a canonical hypergroup,

(ii) $(R, \cdot)$ is a semigroup,

(iii) for any $r, s, t \in R, r \cdot (s + t) = r \cdot s + r \cdot t$ and $(s + t) \cdot r = s \cdot r + t \cdot r$, (iv) for any $r \in R : r \cdot 0 = 0 \cdot r = 0$, i.e. there is a unique element $0 \in R$ is an absorbing element.

A *general hyperring* is an algebraic hypercompositional $(R, +, \cdot)$, where

(i) $(R, +)$ is a hypergroup,

(ii) $(R, \cdot)$ is a semihypergroup and

(iii) for any $x, y, z \in R$: $x \cdot (y+z) = x \cdot y + x \cdot z$ and $(x+y) \cdot z = x \cdot z + y \cdot z$. A $\emptyset \neq I \subseteq R$ is said to be a (*right*)*left hyperideal*, if $(1)(I, +)$ is a subhypergroup of $(R, +)$ and $(2)(R \cdot I \subseteq I)(R \cdot I) \subseteq I$ and a hyperideal if $R \cdot I \subseteq I$ and $R \cdot I \subseteq I$.

3. Construction of a General Hyperring

We introduce two notions of $(+)$-commutative general hyperring as $(\cdot)$-commutative general hyperring and generalize these notions to commutative general hyperrings. In this regards, a general hyperring is constructed on an ideal non-empty set and is determined how to extend a ring to a general hyperring.

Definition 3.1. Assume $(R, +, \cdot)$ is a general hyperring. Then R is:

(i) additive commutative or ((+)-commutative) general hyperring, if it is commutative with respect to the additive hyperoperations "+",

(ii) multiplicative commutative or $((\cdot)$-commutative) general hyperring, if it is commutative with respect to the multiplicative hyperoperation " $\cdot$ ",

(iii) commutative general hyperring, if it is commutative with respect to the both hyperoperations " + " and " $\cdot$ ".

Now, we introduced a manner to construct a general hyperring from a given ring by using a normal subgroup of its multiplicative semigroup.

Theorem 3.2. *Suppose $(R, +, .)$ is a ring and N be a subgroup of its multiplicative semigroup. The there are hyperoperations $+'$ and $\cdot'$ on the quotient space R/N, that the hyperstructure $(R/N, +', \cdot')$ is a general hyperring.*

Proof. For all $xN, yN \in R/N$ define "$+'$" and "$\cdot'$" on R/N by $xN +' yN = \{xN, yN\}$ and $xN.'yN = \{(x.y)N, xN, yN\}$. Now it is easy to see that $(R/N, +', \cdot')$ is a general hyperring. □

Theorem 3.3. *Assume $(R, +, .)$ is a ring and N is a subgroup of its multiplicative semigroup. Then there are hyperoperations "$+'$" and "$\cdot'$" on R/N, that $(R/N, +', \cdot')$ is a general hyperring.*

Proof. For all $xN, yN \in R/N$ define "$+'$" and "$\cdot'$" on R by

$$xN +' yN = \{0N\} \cup \bigcup_{z \in (x+y)N} zN \text{ and } xN.'yN = \begin{cases} \{0N\} \cup \bigcup_{z \in (x.y)N} zN & x \neq y \\ \bigcup_{z \in (x.y)N} zN & x = y \end{cases}.$$

Let $x \in R$. Then $xN +' R/N = \{0N\} \cup \bigcup_{y \in R} (xN +' yN) = \{0N\} \cup \bigcup_{y \in R} \bigcup_{z \in (x+y)N} zN = R/N$. Since $(R, +, .)$ is a ring, we get that $(R/N, +')$ is a hypergroup and $(R/N, \cdot')$ is a semihypergroup. Thus distributive property hold, and hence, $(R/N, +', \cdot')$ is a general hyperring. □

Now, a way to construct a commutative general hyperring on ideal non-empty set are given.

Theorem 3.4. *Suppose R is a non empty set. Then there are hyperoperations "$+$" and "$\cdot$" on R, that $(R, +, \cdot)$ is a commutative general hyperring.*

Proof. For $|R| = 1$, it is done. Let $|R| \geq 2$. Then for any $x, y \in R$ define "$+$" and "$\cdot$" on R by

$$x + y = x \cdot y = \{x, y\}.$$

Obviously, $(R, +, \cdot)$ is a nontrivial commutative general hyperring. □

At the next theorem, a way to construct a (nontrivial)general hyperring on an ideal ring is given.

Theorem 3.5. *Assume $(R, +, 0, \cdot)$ is a commutative ring. Then there are hyperoperations "$+'$" and "$\cdot'$"on R, that $(R, +', 0, \cdot')$ is a general hyperring.*

Proof. If $|R| = 1$, it is clear. Let $|R| \geq 2$. Then for any $x, y \in R$, define "$+'$" and "$\cdot'$" on R by

$$x +' y = \{x, y, x + y\} \qquad \text{and } x \cdot' y = \{x \cdot y, 0\}.$$

One can verify that $(R, +', \cdot')$ is a general hyperring. □

Example 3.6. (i) Let $R = \{Z(a,b) \mid \overline{a}, \overline{b} \in \mathbb{Z}_2\}$, where $Z(a,b) = \overline{a} + \overline{b}i$. Then by Theorem 3.5, $(R, +', \cdot')$ is a commutative general hyperring of cardinal 8 as follows:

$$Z(a,b) +' Z(c,d) = \{Z(a,b), Z(c,d), Z(a+c, b+d) \mid \overline{a}, \overline{b}, \overline{c}, \overline{d} \in \mathbb{Z}_2\}\},$$
$$Z(a,b) \cdot' Z(c,d) = \{Z(ac-bd, ad+bc), \overline{0} \mid \overline{a}, \overline{b}, \overline{c}, \overline{d} \in \mathbb{Z}_2\}\}.$$

(ii) Let $R = \Big\{ \begin{bmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{c} \end{bmatrix} \mid \overline{a}, \overline{b}, \overline{c} \in \mathbb{Z}_2 \Big\}$.

Then in virtue Theorem 3.5, the hyperstructure $(R, +', \cdot')$ is a non-commutative general hyperring of cardinal 16 as follows:

$$\begin{bmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{c} \end{bmatrix} +' \begin{bmatrix} \overline{d} & \overline{e} \\ \overline{0} & \overline{f} \end{bmatrix} = \{ \begin{bmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{c} \end{bmatrix}, \begin{bmatrix} \overline{d} & \overline{e} \\ \overline{0} & \overline{f} \end{bmatrix}, \begin{bmatrix} \overline{a+d} & \overline{b+e} \\ \overline{0} & \overline{c+f} \end{bmatrix} \mid \overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f} \in \mathbb{Z}_2\}\},$$
$$\begin{bmatrix} \overline{a} & \overline{b} \\ \overline{0} & \overline{c} \end{bmatrix} \cdot' \begin{bmatrix} \overline{d} & \overline{e} \\ \overline{0} & \overline{f} \end{bmatrix} = \{ \begin{bmatrix} \overline{ad} & \overline{ae+bf} \\ \overline{0} & \overline{cf} \end{bmatrix}, \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix} \mid \overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f} \in \mathbb{Z}_2\}\}.$$

Now, a way to construct a class of general hyperrings on the cyclic group $\mathbb{Z}_n$, for every $n \in \mathbb{N}$ are given.

