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Optimal Solution for the System of Differential Inclusion in Hilbert Space

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Abstract

In this paper, we study the existence of the following optimal solution for the system of differential inclusion

$$y' \in \Phi(t, y(t)) \quad a.e.\ t \in I = [t_0, b] \quad and \quad y(t_0) = u_2,$$
$$y' \in \Psi(t, y(t)) \quad a.e.\ t \in I = [t_0, b] \quad and \quad y(t_0) = u_1.$$

in a Hilbert space, where Φ and Ψ are multivalued maps. Our existence result is obtained via selection technique and the best proximity point methods reducing the problem to a differential inclusion.

Keywords: differential inclusion, best proximity point, selection theorem.

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1 Introduction and preliminaries

It is know that the fixed point theory is a principal tools in the study of the existence of solution of differential equation and differential inclusion. In 2014,

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Veeramani and Rajesh [11], introduced the following system of differential equations

$$(1) \qquad y' = \varphi(t, y(t)) \qquad t \in I = [t_0, b] \quad and \quad y(t_0) = u_2,$$
$$(2) \qquad y' = \psi(t, y(t)) \qquad t \in I = [t_0, b] \quad and \quad y(t_0) = u_1.$$

where f, g are real-valued functions defined on

$$T = \{(t, y) \in \mathbb{R}^2 : |t - t_0| \leq a, |y - y_0| \leq b\}$$

for some $a, b > 0$, $(t_0, y_0) \in \mathbb{R}^2$ and $(t, u_1), (t, u_2) \in T$. Veeramani and Rajesh investigated the existence of pair (y, z) under suitable condition on f and g, that is, y and z are solution of system of differential equation given in (1) and (2), respectively and $\|y - z\| = \|u_1 - u_2\|$. The pair (y, z) is called an optimal solution for the system of differential equations. They proved the existence optimal solution of system of differential equation in real space by the well known best proximity point theorems.

Now we present the following system of differential inclusion

$$(3) \qquad y' \in \Phi(t, y(t)) \qquad a.e.\ t \in I = [t_0, b] \quad and \quad y(t_0) = u_2,$$
$$(4) \qquad y' \in \Psi(t, y(t)) \qquad a.e.\ t \in I = [t_0, b] \quad and \quad y(t_0) = u_1.$$

in a Hilbert spaces in which $\Phi, \Psi : I \times H \rightrightarrows H$ are multivalued maps and two distinct elements $u_1, u_2 \in H$. We establish the existence optimal solution of the above system of differential inclusion by the best proximity point theorem which was proved by Fakhar et al. [4]. We introduce first some definitions and facts that will be used in the paper.

Let $S : M \rightrightarrows N$ be a multivalued map. We say that

(i) S is upper-semi-continuous (usc), if for every closed set $D \subseteq N$, $S^-(D) = \{x \in M : S(x) \cap D \neq \emptyset\}$ is closed in M.

(ii) S is lower-semi-continuous (lsc), if for every open set $U \subseteq N$, $S^-(U) = \{x \in M : S(x) \cap U \neq \emptyset\}$ is open in M.

(iii) S is continuous if it's lsc and usc.

Suppose that $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space with the norm $\|.\|$. Assume that $C(I, H)$ is the Banach space of all continuous functions $y : I \to H$ equipped with the uniform norm $\|y\|_{C(I,H)} := \sup_{t \in I} \|y(t)\|$. Let (Ω, Σ, μ) be measurable space then $L^1(\Omega, H)$ denotes the Banach space of all Bochner integrable functions $y : \Omega \to H$, i.e. $y \in L^1(\Omega, H)$, such that y is measurable and $\|y\|_{L^1(\Omega,H)} := \int_\Omega \|y(x)\| d\mu(x) < \infty$.

Definition 1.1. Let (M, d) be a metric space. The Pompeiu-Hausdorff metric on nonempty bounded and closed subsets of M is given by

$$H(C, D) = \max\{e(C, D), e(D, C)\},$$

where $e(C, D) = \sup_{c \in C} d(c, D)$ and $d(c, D) = \inf_{d \in D} d(c, d)$.

Definition 1.2. Let C and D are nonempty subsets of metric space (M, d). Thus a multivalued map $S : C \cup D \rightrightarrows C \cup D$ is called:

(i) cyclic if $S(C) \subset D$ and $S(D) \subset C$.

(ii) cyclic contraction if there exists $\gamma \in (0, 1)$ such that

$$H(S(x), S(y)) < \gamma d(x, y) + (1 - \gamma)d(C, D),$$

for every $x \in C$ and $y \in D$.

(iii) cyclic Meir–Keeler contraction if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < d(C, D) + \varepsilon + \delta \quad \Rightarrow \quad H(S(x), S(y)) < d(C, D) + \varepsilon,$$

for every $x \in C$ and $y \in D$.

where $d(C, D) = \inf\{d(x, y) : x \in C, y \in D\}$.

It's easy to check that every multivalued cyclic contraction map is multivalued cyclic Meir–Keeler contraction map.

Definition 1.3. Let S be a cyclic multivalued map on $C \cup D$, then a point $y \in C \cup D$ is said to be a best proximity point of S if $d(y, S(y)) = d(C, D)$.

Definition 1.4. ([9]) Suppose that (M, d) is a metric space and C and D are nonempty subsets of a M. Then the pair (C, D) satisfies the property UC if sequences $\{a_n\}$ and $\{b_n\}$ in C and sequence $\{d_n\}$ in D, respectively, such that $\lim_n d(a_n, d_n) = d(C, D)$ and $\lim_n d(b_n, d_n) = d(C, D)$, then $\lim_n d(a_n, b_n) = 0$ holds.

