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A Remark on the Factorization of Factorials

Mehdi Hassani ⋆ *and Mahmoud Marie*

Abstract

The subject of this paper is to study distribution of the prime factors p and their exponents, which we denote by $v_p(n!)$, in standard factorization of $n!$ into primes. We show that for each $\theta > 0$ the primes p not exceeding n^θ eventually assume almost all value of the sum $\sum_{p\leqslant n} v_p(n!)$. Also, we introduce the notion of θ-truncated factorial, defined by $n!_\theta = \prod_{p\leqslant n^\theta} p^{v_p(n!)}$, and we show that the growth of $\log n!_{\frac{1}{2}}$ is almost half of growth of $\log n!_1$.

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1. Introduction

Letting $n! = \prod_{p\leqslant n} p^{v_p(n!)}$, Legendre's theorem asserts that

$$v_p(n!) = \sum_{\alpha=1}^{\infty} \left[\frac{n}{p^\alpha}\right]. \tag{1}$$

Here $[x] = \max\{k \in \mathbb{Z} : k \leqslant x\}$. This relation enables us to observe several facts about the distribution of the exponents. It implies that for primes p and q with $p < q$ we have $v_p(n!) \geqslant v_q(n!)$. Primes p satisfying $\frac{n}{2} < p \leqslant n$ assume the minimum possible exponent $v_p(n!) = 1$. Also, for primes p with $p > \sqrt{n}$ Legendre's sum in (1) assumes only its first term.

⋆Corresponding author (E-mail: mehdi.hassani@znu.ac.ir)
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These properties show that small primes get larger exponents than large primes. Our first result yields information about clustering of exponents on small primes.

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Theorem 1.1. For given $\theta \in (0,1]$ let

$$\mathfrak{S}_\theta(n) = \sum_{p \leqslant n^\theta} v_p(n!), \quad \text{and} \quad \mathfrak{A}_\theta(n) = \frac{1}{\mathfrak{S}_1(n)} \sum_{p \leqslant n^\theta} v_p(n!).$$

For each integer $k \geqslant 0$, which we keep it fixed, as $n \to \infty$, we have

$$\mathfrak{S}_{\frac{1}{k+1}}(n) = n \log\log n + (M' - \log(k+1))\, n + O\Big(\frac{n}{\log n}\Big), \tag{2}$$

where

$$M' = M + \sum_p (p(p-1))^{-1}, \quad M = \gamma + \sum_p \left(\log\left(1 - p^{-1}\right) + p^{-1}\right), \tag{3}$$

and $\gamma = \lim_{n\to\infty} H_n - (\log n)$, with $H_n = \sum_{j=1}^n j^{-1}$. Also, for each fixed integer $m \geqslant 1$ we have

$$\mathfrak{A}_{\frac{1}{k+1}}(n) = 1 + \sum_{j=1}^{m} \frac{(-1)^j \,(M')^{j-1} \log(k+1)}{(\log\log n)^j} + O\Big(\frac{1}{(\log\log n)^{m+1}}\Big). \tag{4}$$

To study clustering of exponents on small primes, we introduce the notion of *θ-truncated factorial*, defined by

$$n!_\theta = \prod_{p \leqslant n^\theta} p^{v_p(n!)}.$$

Naturally we ask about the growth of $n!_\theta$, more precisely for the values of θ close to 0. The following result asserts that the growth of $n!_{\frac{1}{2}}$ is almost half of growth of $n!_1$ in logarithmic scale.

Theorem 1.2. Let $\mathfrak{L}_\theta(n) = \log(n!_\theta)$. As $n \to \infty$ we have

$$\mathfrak{L}_{\frac{1}{2}}(n) = \log\left(n^{\frac{n}{2}}\right) - \gamma n + O\Big(\frac{n}{\log n}\Big). \tag{5}$$

A natural question, for which we have no solution yet, deals with the approximation of $\mathfrak{L}_\theta(n)$ for the values of θ close to 0. If for $\alpha, \beta \in (0,1]$ we let $\mathfrak{Q}(\alpha,\beta;n) = \mathfrak{L}_\alpha(n)/\mathfrak{L}_\beta(n)$, then (5) and Stirling's approximation imply

$$\mathfrak{Q}\Big(\frac{1}{2}, 1; n\Big) = \frac{1}{2} - \frac{\gamma - \frac{1}{2}}{\log n} + O\Big(\frac{1}{\log^2 n}\Big).$$