Theorem 3.7. *Let $n \in \mathbb{N}$. Then there is a hyperoperation "$\oplus$" on $\mathbb{Z}_n$, that $(\mathbb{Z}_n, \oplus)$ is a hypergroup.*

Proof. Fix $\overline{a} \in \mathbb{Z}_n$, where $\overline{a} \neq \overline{0}$. Then for any $\overline{x}, \overline{y} \in \mathbb{Z}_n$, define "$\oplus$" as follows:

$$\overline{x} \oplus \overline{y} = \overline{x} +_{\overline{a}} \overline{y} = \{\overline{x+y}, \overline{x+y+a}\}.$$

Let $\overline{x} \in \mathbb{Z}_n$. Then $\overline{x} \oplus \mathbb{Z}_n = \bigcup_{\overline{y} \in \mathbb{Z}_n} (\overline{x} \oplus \overline{y}) = \bigcup_{\substack{\overline{y} \in \mathbb{Z}_n \\ \overline{a} \in \mathbb{Z}_n}} \{\overline{x+y}, \overline{x+y+a}\} = \mathbb{Z}_n$. Since $(\mathbb{Z}_n, +)$ is an abelian group, it get that $(\mathbb{Z}_n, \oplus)$ is a hypergroup. □

By Theorem 3.5, a general way to constructed a general hyperring from an ideal ring was introduced. In the following, a different way from that in Theorem 3.5, to construct a general hyperring for the finite ring $(\mathbb{Z}_n, +, \cdot)$, whence $n \in \mathbb{N}$ are presented.

Theorem 3.8. *Assume n is an even. Then there are hyperoperations "$\oplus$" and "$\odot$", that $(\mathbb{Z}_n, \oplus, \odot)$ is a general hyperring.*

Proof. Fix $\overline{0} \neq \overline{a} \in \mathbb{Z}_n$, where $\overline{2a} = \overline{0}$. By Theorem 3.7, $(\mathbb{Z}_n, \oplus)$ is a hypergroup. Then for any $\overline{x}, \overline{y} \in \mathbb{Z}_n$. Define $\odot$ as follows:

$$\overline{x} \odot \overline{y} = \overline{x} \cdot_{\overline{a}} \overline{y} = \{\overline{xy}, \overline{xy+a}\}.$$

Let $\overline{x}, \overline{y}, \overline{z} \in \mathbb{Z}_n$. Then

(Associative):

$\overline{x} \odot (\overline{y} \odot \overline{z}) = \overline{x} \odot \{\overline{yz}, \overline{yz+a}\} = \{\overline{xyz}, \overline{xyz+a}, \overline{x(yz+a)}, \overline{x(yz+a)+a}\}$ and

$$(\overline{x} \odot \overline{y}) \odot \overline{z} = \{\overline{xy}, \overline{xy+a}\} \odot \overline{z} = \{\overline{xyz}, \overline{xyz+a}, \overline{(xy+a)z}, \overline{(xy+a)z+a}\}.$$

Now, $\overline{x} \odot (\overline{y} \odot \overline{z}) = (\overline{x} \odot \overline{y}) \odot \overline{z}$, implies $\overline{xa} = \overline{za}$ or $\overline{xa} = \overline{za+a}$ and $\overline{za} = \overline{xa+a}$. Thus $(\mathbb{Z}_n, \odot)$ is a semihypergroup if and only if $\overline{2a} = \overline{0}$.

(Distributive):

$$\begin{aligned} \overline{x} \odot (\overline{y} \oplus \overline{z}) &= \overline{x} \odot \{\overline{y+z}, \overline{y+z+a}\} \\ &= \{\overline{x(y+z)}, \overline{x(y+z)+a}, \overline{x(y+z+a)}, \overline{x(y+z+a)+a}\}, \end{aligned}$$

and

$$\begin{aligned} \overline{x} \odot \overline{y} \oplus \overline{x} \odot \overline{z} &= \{\overline{xy}, \overline{xy+a}\} + \{\overline{xz}, \overline{xz+a}\} \\ &= \{\overline{xy+xz}, \overline{xy+xz+a}, \overline{xy+xz+2a}, \overline{xy+xz+3a}\}. \end{aligned}$$

Now, $\overline{2a} = \overline{0}$ implies that $\overline{ka} = \overline{0}$ and $\overline{k'a} = \overline{a}$, where k is an even and k' is an odd. Thus

$$\overline{x} \odot (\overline{y} \oplus \overline{z}) = \overline{x} \odot \overline{y} \oplus \overline{x} \odot \overline{z}.$$

Therefore, $(\mathbb{Z}_n, \oplus, \odot)$ is a general hyperring. □

Remark 1. In Theorem 3.8, if n is odd, then $(\mathbb{Z}_n, \odot)$ is not a semihypergroup, and so $(\mathbb{Z}_n, \oplus, \odot)$ is not a general hyperring.

Example 3.9. By Theorem 3.8, $(\mathbb{Z}_6, \oplus, \odot)$ is a general hyperring by the following hyperoperations:

$+_3$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\{\overline{0},\overline{3}\}$	$\{\overline{1},\overline{4}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{5},\overline{2}\}$
$\overline{1}$	$\{\overline{1},\overline{4}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{5},\overline{2}\}$	$\{\overline{0},\overline{3}\}$
$\overline{2}$	$\{\overline{2},\overline{5}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{5},\overline{2}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{1},\overline{4}\}$
$\overline{3}$	$\{\overline{3},\overline{0}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{5},\overline{2}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{1},\overline{4}\}$	$\{\overline{2},\overline{5}\}$
$\overline{4}$	$\{\overline{4},\overline{1}\}$	$\{\overline{5},\overline{2}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{1},\overline{4}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{3},\overline{0}\}$
$\overline{5}$	$\{\overline{5},\overline{2}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{1},\overline{4}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{1},\overline{4}\}$

,

$\cdot_3$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\{\overline{0},\overline{3}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{0},\overline{3}\}$
$\overline{1}$	$\{\overline{0},\overline{3}\}$	$\{\overline{1},\overline{4}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{5},\overline{2}\}$
$\overline{2}$	$\{\overline{0},\overline{3}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{4},\overline{1}\}$
$\overline{3}$	$\{\overline{3},\overline{0}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{0},\overline{3}\}$
$\overline{4}$	$\{\overline{0},\overline{3}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{0},\overline{3}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{2},\overline{5}\}$
5	$\{\overline{0},\overline{3}\}$	$\{\overline{5},\overline{2}\}$	$\{\overline{4},\overline{1}\}$	$\{\overline{3},\overline{0}\}$	$\{\overline{2},\overline{5}\}$	$\{\overline{1},\overline{4}\}$

.

Let $m \in \mathbb{N}$ and $R = (\mathbb{Z}_n, \oplus, \odot)$ be the general hyperring in Theorem 3.8. Define $\overline{x} \in R, 1\overline{x} = \{\overline{x}\}, m\overline{x} = \underbrace{\overline{x} \oplus \overline{x} \oplus \ldots \overline{x} \oplus \overline{x}}_{m-times}$, $\overline{x}^1 = \{\overline{x}\}$, $\overline{x}^m = \underbrace{\overline{x} \odot \overline{x} \odot \ldots \overline{x} \odot \overline{x}}_{m-times}$ and $(-m)\overline{x} = m(\overline{n-x})$ and $(\overline{x})^{-m} = (\overline{n-x})^m$.

So, the following theorem follows.

Theorem 3.10. *Suppose $p = 2$ and $2 \leq n$. Then in the general hyperring $(\mathbb{Z}_{p^n}, \oplus, \odot)$:*

(i) *if $\overline{2a} = \overline{0}$, then for any $k \in \mathbb{N}$, we have $k\overline{a} = \{\overline{0}, \overline{a}\}$,*

(ii) *if $\overline{2a} = \overline{0}$, then for any $k \in \mathbb{N}$, we have $\overline{a}^k = \{\overline{0}, \overline{a}\}$.*

Proof. (i) By induction on $k \in \mathbb{N}$, if k is odd, then $\overline{ka} = \overline{a}$. If k is even, then $\overline{ka} = \overline{0}$.

(ii) By Theorem 3.8, $\overline{a}^2 = \{\overline{a^2}, \overline{a^2 + a}\}$. If $\overline{a^2} = \overline{x}$, then $a^2 \equiv x \pmod{p^n})\}$. Since $\overline{2a} = \overline{0}$, there is $t \in \mathbb{N}$, that $x = a(a+t)$ and so $a \mid x$ (a mids k). So by item (i), $\overline{x} = \overline{0}$ or $\overline{x} = \overline{a}$. Then by induction for any $k \in \mathbb{N}$, we get that $\overline{a}^k = \{\overline{0}, \overline{a}\}$.

□

Theorem 3.11. *Let p be a prime. Then there are hyperoperations "$+_p$" and "$\cdot_p$", that $(\mathbb{Z}_p, +_p, \cdot_p)$ is a general hyperring.*

Proof. Fix $\overline{0} \neq \overline{a} \in \mathbb{Z}_n$. For $\overline{x}, \overline{y} \in \mathbb{Z}_n$, define "$+_p$" and "$\cdot_p$" as follows:

$$\overline{x} +_{\{p,\overline{a}\}} \overline{y} = \begin{cases} \overline{0} & \overline{x} = \overline{y} = \overline{0} \\ \{\overline{x+y}, \overline{x+y+a}\} & \text{otherwise} \end{cases} \text{ and } \quad \overline{x} \cdot_{\{p,\overline{a}\}} \overline{y} = \begin{cases} \overline{0} & \overline{x} = \overline{0} \text{ or } \overline{y} = \overline{0} \\ \mathbb{Z}_p & \text{otherwise} \end{cases}.$$

Let $\overline{x} \in \mathbb{Z}_p$. Then

$$\overline{x} +_{\{p,\overline{a}\}} \mathbb{Z}_p = \{\overline{0}\} \cup \bigcup_{\overline{y} \in \mathbb{Z}_p} (\overline{x} +_{\{p,\overline{a}\}} \overline{y}) = \{\overline{0}\} \cup \bigcup_{\substack{\overline{y} \in \mathbb{Z}_p \\ \overline{a} \in \mathbb{Z}_p}} \{\overline{x+y}, \overline{x+y+a}\} = \mathbb{Z}_p.$$

Since $(\mathbb{Z}_p, +, \cdot)$ is a commutative ring, we get that $(\mathbb{Z}_p, +_p)$ is a hypergroup. In addition, $\overline{x} \cdot_{\{p,\overline{a}\}} \mathbb{Z}_p$ is equal to $\mathbb{Z}_p$ or $\{\overline{0}\}$, so $(\mathbb{Z}_p, \cdot_p)$ is a semihypergroup. The distributive property is valid, thus $(\mathbb{Z}_p, +_p, \cdot_p)$ is a general hyperring. □

4. Hyperideals of General Hyperrings

Now, we present hyperideals of a general hyperring. In particular, we determine hyperideals of finite commutative general hyperrings.

Theorem 4.1. *Suppose $(R, +, \cdot)$ is a general hyperring and $\emptyset \neq I \subseteq R$. Then I is a hyperideal of R if and only if the following hold:*

(i) *for any* $x \in I, x + I = I + x = I$,

(ii) *for any* $r \in R$ *and* $x \in I$, *we have* $(r \cdot x) \cup (x \cdot r) \subseteq I$.

Proof. Immediate by definition. □

Assume $(R, +, \cdot)$ is a general hyperring. We symbolize the set hyperideals of R by $\mathcal{I}(R)$. Clearly, $R \in \mathcal{I}(R) \neq \emptyset$ and will call R as a non-proper hyperideal of any general hyperring.

Example 4.2. (i) Consider general hyperrings in Example 3.6 (i). Then

$$\mathcal{I}(R) = \{I_1 = \{0\}, I_2 = \{0, 1, i, 1+i\}, I_3 = \{0, 1+i\}\}.$$

(ii) Consider general hyperrings R, in Example 3.6 (ii). Then

$$\begin{aligned} \mathcal{I}(R) &= \big\{I_1 = \{A_1\}, I_2 = \{A_1, A_2\}, I_3 = \{A_1, A_2, A_3, A_7\}, I_4 \\ &= \{A_1, A_2, A_4, A_5\}, , I_5 = R\big\}, \end{aligned}$$

where,

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$
$$A_6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_7 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \textit{and } A_8 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Easily, $I_3 \cup I_4 = \{A_1, A_2, A_3, A_4, A_5, A_7\} \notin \mathcal{I}(R)$.

Theorem 4.3. *Assume* $(R, +, \cdot)$ *is a commutative general hyperring and* $I, I' \in \mathcal{I}(R)$. *Then*

(i) $I + I' \in \mathcal{I}(R)$.

(ii) *if* $I \cap I' \neq \emptyset$, *then* $I \cap I' \in \mathcal{I}(R)$.

Proof. (i) Clearly, $I + I' \neq \emptyset$. Let $a \in I$ and $a' \in I'$. Then for any $z \in ((a + a') + I + I')$, there is $b \in I, b' \in I'$, that $z \in (a + a') + (b + b') = (a+b)+(a'+b') \subseteq I+I'$. If $z \in I+I'$ be an arbitrary element in $I+I'$, then there are $a, b, c \in I$, and $a', b', c' \in I'$, that $z \in c + c' \subseteq (a + b) + (a' + b') = (a+a')+(b+b') \subseteq (a+a')+(I+I')$. Hence $(a+a')+(I+I') = I+I'$. Now, for any $r \in R, a \in I$ and $a' \in I'$, one obtains $(r \cdot (a + a')) \cup ((a + a') \cdot r) = (r \cdot a + r \cdot a') \cup (a \cdot r + a' \cdot r) = r \cdot a + r \cdot a' \subseteq I + I'$. Hence $I + I' \in \mathcal{I}(R)$.