For $\alpha, \beta \in (0,1]$ we conjecture that $\lim_{n\to\infty} \mathfrak{Q}(\alpha,\beta;n) = \alpha/\beta$. Also, if we let $\mathfrak{G}_\theta(n) = (n!_\theta)^{\frac{1}{n}}$, then Stirling's approximation and (5) imply respectively

$$\mathfrak{G}_1(n) = \mathrm{e}^{-1}n + O(\log n), \quad \text{and} \quad \mathfrak{G}_{\frac{1}{2}}(n) = \mathrm{e}^{-\gamma}\sqrt{n} + O\Big(\frac{\sqrt{n}}{\log n}\Big).$$

Hence $\lim_{n\to\infty} \mathfrak{G}_{\frac{1}{2}}(n)/\mathfrak{G}_1(n)^{\frac{1}{2}} = \mathrm{e}^{\frac{1}{2}-\gamma}$. We ask about the existence and possible value of $\lim_{n\to\infty} \mathfrak{G}_\theta(n)/\mathfrak{G}_1(n)^\theta$ for given $\theta \in (0,1]$.

2. Auxiliary Sums over Primes

In this section we approximate several summations running over prime numbers, for which as a key idea, we relate such summations with the function $\pi(x)$, the number of primes $\leqslant x$.

Proposition 2.1. Let f be a positive, strictly decreasing, and continuously differentiable function on $[2,\infty)$, and $f(t) = o(\frac{1}{t})$ as $t \to \infty$. Then, for each $z > 1$ we have

$$\sum_{p>z} f(p) < \frac{1 + \frac{3}{2\log z}}{\log z} \int_z^\infty f(t)\,\mathrm{d}t + \frac{9z f(z)}{4\log^2 z}. \tag{6}$$

Proof. If we let $\varpi(n) = 1$ when n is prime and 0 otherwise, then

$$\pi(x) = \sum_{n\leqslant x} \varpi(n). \tag{7}$$

This representation allows us to write summations running over prime numbers an a Stieltjes integral as follows

$$\sum_{p\leqslant x} f(p) = \sum_{2\leqslant n\leqslant x} \varpi(n) f(n) = \int_{2^-}^x f(t)\,\mathrm{d}\pi(t).$$

Integration by parts we get

$$\sum_{p\leqslant x} f(p) = \pi(x)f(x) - \int_2^x \pi(t)\Big(\frac{\mathrm{d}f(t)}{\mathrm{d}t}\Big)\,\mathrm{d}t.$$

Hence

$$\sum_{p>z} f(p) = \lim_{b\to\infty} \sum_{z<p\leqslant b} f(p) = -\pi(z)f(z) - \int_z^\infty \pi(t)\Big(\frac{\mathrm{d}f(t)}{\mathrm{d}t}\Big)\,\mathrm{d}t.$$

Rosser and Schoenfeld [4, Theorem 1] proved that

$$\pi(x) < \frac{x}{\log x}\Big(1 + \frac{3}{2\log x}\Big) \qquad (x > 1).$$

Since $f>0$ is strictly decreasing and $f(b)=o(\frac{1}{b})$ as $b\to\infty$, by using this bound we obtain

$$\begin{aligned}\int_z^\infty \pi(t)\Big(-\frac{\mathrm{d}f(t)}{\mathrm{d}t}\Big)\,\mathrm{d}t &< \int_z^\infty \frac{t(1+\frac{3}{2\log t})}{\log t}\Big(-\frac{\mathrm{d}f(t)}{\mathrm{d}t}\Big)\,\mathrm{d}t\\ &\leqslant \frac{1+\frac{3}{2\log z}}{\log z}\int_z^\infty t\Big(-\frac{\mathrm{d}f(t)}{\mathrm{d}t}\Big)\,\mathrm{d}t\\ &= \frac{1+\frac{3}{2\log z}}{\log z}\left(zf(z)+\int_z^\infty f(t)\,\mathrm{d}t\right).\end{aligned}$$