(ii) It is straightforward.

□

Theorem 4.4. *Assume* $(R, +, \cdot)$, $(S, +, \cdot)$ *are general hyperrings,* $f : R \to S$ *be a homomorphism, and* $I \in \mathcal{I}(R)$ *and* $J \in \mathcal{I}(S)$.

(i) *If f is an epimorphism, then $f(I) \in \mathcal{I}(S)$.*

(ii) $f^{-1}(J) \in \mathcal{I}(R)$.

Proof. (i) Since $\emptyset \neq I$, we have $f(I) \neq \emptyset$. Let $f(a) \in f(I)$. Then for every $f(b) \in f(I)$, there is $a' \in I$, that $b \in a + a'$, and so $f(b) \in f(a+a') = f(a) + f(a') \subseteq f(a)+I$. Hence $f(I) \subseteq f(a)+f(I)$. If $c \in f(a)+f(I)$ is an arbitrary element, then there is $a' \in I$, that $c \in f(a)+f(a') = f(a+a') \subseteq f(I)$. Hence, $f(a) + f(I) = f(I)$. Now, for any $s \in S$ and $f(a) \in f(I)$, there is $r \in R$, that

$$\begin{aligned}(s \cdot f(a)) \cup (f(a) \cdot s) &= (f(r) \cdot f(a)) \cup (f(a) \cdot f(r)) = (f(r \cdot a)) \cup (f(a \cdot r)) \\ &= f(r \cdot a \cup a \cdot r) \subseteq f(I).\end{aligned}$$

(ii) It is straightforward.

□

4.1 Hyperideals of $R = (\mathbb{Z}_n, \oplus, \odot)$

At the following the hyperideals of a commutative general hyperring $R = (\mathbb{Z}_n, \oplus, \odot)$ are obtained and it is shown that they are depended to the divisors elements, say a, where $\overline{a} \in \mathbb{Z}_n$ and $\overline{2a} = \overline{0}$.

Theorem 4.5. *Assume p is a prime. Then $\mathcal{I}(\mathbb{Z}_p, +_p, \cdot_p) = \{\{\overline{0}\}, \mathbb{Z}_p\}$.*

Proof. Obviously, $I = \{\overline{0}\} \in \mathcal{I}((\mathbb{Z}_p, +_p, \cdot_p)$. Let $\{\overline{0}\} \neq I \in \mathcal{I}((\mathbb{Z}_p, +_p, \cdot_p)$ be a hyperideal and $\overline{0} \neq \overline{x} \in I$. Since for any $\overline{0} \neq \overline{y} \in \mathbb{Z}_p, \mathbb{Z}_p = \overline{x}._p\overline{y} \subseteq I$, it obtain that $I = \mathbb{Z}_p$. Thus $|\mathcal{I}(\mathbb{Z}_p, +_p, \cdot_p)| = 2$. □

Example 4.6. Let $\overline{a} = \overline{2}$. Then for $n = 4$, $R = (\mathbb{Z}_n, +_n, \cdot_n)$ is a general hyperrings and $\mathcal{I}(\mathbb{Z}_n, +_n, \cdot_n) = \{\{\overline{0}\}, \mathbb{Z}_n\}$ and otherwise $R = (\mathbb{Z}_n, +_n, \cdot_n)$ is not a hyperring.

Example 4.6, disprove the converse of Theorem 4.5, necessarily is not true.

Assume $R = (\mathbb{Z}_n, \oplus, \odot)$ is the general hyperring in Theorem 3.8 and $\overline{x} \in R$. Define

$$\langle \overline{x} \rangle = \bigcup_{k \in \mathbb{N}} k\overline{x}.$$

Thus have next results.

Theorem 4.7. *Suppose $2 \leq n$ is an even, $\overline{a} \in R$ and $\overline{x} \in R$. If $\overline{2a} = \overline{0}$, then*

(i) $\langle \overline{x} \rangle \in \mathcal{I}(\mathbb{Z}_n, \oplus, \odot)$,

(ii) $\langle \overline{0} \rangle = \langle \overline{a} \rangle$,

(iii) *if $x \neq a$ and $\gcd(x, a) = d$, we have $\langle \overline{x} \rangle = \langle \overline{d} \rangle$,*

(iv) *$I \in \mathcal{I}(\mathbb{Z}_n, \oplus, \odot) \Leftrightarrow$ there is $\overline{x} \in R$, that $I = \langle \overline{x} \rangle$.*

Proof. (i) Let $\overline{x} \in R$. By definition, we have

$$\langle \overline{x} \rangle = \bigcup_{k\in\mathbb{N}} \{\overline{kx}, \overline{kx+a}\}.$$

This shows that it is a hyperideal of R. Let $\overline{y} \in \langle \overline{x} \rangle$ and $\overline{z} \in \overline{y} \oplus \langle \overline{x} \rangle$. Thus there is $k, k' \in \mathbb{N}$ and $\overline{w} \in \langle \overline{x} \rangle$, that $\overline{z} \in \overline{y} \oplus \overline{w}$ and hence

$$\overline{z} \in \overline{kx} \oplus \overline{k'x}, \overline{z} \in \overline{kx} \oplus \overline{k'x+a} \text{ and } \overline{z} \in \overline{kx+a} \oplus \overline{k'x}.$$

Since $\overline{2a} = \overline{0}$, there is $k'' \in \mathbb{N}$, that

$$\overline{z} \in \{\overline{k''x}, \overline{k''x+a}\} \subseteq \langle \overline{x} \rangle.$$

Similarly, it implies that $\overline{r} \in \mathbb{Z}_n$ and $\overline{y} \in \langle \overline{x} \rangle$. Thus

$$\overline{r} \odot \langle \overline{x} \rangle \subseteq \langle \overline{x} \rangle.$$

Therefore,

$$\langle \overline{x} \rangle \in \mathcal{I}(\mathbb{Z}_n, \oplus, \odot).$$

(ii) By Theorem 3.10, we get that

$$\langle \overline{0} \rangle = \langle \overline{a} \rangle = \{\overline{0}, \overline{a}\}.$$

Therefore, I is a hyperideal as desired.

(iii) Let $\overline{y} \in \langle \overline{x} \rangle$. Then there is $k \in \mathbb{N}$, that $\overline{y} = \overline{kx}$ or $\overline{y} = \overline{kx+a}$. Since $\gcd(x, a) = d$ and $\overline{2x} \neq \overline{0}$, by item (i), there is $k' \in \mathbb{Z}$, that $x = k'd$.

If $\overline{y} = \overline{kx}$, then

$$\overline{y} = \overline{kx} = \overline{kk'd} \in \langle \overline{x} \rangle,$$

and if $\overline{y} = \overline{kx+a}$, then $\overline{y} = \overline{kx+a} = \overline{kk'd+a} \in \langle \overline{x} \rangle$. Hence

$$\langle \overline{x} \rangle \subseteq \langle \overline{d} \rangle.$$

Let $\overline{y} \in \langle \overline{d} \rangle$. Then there is $k \in \mathbb{N}$, that $\overline{y} = \overline{kd}$ or $\overline{y} = \overline{kd+a}$. Since $\gcd(x, a) = d$ and $\overline{2x} \neq \overline{0}$ by (i), there is $r, s \in \mathbb{Z}$, that $rx + as = d$, and so $\overline{rkx+aks} = \overline{kd}$. Applying Theorem 3.10, we get that $\overline{y} = \overline{krx}$ or $\overline{y} = \overline{krx+a}$. Hence

$$\langle \overline{d} \rangle \subseteq \langle \overline{x} \rangle.$$

(iv) Let $I \in \mathcal{I}(\mathbb{Z}_n, \oplus, \odot)$. Then $I \neq \emptyset$ and so there is $\overline{x} \in I$. Since I is a hyperideal, we get that $\overline{x} \oplus \overline{x} \subseteq I$. By induction on $k \in \mathbb{N}$ it conclude that $k\overline{x} \subseteq I$, and hence

$$I = \bigcup_{k\in\mathbb{N}} \{\overline{kx}, \overline{kx+a}\} = \langle \overline{x} \rangle.$$

This complete the proof.

□

Example 4.8. Consider the general hyperring $R = (\mathbb{Z}_{12}, \oplus, \odot)$. By Theorem 4.7,

$\mathcal{I}(\mathbb{Z}_{12}, \oplus, \odot) = \{I_1 = \{\overline{0}\}, I_2 = \{\overline{0}, \overline{2}, \overline{4}, \ldots, \overline{10}\}, I_3 = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}\}, I_4 = \{\overline{0}, \overline{6}\}, I_5 = \mathbb{Z}_{12}\}$.

Theorem 4.9. *Let $2 \leq n$ and $\overline{a} \in R$. If $\overline{2a} = \overline{0}$, then*

(i) $|\mathcal{I}(\mathbb{Z}_n, \oplus, \odot)| = |Div(a)| + 1$,

(ii) *if for any $\overline{x} \in R, x \mid a$, then $\overline{a} \in \langle \overline{x} \rangle$.*

Proof. (i) Clearly $\{\overline{0}\} \in \mathcal{I}$. By Theorem 4.7, $\{\overline{0}\} \neq I \in \mathcal{I}$ if and only if there is $\overline{x} \in R$ that $I = \langle \overline{x} \rangle$. Also for any $\overline{x} \in R, gcd(x, a) = d$ if and only if $\langle \overline{x} \rangle = \langle \overline{d} \rangle$. So $|\mathcal{I}(\mathbb{Z}_n, \oplus, \odot)| = |Div(a)| + 1$.

(ii) Let $\overline{x} \in R$. By definition, we have $\langle \overline{x} \rangle = \bigcup_{k \in \mathbb{N}} \{\overline{kx}, \overline{kx + a}\}$. If $x \mid a$ (x mids a), then there is $k \in \mathbb{N}$, that $a = kx$ and so $\overline{a} \in \langle \overline{x} \rangle$.

□

Corollary 4.10. *Let $2 = p_1, p_2, \ldots, p_k$ be primes, $k, \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$ and $n = \prod_{i=1}^{k} p_i^{\alpha_i}$. Then*

$(\mathbb{Z}_n, \oplus, \odot) = \{\overline{0}\} \cup \{\langle \overline{p_1^{t_1} p_2^{t_2} \ldots p_j^{t_j}} \rangle \mid 0 \leq t_1 \leq \alpha_1 - 1,$ *and for all* $j \neq 1,\ 0 \leq t_j \leq \alpha_i\}\rangle$.

Example 4.11. Let $2 = p$ and q, r be odd primes, $m, l, k \in \mathbb{N}$ and $n = p^m q^l r^k$. Then

$$\mathcal{I}(\mathbb{Z}_n, \oplus, \odot) = \{\overline{0}\} \cup \{\langle \overline{p^{t_1} q^{t_2} r^{t_3}} \rangle \mid 0 \leq t_1 \leq m - 1, 0 \leq t_2 \leq l, 0 \leq t_3 \leq k\}\rangle.$$

Corollary 4.12. *Let $2 = p_1$ and $p_2, \ldots, p_k$ be primes, $k, \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$ and $n = \prod_{i=1}^{k} p_i^{\alpha_i}$. Then $|\mathcal{I}(\mathbb{Z}_n, \oplus, \odot)| = 1 + \alpha_1 \prod_{i=2}^{k+1} (\alpha_i + 1)$.*

Example 4.13. Consider the general hyperring $R = (\mathbb{Z}_{36}, \oplus, \odot)$. Then we have

$$\mathcal{I}(\mathbb{Z}_{36}, \oplus, \odot) = \{I_1 = \{\overline{0}\}, I_2 = \{\overline{0}, \overline{2}, \overline{4}, \ldots, \overline{34}\}, I_3 = \{\overline{0}, \overline{3}, \overline{6}, \ldots, \overline{33}\},$$
$$I_4 = \{\overline{0}, \overline{6}, \overline{12}, \ldots, \overline{30}\}, I_5 = \{\overline{0}, \overline{9}, \overline{18}, \overline{27}\}, I_6 = \{\overline{0}, \overline{18}\} \text{ and } I_7 = \mathbb{Z}_{36}\}.$$

4.2 Hyperideals of General Hyperring $R = (\mathbb{Z}_n, +', \cdot')$

In this part, all hyperideals of finite commutative general hyperring $R = (\mathbb{Z}_n, +', \cdot')$ are computed and it is proved that every hyperideal of the general hyperring $R = (\mathbb{Z}_n, +', \cdot')$ is characterized by the divisors of cardinal of $\mathbb{Z}_n$.

Let $m \in \mathbb{N}$ and $R = (\mathbb{Z}_n, +', \cdot')$ be the general hyperring in Theorem 3.5. Then for any $\overline{x} \in R$ define, $1\overline{x} = \{\overline{x}\}, m\overline{x} = \underbrace{\overline{x} +' \overline{x} +' \ldots \overline{x} +' \overline{x}}_{m-times}$, $\overline{x}^1 = \{\overline{x}\}$, $\overline{x}^m = \underbrace{\overline{x} \cdot' \overline{x} \cdot' \ldots \overline{x} \cdot' \overline{x}}_{m-times}$ and $\langle \overline{x} \rangle = \bigcup\limits_{k \in \mathbb{N}} k\overline{x}$. With this regard to these notations and definitions, We have the following theorem.

Theorem 4.14. *Let $n, d \in \mathbb{N}$ and $\overline{x}, \overline{y} \in R$. Then*

(i) $\langle \overline{x} \rangle \in \mathcal{I}(\mathbb{Z}_n, +', \cdot')$,

(ii) $\langle \overline{0} \rangle = \{\overline{0}\}$,

(iii) $\langle \overline{x} \rangle = \langle \overline{y} \rangle \Leftrightarrow \gcd(x, n) = \gcd(y, n) = d$.