Rosser and Schoenfeld [4, Theorem 1] also proved that

$$\frac{x}{\log x}\Big(1+\frac{1}{2\log x}\Big)<\pi(x), \qquad (x\geqslant 59).$$

This implies that

$$\frac{x}{\log x}\Big(1-\frac{3}{4\log x}\Big)<\pi(x), \qquad (x>1).$$

Thus, $-\pi(z)f(z) < -\frac{z}{\log z}\big(1-\frac{3}{4\log z}\big)f(z)$ for each $z>1$. Combining the above bounds we obtain (6). This finishes the proof. □

Corollary 2.2. For $s>1$ and $x\geqslant 2$ let $\mathfrak{B}_s(x)=\sum_{p\leqslant x}\frac{1}{p^s}$. Then

$$\mathfrak{B}_s(x)=P(s)+O\Big(\frac{1}{x^{s-1}\log x}\Big), \tag{8}$$

where $P(s)=\sum_p \frac{1}{p^s}$ is an absolute constant, known as the prime zeta function.

Proof. Let $\mathfrak{B}_s(x)=P(s)-\sum_{p>x}\frac{1}{p^s}$. Approximation of the last sum is straightforward by using Proposition 2.1. □

Proposition 2.3. For each $n\geqslant 2$,

$$\sum_{p\leqslant n}\Big\{\frac{n}{p}\Big\}(\log p)=(1-\gamma)n+O\Big(\frac{n}{\log n}\Big). \tag{9}$$

Proof. Let

$$\mathfrak{F}(n)=\sum_{p\leqslant n}\Big\{\frac{n}{p}\Big\}.$$

Lee [3, Lemma 3] obtained the following approximation

$$\sum_{p^\alpha\leqslant n}\Big\{\frac{n}{p^\alpha}\Big\}=(1-\gamma)\frac{n}{\log n}+O\Big(\frac{n}{\log^2 n}\Big).$$

We observe that

$$\sum_{p^\alpha \leqslant n} \left\{ \frac{n}{p^\alpha} \right\} - \mathfrak{F}(n) = \sum_{\substack{p^\alpha \leqslant n \\ \alpha \geqslant 2}} \left\{ \frac{n}{p^\alpha} \right\} < \sum_{\substack{p^\alpha \leqslant n \\ \alpha \geqslant 2}} 1 = \sum_{\substack{p \leqslant n^{\frac{1}{\alpha}} \\ \alpha \geqslant 2}} 1 = \sum_{2 \leqslant \alpha \leqslant \frac{\log n}{\log 2}} \pi(n^{\frac{1}{\alpha}}).$$

By using the approximation $\pi(x) = O(\frac{x}{\log x})$ we get

$$\sum_{2 \leqslant \alpha \leqslant \frac{\log n}{\log 2}} \pi(n^{\frac{1}{\alpha}}) \ll \sum_{2 \leqslant \alpha \leqslant \frac{\log n}{\log 2}} \frac{n^{\frac{1}{\alpha}}}{\log n^{\frac{1}{\alpha}}} \leqslant \frac{n^{\frac{1}{2}}}{\log n} \sum_{2 \leqslant \alpha \leqslant \frac{\log n}{\log 2}} \alpha \ll \sqrt{n} \log n.$$

Hence,

$$\mathfrak{F}(n) = (1-\gamma) \frac{n}{\log n} + O\Big(\frac{n}{\log^2 n}\Big). \tag{10}$$

By using Abel summation we get

$$\begin{aligned}\sum_{p \leqslant n} \left\{ \frac{n}{p} \right\} (\log p) &= \sum_{k=2}^{n} \left\{ \frac{n}{k} \right\} \varpi(k) \log k \\ &= \mathfrak{F}(n)\, \log n - \mathfrak{F}(2^-)\, \log 2 - \int_2^n \eta(t) \frac{\mathrm{d}t}{t},\end{aligned}$$

where $\eta(t) = \sum_{p \leqslant t} \left\{ \frac{n}{p} \right\}$. Note that $0 \leqslant \eta(t) \leqslant \pi(t) \ll \frac{t}{\log t}$. Combining this approximation with (10) we deduce

$$\sum_{p \leqslant n} \left\{ \frac{n}{p} \right\} (\log p) = (1-\gamma) n + O\Big(\frac{n}{\log n}\Big) - \int_2^n O\Big(\frac{t}{\log t}\Big) \frac{\mathrm{d}t}{t}.$$

This is the desired conclusion. □

3. Proof of the Main Results

Proof of Theorem 1.1. First we consider the case $k = 0$. Approximation of $\mathfrak{S}_1(n)$ is related to the average of the function $\Omega(n)$, which denotes the total number of prime factors of positive integer n. The function Ω is completely additive. We recall the known approximation of the sum $\sum_{k \leqslant n} \Omega(k)$ due to Hardy and Ramanujan [1] to obtain

$$\mathfrak{S}_1(n) = \Omega(n!) = \sum_{k \leqslant n} \Omega(k) = n \, \log\log n + M' \, n + O\Big(\frac{n}{\log n}\Big). \tag{11}$$

Let $k \geqslant 1$. For primes p satisfying $n^{\frac{1}{k+1}} < p \leqslant n$ we have $n < p^{k+1}$. Thus, by using the relation (1) we obtain

$$v_p(n!) = \sum_{j=1}^{k} \left[\frac{n}{p^j} \right] = n \sum_{j=1}^{k} \frac{1}{p^j} + O(1).$$

Thus by using the approximation $\pi(x) \asymp \frac{x}{\log x}$ we obtain

$$\begin{aligned}
\mathfrak{S}_1(n) - \mathfrak{S}_{\frac{1}{k+1}}(n) &= \sum_{n^{1/(k+1)}<p\leqslant n} v_p(n!) \\
&= n \sum_{n^{1/(k+1)}<p\leqslant n} \sum_{j=1}^{k} \frac{1}{p^j} + O\Big(\sum_{n^{1/(k+1)}<p\leqslant n} 1\Big) \\
&= n \sum_{j=1}^{k} \sum_{n^{1/(k+1)}<p\leqslant n} \frac{1}{p^j} + O(\pi(n)) \\
&= n \sum_{j=1}^{k} \Big(\mathfrak{B}_j(n) - \mathfrak{B}_j(n^{\frac{1}{k+1}})\Big) + O\Big(\frac{n}{\log n}\Big),
\end{aligned}$$

where $\mathfrak{B}_1(x) = \sum_{p\leqslant x} 1/p$, and $\mathfrak{B}_j$ for $j \geqslant 2$ is defined in Corollary 2.2. By using Mertens' approximation $\mathfrak{B}_1(x) = \log\log x + M + O(\frac{1}{\log x})$ with M as in (3) we obtain

$$\mathfrak{B}_1(n) - \mathfrak{B}_1(n^{1/(k+1)}) = \log(k+1) + O\Big(\frac{1}{\log n}\Big).$$

Also, for $s > 1$ we use the approximation (8) to get

$$\mathfrak{B}_s(n) - \mathfrak{B}_s(n^{1/(k+1)}) = O\Big(\frac{1}{n^{(s-1)/(k+1)}\log n}\Big).$$

Hence

$$\mathfrak{S}_1(n) - \mathfrak{S}_{\frac{1}{k+1}}(n) = n\log(k+1) + O\Big(\frac{n}{\log n}\Big).$$

This approximation and (11) imply (2). To obtain (4) we divide the right hand side of (2) by the right hand side of (11). Let $z = \log n$ and $M'' = M' - \log(k+1)$. Hence

$$\mathfrak{A}_{\frac{1}{k+1}}(n) = \frac{1 + \frac{M''}{\log z} + O(\frac{1}{z\log z})}{1 + \frac{M'}{\log z} + O(\frac{1}{z\log z})}.$$

To deal with the above fraction, we consider the expansion

$$\frac{1}{1+t} = \sum_{j=0}^{m} (-1)^j\, t^j + O(t^{m+1}),$$

which is valid for each fixed integer $m \geqslant 1$, as $t \to 0$. If we let $t = \frac{M'}{\log z} + O\big(\frac{1}{z\log z}\big)$, then

$$\frac{1}{1 + \frac{M'}{\log z} + O(\frac{1}{z\log z})} = \sum_{j=0}^{m} \frac{(-1)^j\,(M')^j}{(\log z)^j} + O\Big(\frac{1}{(\log z)^{m+1}}\Big).$$