Proof. (i) Let $\overline{x} \in R$. By definition, we have $\langle \overline{x} \rangle = \bigcup\limits_{k \in \mathbb{N}} \{\overline{kx}, \overline{0}\}$ and show that it is a hyperideal of R. Let $\overline{y} \in \langle \overline{x} \rangle$ and $\overline{z} \in \overline{y} +' \langle \overline{x} \rangle$. Thus there is $k, k' \in \mathbb{N}$ and $\overline{w} \in \langle \overline{x} \rangle$ that $\overline{z} \in \overline{y} +' \overline{w}$ and so $\overline{z} \in \overline{kx} +' \overline{k'x}, \overline{z} \in \overline{kk'x}$ and $\overline{z} \in \langle \overline{0} \rangle$. There is $k'' \in \mathbb{N}$ that $\overline{z} \in \{\overline{k''x}, \overline{0}\} \subseteq \langle \overline{x} \rangle$. In a similar way, for any $\overline{r} \in \mathbb{Z}_n$ and $\overline{y} \in \langle \overline{x} \rangle$, we have $\overline{r} +' \langle \overline{x} \rangle \subseteq \langle \overline{x} \rangle$. Hence $\langle \overline{x} \rangle \in \mathcal{I}(\mathbb{Z}_n, +', \cdot')$.

(ii) It is straightforward by definition.

(iii) Let $\overline{z} \in \langle \overline{x} \rangle$. Then there is $k \in \mathbb{N}$, that $\overline{z} = \overline{kx}$ or $\overline{z} = \overline{0}$. Since $\gcd(x, n) = d$ and by item (i), there is $k' \in \mathbb{Z}$, that $x = k'd$. If $\overline{z} = \overline{kx}$, then $\overline{z} = \overline{kx} = \overline{kk'd} \in \langle \overline{x} \rangle$ and if $\overline{z} = \overline{0}$, then $\overline{z} = \overline{k0} = \overline{0} \in \langle \overline{x} \rangle$. Hence $\langle \overline{x} \rangle \subseteq \langle \overline{d} \rangle$. Let $\overline{z} \in \langle \overline{d} \rangle$. Then there is $k \in \mathbb{N}$, that $\overline{z} = \overline{kd}$ or $\overline{z} = \overline{0}$. Since $\gcd(x, n) = d$ and by item (i), there is $r, s \in \mathbb{Z}$ that $rx + ns = d$, and so $\overline{rkx} + \overline{nks} = \overline{kd}$. Applying Theorem 3.10, we get that $\overline{z} = \overline{krx}$ or $\overline{z} = \overline{0}$. Hence $\langle \overline{d} \rangle \subseteq \langle \overline{x} \rangle$. Also for $\gcd(y, n) = d$ the proof is similarly, then $\langle \overline{d} \rangle = \langle \overline{y} \rangle$, there for $\langle \overline{x} \rangle = \langle \overline{y} \rangle$.

□

Example 4.15. Consider the general hyperring $R = (\mathbb{Z}_{45}, +', \cdot')$. By Theorem 4.14, we have

$$\mathcal{I}(\mathbb{Z}_{45}, +', \cdot') = \{I_1 = \{\overline{0}\}, I_2 = \{\overline{0}, \overline{3}, \overline{6}, \ldots, \overline{42}\}, I_3 = \{\overline{0}, \overline{5}, \overline{10}, \ldots, \overline{40}\},$$
$$I_4 = \{\overline{0}, \overline{9}, \overline{18}, \overline{27}, \overline{36}\}, I_5 = \{\overline{0}, \overline{15}, \overline{30}\} \textit{ and } I_6 = \mathbb{Z}_{45}\}.$$

Theorem 4.16. *Let $n \in \mathbb{N}$. Then*

(i) $|\mathcal{I}(\mathbb{Z}_n, +', \cdot')| = |Div(n)|$.

(ii) *for any* $\overline{x}, \overline{y} \in R, \langle \overline{x} \rangle \cap \langle \overline{y} \rangle = \langle \overline{lcm(x, y)} \rangle$.

Proof. (i) By Theorem 4.14, $I \in \mathcal{I}$ if and only if there is $s\overline{x} \in R$, that $I = \langle \overline{x} \rangle$. Also for any $\overline{x} \in R, gcd(x, n) = d$ if and only if $\langle \overline{x} \rangle = \langle \overline{d} \rangle$. Thus $|\mathcal{I}(\mathbb{Z}_n, +', \cdot')| = |Div(n)|$.

(ii) Let $\overline{x} \in R$. By definition, we have $\langle \overline{x} \rangle = \bigcup_{k \in \mathbb{N}} \{\overline{kx}, 0\}$. Clearly, there is $k_1, k_2 \in \mathbb{N}$ that $lcm(x,y) = k_1 x$ and $lcm(x,y) = k_2 y$. Hence $\overline{lcm(x,y)} \in \langle \overline{x} \rangle \cap \langle \overline{y} \rangle$ and so $\langle \overline{lcm(x,y)} \rangle \subseteq \langle \overline{x} \rangle \cap \langle \overline{y} \rangle$. Conversely. let $\overline{a} \in \langle \overline{x} \rangle \cap \langle \overline{y} \rangle$. Then $\overline{a} = \overline{0}$ or there is $k_1, k_2 \in \mathbb{N}$ that $\overline{a} = \overline{k_1 x} = \overline{k_2 y}$. Thus $x \mid a$ and $y \mid a$ and so $lcm(x,y) \mid a$. Hence there is $k \in \mathbb{N}$ that $\overline{a} = \overline{k \times lcm(x,y)}$ and so $\langle \overline{a} \rangle \subseteq \langle \overline{x} \rangle \cap \langle \overline{y} \rangle$.

□

Corollary 4.17. *Let $n \in \mathbb{N}$. Then $\mathcal{I}(\mathbb{Z}_n, +', \cdot') = \{\langle \overline{d} \rangle \mid d \in Div(n)\}$.*

Example 4.18. Let p, q, r be primes, $m, l, k \in \mathbb{N}$ and $n = p^m q^l r^k$. Then

$$\mathcal{I}(\mathbb{Z}_n, +', \cdot') = \{\langle \overline{p^{t_1} q^{t_2} r^{t_3}} \rangle \mid 0 \leq t_1 \leq m, 0 \leq t_2 \leq l, 0 \leq t_3 \leq k\}\rangle.$$

Corollary 4.19. *Assume $p_1, p_2, \ldots, p_k$ are primes, $k, \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$ and $n = \prod_{i=1}^{k} p_i^{\alpha_i}$. Then $|\mathcal{I}(\mathbb{Z}_n, +', \cdot')| = \sum_{i=1}^{k} (\alpha_i + 1)$.*