Thus,

$$\mathfrak{A}_{\frac{1}{k+1}}(n) = \sum_{j=0}^{m} \frac{(-1)^j\,(M')^j}{(\log z)^j} + \sum_{j=0}^{m} \frac{(-1)^j\,(M')^j\,M''}{(\log z)^{j+1}} + O\Big(\frac{1}{(\log z)^{m+1}}\Big).$$

We simplify to get

$$\mathfrak{A}_{\frac{1}{k+1}}(n) = 1 + \sum_{j=1}^{m} \frac{(-1)^j\,(M')^j + (-1)^{j-1}\,(M')^{j-1}\,M''}{(\log z)^j} + O\Big(\frac{1}{(\log z)^{m+1}}\Big),$$

and then

$$\mathfrak{A}_{\frac{1}{k+1}}(n) = 1 + \sum_{j=1}^{m} \frac{(-1)^j\,(M')^{j-1}\,(M' - M'')}{(\log z)^j} + O\Big(\frac{1}{(\log z)^{m+1}}\Big).$$

Note that $M' - M'' = \log(k+1)$. This completes the proof. □

Proof of Theorem 1.2. We have

$$\mathfrak{L}_{\frac{1}{2}}(n) = \mathfrak{L}_1(n) - \sum_{\sqrt{n}<p\leqslant n} v_p(n!)\,(\log p)\,.$$

Stirling's approximation for $n!$ asserts that

$$\log(n!) = \mathfrak{L}_1(n) = n \log\Big(\frac{n}{\mathrm{e}}\Big) + \log\sqrt{2\pi n} + O\Big(\frac{1}{n}\Big). \tag{12}$$

Also, we have

$$\sum_{\sqrt{n}<p\leqslant n} v_p(n!)\log p = \sum_{\sqrt{n}<p\leqslant n} \left[\frac{n}{p}\right](\log p) = \sum_{p\leqslant n} \left[\frac{n}{p}\right](\log p) - \sum_{p\leqslant\sqrt{n}} \left[\frac{n}{p}\right](\log p)\,.$$

Let us write

$$\sum_{p\leqslant n} \left[\frac{n}{p}\right]\log p = n\,\mathfrak{K}(n) - \sum_{p\leqslant n} \left\{\frac{n}{p}\right\}(\log p)\,, \tag{13}$$

where

$$\mathfrak{K}(n) = \sum_{p\leqslant n} \frac{\log p}{p}.$$

Landau [2, p. 198] proved that

$$\mathfrak{K}(n) = \log n + E + O\Big(\frac{1}{\log n}\Big). \tag{14}$$

where

$$E = -\gamma - \sum_{j=2}^{\infty}\sum_{p} \frac{\log p}{p^j},$$

and $c > 0$ is a computable constant. Note that the double series defining the constant E is absolutely convergent. Approximations (9) and (14) imply

$$\sum_{p\leqslant n}\left[\frac{n}{p}\right](\log p) = n\log n + (\gamma + E - 1)\,n + O\Big(\frac{n}{\log n}\Big). \tag{15}$$

By using (14) and applying the Chebyshev type approximation $\sum_{p\leqslant z}\log p \asymp z$ with $z = \sqrt{n}$ we deduce that

$$\sum_{p\leqslant\sqrt{n}}\left[\frac{n}{p}\right](\log p) = \frac{1}{2}\,n\log n + E\,n + O\left(\sqrt{n}\right).$$

Combining the relations (12), (15) and the last approximation we get (5). This finishes the proof. □

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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Mehdi Hassani
Department of Mathematics,
University of Zanjan,
University Blvd., 45371-38791
Zanjan, I. R. Iran
e-mail: mehdi.hassani@znu.ac.ir

Mahmoud Marie
Department of Mathematics,
University of Zanjan,
University Blvd., 45371-38791
Zanjan, I. R. Iran
e-mail: mahmoud.marie@znu.ac.ir