Example 4.20. Consider the finite general hyperring $R = (\mathbb{Z}_{90}, +', \cdot')$. Based on corollary 4.17,

$$\begin{aligned}
&\mathcal{I}(\mathbb{Z}_{90}, +', \cdot') = \big\{I_1 = \{\overline{0}\}, I_2 = \{\overline{0}, \overline{2}, \overline{4}, \ldots, \overline{88}\}, I_3 = \{\overline{0}, \overline{3}, \overline{6}, \ldots, \overline{87}\},\\
&I_4 = \{\overline{0}, \overline{5}, \overline{10}, \ldots, \overline{85}\}, I_5 = \{\overline{0}, \overline{9}, \overline{18}, \ldots, \overline{81}\}, I_6 = \{\overline{0}, \overline{10}, \overline{20}, \ldots, \overline{80}\},\\
&I_7 = \{\overline{0}, \overline{6}, \overline{12}, \ldots, \overline{84}\}, I_8 = \{\overline{0}, \overline{15}, \overline{30}, \ldots, \overline{75}\}, I_9 = \{\overline{0}, \overline{18}, \overline{36}, \ldots, \overline{72}\},\\
&I_{10} = \{\overline{0}, \overline{45}\}, I_{11} = \{\overline{0}, \overline{30}, \overline{60}\} \textit{ and } I_{12} = \mathbb{Z}_{90}\big\}.
\end{aligned}$$

4.3 Restriction of General Hyperring to Krasner Hyperring

We consider an especial of general hyperring as Krasner hyperring and construct this class upon previously section. Now, we construct a Krasner hyperrings based on quotient of a ring on normal subgroup of its multiplicative semigroup in different to Krasner way.

Theorem 4.21. *Assume $(R, +, .)$ is a ring and N is a normal subgroup of its multiplicative semigroup. Then there are hyperoperations "$+'$" and "$\cdot'$" on R/N, that $(R/N, +', \cdot')$ is a Krasner hyperring.*

Proof. For all $xN, yN \in R/N$ define "$+'$" and "$\cdot'$" on R by

$$xN +' yN = \begin{cases} \{xN\} \cup \bigcup_{z \in (x+y)N} zN & x \neq y \\ R/N & x = y \\ xN & y = 0. \end{cases} \quad \text{and } xN \cdot' yN = (xy)N.$$

It is easy to verify that $(R/N, +', \cdot')$ is a commutative Krasner hyperring. □

Theorem 4.22. *Let $\mathbb{Z}_\xi = \mathbb{Z} \cup \{\xi\}$, where $\xi \notin \mathbb{Z}$. Then there are hyperoperation "$+_\xi$" and an operation "$\cdot_\xi$", that $(\mathbb{Z}_\xi, +_\xi, 0, \cdot_\xi, 1)$ is a Krasner hyperring.*

Proof. Let $x, y \in \mathbb{Z}$. Define hyperoperation "$+_\xi$" and operation "$\cdot_\xi$" on $\mathbb{Z}_\xi$ as follows:

$$x +_\xi y = \begin{cases} \{0,\xi\} & x = -y \\ x+y & x, y \in \mathbb{Z}, x \neq -y \\ y & x = 0 \\ \xi & x = y = \xi \end{cases} \text{ and } x \cdot_\xi y = \begin{cases} x.y & x, y \in \mathbb{Z} \\ \xi & y = \xi. \end{cases},$$

where for any $x, y \in \mathbb{Z}_\xi$, we have $x +_\xi y = y +_\xi x, x \cdot_\xi y = y \cdot_\xi x$ and $0 \cdot_\xi x = 0$. Some modifications and computations show that $(\mathbb{Z}_\xi, +_\xi, 0, \cdot_\xi, 1)$ is a Krasner hyperring. □

Theorem 4.23. *Let $\mathbb{Z}_{n_\xi} = \mathbb{Z}_n \cup \{\xi\}$, where $\xi \notin \mathbb{Z}_n$. Then there are hyperoperation "$+_\xi$" and an operation "$\cdot_\xi$" that $(\mathbb{Z}_{n_\xi}, +_\xi, 0, \cdot_\xi, 1)$ is a Krasner hyperring.*

Proof. Let $x, y \in \mathbb{Z}_n$. Define hyperoperation "$+_\xi$" and operation "$\cdot_\xi$" on $\mathbb{Z}_{n_\xi}$ as follows:
$$x +_\xi y = \begin{cases} \{0,\xi\} & x = -y \\ x+y & x, y \in \mathbb{Z}_n, x \neq -y \\ y & x = 0 \text{ or } (\ x = \xi \text{ and } y \notin \{\xi\}) \\ \xi & x = y = \xi \end{cases} \text{ and}$$
$$x \cdot_\xi y = \begin{cases} x.y & x, y \in \mathbb{Z}_n \\ \xi & (y = \xi, gcd(x,n) \neq 1) \text{ or } (x = y = \xi) \\ 0 & x = \xi, (x,n) = 1 \end{cases},$$
where for any $x, y \in \mathbb{Z}_{n_\xi}$, we have $x +_\xi y = y +_\xi x, x \cdot_\xi y = y \cdot_\xi x$ and $0 \cdot_\xi x = 0$. Some modifications and computations show that $(\mathbb{Z}_{n_\xi}, +_\xi, 0, \cdot_\xi, 1)$ is a Krasner hyperring. □

4.4 Hyperideals of Krasner Hyperrings

At the following we compute the hyperideals of some class of Krasner hyperrings based on way introduced at the previous section.

Theorem 4.24. *Let $\mathbb{Z}_{n_\xi} = \mathbb{Z}_n \cup \{\xi\}$, where $\xi \notin \mathbb{Z}_n$ and $n \in \mathbb{N}$. Then*

(i) *if n is a prime, then $\mathcal{I}(\mathbb{Z}_{n_\xi}, +_\xi, \cdot_\xi) = \{I_1 = \{0,\xi\}, I_2 = \mathbb{Z}_{n_\xi}\}$,*

(ii) *if n is an odd non-prime, then $\mathcal{I}(\mathbb{Z}_{n\xi}, +_\xi, \cdot_\xi) = \{I_1 = \{0,\xi\}, I_2 = \{a_i \mid gcd(a_i ,n) \neq 1\} \cup \xi, I_3 = \mathbb{Z}_{n\xi}\}$,*

(iii) *if n is an even, then $\mathcal{I}(\mathbb{Z}_{n_\xi}, +_\xi, \cdot_\xi) = \{I_1 = \{0,\xi\}, I_2 = \{a_i \mid gcd(a_i,n) \neq 1\} \cup \xi, I_3 = \{0, \frac{n}{2}, \xi\}, I_4 = \mathbb{Z}_{n\xi}\}$.*

Proof. (i) Obviously $I = \{0, \xi\} \in \mathcal{I}(\mathbb{Z}_{n_\xi}, +_\xi, \cdot_\xi)$. Let $\{0, \xi\} \neq I \in \mathcal{I}(\mathbb{Z}_{n_\xi}, +_\xi, \cdot_\xi)$ be a hyperideal and $\overline{0} \neq \overline{x} \in I$. Since for any $\overline{0} \neq \overline{y} \in \mathbb{Z}_n$ and $\mathbb{Z}_n = \overline{x} \cdot_\xi \overline{y} \subseteq I$, it obtains that $I = \mathbb{Z}_n$.

(ii) Obviously $I = \{0, \xi\} \in \mathcal{I}(\mathbb{Z}_{n_\xi}, +_\xi, ._\xi)$. Let $\{0, \xi\} \neq I = \{a_i \mid gcd(a_i, n) \neq 1\} \in \mathcal{I}(\mathbb{Z}_{n_\xi}, +_\xi, \cdot_\xi)$ be a hyperideal then $gcd(a_i, n) = d$ concludes that $a_i\mathbb{Z} + n\mathbb{Z} = d\mathbb{Z}$. Because there are $x, y \in \mathbb{Z}$ and $d = a_i x + ny$, we get $d \in a_i\mathbb{Z} + n\mathbb{Z}$ and hence $d\mathbb{Z} \subseteq a_i\mathbb{Z} + n\mathbb{Z}$. Also $d \mid a_i$ (d mids a_i) and $d \mid n$ (d mids n), then $a_i\mathbb{Z} \subseteq d\mathbb{Z}$ and $n\mathbb{Z} \subseteq d\mathbb{Z}$ imply that $a_i\mathbb{Z} + n\mathbb{Z} \subseteq d\mathbb{Z}$ and hence $a_i\mathbb{Z} + n\mathbb{Z} = d\mathbb{Z}$.

(iii) It's similar to (ii).

□

Example 4.25. For prime, odd non-prime and even numbers $7, 9, 6$, we have:

(i) if $n = 7$, then $\mathcal{I}(\mathbb{Z}_{7_\xi}, +_\xi, \cdot_\xi) = \{I_1 = \{0, \xi\}, I_2 = \mathbb{Z}_{7_\xi}\}$,

(ii) if $n = 9$, then $\mathcal{I}(\mathbb{Z}_{9\xi}, +_\xi, \cdot_\xi) = \{I_1 = \{0, \xi\}, I_2 = \{0, 3, 6, \xi\}, I_3 = \mathbb{Z}_{9\xi}\}$,

(iii) if $n = 6$, then $\mathcal{I}(\mathbb{Z}_{6_\xi}, +_\xi, \cdot_\xi) = \{I_1 = \{0, \xi\}, I_2 = \{0, 2, 3, 4, \xi\}, I_3 = \{0, 3, \xi\}, I_4 = \mathbb{Z}_{6\xi}\}$.

Theorem 4.26. *Let $\mathbb{Z}_\xi = \mathbb{Z} \cup \{\xi\}$, where $\xi \notin \mathbb{Z}$. Then $\mathcal{I}(\mathbb{Z}_\xi, +_\xi, \cdot_\xi) = \{m\mathbb{Z}_\xi \mid m \in \mathbb{N}\}$.*

Proof. Obviously for any $m \in \mathbb{N}$ and $m\mathbb{Z}$ is a hyperideal of $\mathbb{Z}_\xi$, because of for any $mk \in m\mathbb{Z}, n \in \mathbb{N}$, have $nmk = mkn \in m\mathbb{Z}_\xi$. □

Example 4.27. Consider the Krasner hyperring $R = (\mathbb{Z}_{10}, +_\xi, \cdot_\xi)$ by Theorem 4.22. So we have

$$\mathcal{I}(\mathbb{Z}_{10}, +_\xi, ._\xi) = \{I_1 = \{\overline{0}, \xi\}, I_2 = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \xi\}, I_3 = \{\overline{0}, \overline{5}, \xi\}, I_4 = \mathbb{Z}_{10_\xi}\}.$$

5. Conclusion

The notion of a general hyperring, as generalization of hyperring as well as classical ring are introduced. In this regards some ways to extend a classical ring to a general hyperring was given, also a way to construct a general hyperring from every set was introduced. Also, the hyperideals of some special classes of general hyperring such (finite) Krasner hyperring was computed. Moreover, under some certain condition hyperideals in finite general hyperrings was characterized. In particular the hyperideals of some finite Krasner hyperrings was obtained. Accordingly, This paper has been provided a good introduction to construct general hyperrings as well as to computing its hyperideals, especially for finite case. At the end, the authors propose to use a computer programming way to construct finite general hyperring and computing their hyperideals. for future works in this area.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] R. Ameri and I. G. Rosenberg, Congruences of multialgebras, *J. Multiple Valued Log. Soft Comput.* **15** (5-6) (2009) 525 – 536.

[2] R. Ameri and M. M. Zahedi, Hyperalgebraic systems, *Italian J. Pure Appl. Math.* **6** (1999) 21 – 32.

[3] R. Ameri, M. Hamidi and A. A. Tavakoli, Boolean rings based on multirings, *J. Sci. I. R. Iran* **32** (2) (2021) 159 – 168.

[4] H. Bordbara and I. Cristea, Height of prime hyperideals in Krasner hyperrings, *Filomat* **31** (19) (2017) 6153 – 6163.

[5] P. Corsini, *Prolegomena of Hypergroup Theory*, 2nd ed., Aviani Editor, Tricesimo, Italy, (1993).

[6] J. Chvalina, S. H. Mayerova and A. D. Nezhad, General actions of hyperstructures and some applications, *Analele Stiintifice ale Univ. Ovidius Constanta, Ser. Mat.* **21** (1) (2013) 59 – 82.

[7] P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory,* Kluwer Academic Publishers, Dordrecht, 2002.

[8] B. Davvaz and V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, USA, 2007.

[9] M. Krasner, Approximation des corps values complets de caracteristique p, $p > 0$, par ceux de caracter- istique zero, Colloque d'Algebre Superieure (Bruxelles, Decembre 1956), CBRM, Bruxelles, 1957.

[10] F. Marty, Sur une generalization de la notion de groupe, 8*th Congres Math. Scandinaves, Stockholm, Sweden* (1934) 45 – 49.

[11] S. Omidi and B. Davvaz, Contribution to study special kinds of hyperideals in ordered semihyperrings, *J. Taibah Univ. Sci.* **11** (2017) 1083 – 1094.

[12] R. Rota, Sugli iperanelli moltiplicativi, *Rend. Di Mat. Series VII* **2** (4) (1982) 711 – 724.

[13] T. Vougiouklis, The fundamental relation in hyperrings, *The general hypereld, Proc. Fourth Int. Congress on Algebraic Hyperstructures and Applications* (*AHA* 1990), *World Scientic* (1991) 203 – 211.

Reza Ameri
School of Mathematics,
Statistic and Computer Sciences,
University of Tehran,

Tehran, I. R. Iran
e-mail: rameri@ut.ac.ir

Mohammad Hamidi
Department of Mathematics,
Payame Noor University,
Tehran, I. R. Iran
e-mail: m.hamidi@pnu.ac.ir

Hoda Mohammadi
Department of Mathematics,
Payame Noor University,
Tehran, I. R. Iran
e-mail: Hodamohamadi@student.pnu.ac.ir