Remark 1. Notice that if M is a uniformly convex Banach space, then every pairs of nonempty subsets C, D which C is convex satisfy the property UC.

2. Main Results

In this section, we prove an existence result of optimal solution for the system of differential inclusion given in (3) and (4).

First we define $C = \{y \in C(I, H) : \ y(t_0) = u_1\}$ and $D = \{y \in C(I, H) : \ y(t_0) = u_2\}$. For any $y \in C$ and $z \in D$, we have $\|y - z\| \geq \|u_1 - u_2\|$ if $d(C, D) = \|u_1 - u_2\|$, where $d(C, D) = \inf\{\|y - z\| : \ y \in C, z \in D\}$.
We define the multivalued map $S : C \cup D \rightrightarrows C(I, H)$ as

$$S(y) := \{h \in C(I, H) \mid h(t) = u_2 + \int_{t_0}^{t} \vartheta(s)ds, \ \ t \in I\} \qquad y \in C,$$

where

$$\vartheta \in T_{\Psi,y} = \{\vartheta \in L^1(I,H) \mid \vartheta(t) \in \Psi(t,y(t)) \quad for\ a.e.\ t \in I\}.$$

and

$$S(z) := \{g \in C(I,H) \mid g(t) = u_1 + \int_{t_0}^{t} \omega(s)ds, \quad t \in I\}, \qquad z \in D,$$

where

$$\omega \in T_{\Phi,z} = \{\omega \in L^1(I,H) \mid \omega(t) \in \Phi(t,z(t)) \quad for\ a.e.\ t \in I\}.$$

Theorem 2.1. *Let H be a real separable Hilbert space. Let S, C, and D are defined as above. Suppose that $\Phi, \Psi : I \times H \rightrightarrows H$ are compact convex valued satisfy the following conditions*

($\mathcal{A}_1$) *Φ and Ψ are measurable in the first argument and continuous in the second argument;*

($\mathcal{A}_2$) *there is a function $l \in L^1(I,[0,\infty))$ such that*

$$H(\Phi(t,z),\Psi(t,y)) \leq l(t)(\|y-z\| - \|u_1-u_2\|),$$

for each $t \in I$ and $y \in C$ and $z \in D$;

($\mathcal{A}_3$) *there is a continuous nondecreasing function $\xi_1, \xi_2 : [t_0,\infty) \to [0,\infty)$ and $\rho_1, \rho_2 \in L^1(I,\mathbb{R}^+)$ such that for each $t \in I$ and $u \in H$,*

$$\|\Phi(t,z)\| := \sup\{\|\omega\| : \ \omega \in \Phi(t,z(t))\} \leq \rho_1(t)\xi_1(\|z\|),$$

and

$$\|\Psi(t,y)\| = \sup\{\|\vartheta\| : \ \vartheta \in \Psi(t,y(t))\} \leq \rho_2(t)\xi_2(\|y\|).$$

Then there exists $y \in C \cup D$ such that $d(y,Sy) = d(C,D)$. Therefore, the pair (y,z) such that $z \in Sy$ and $d(y,z) = d(C,D)$, is an optimal solution for the system of differential inclusion given in (3) *and* (4).

Proof. First, notice that since H is separable and Φ satisfies condition ($\mathcal{A}_1$), then by Theorem 7.25 of [5], $S(y)$ is nonempty so S is well-defined. It is easy to check that $S(C) \subset D$ and $S(D) \subset C$. Also, It's obvious that C is closed and convex. Since H is a Hilbert space so it's uniformly convex space then by Proposition 23.2 of [10] and Remark 1, pair (C,D) satisfies the property UC.

Now, we show that $S(K)$ is compact for every compact set $K \subset C \cup D$. Let $\{h_n\}$ in $S(K)$, then there exists $y_n \in K$ such that $h_n \in S(y_n)$, but since K is compact, so we can assume that y_n converges to a point $y \in K$. If $y \in C$, since C

and D are closed and $C \cap D = \emptyset$, then we can conclude that there exists $N_0 \in \mathbb{N}$ such that for $n > N_0$, $y_n \in A$ so there is $\vartheta_n \in T_{\Psi,y_n}$ such that

$$h_n(t) = u_2 + \int_{t_0}^{t} \vartheta_n(r)dr \qquad t \in I.$$

Since, the sequence $\{y_n\}$ is convergent and ξ_2 is continuous, from $(\mathcal{A}_3)$ we have

$$\|\vartheta_n\| \leq \rho_2(t)\xi_2(\|y_n\|) \leq k\rho_2(t),$$

thus, the set $\{\vartheta_n\}$ is bounded. Also, since Ψ and Φ are compact valued so $\{\vartheta_n(t)\}$ is relatively compact in H for each $t \in I$. Therefore, the sequence $\{\vartheta_n\}$ satisfies the hypotheses of Proposition 4.2.1 in [6], then it is weakly compact in $L^1(I, H)$, hence we assume $\vartheta_n \rightharpoonup \vartheta$ in $L^1(I, H)$. Then by Theorem 5.1.1 of [6],

$$h_n(t) = u_2 + \int_{t_0}^{t} \vartheta_n(r)dr \to u_2 + \int_{t_0}^{t} \vartheta(r)dr.$$

We set $h(t) := u_2 + \int_{t_0}^{t} \vartheta(r)dr$. Also, by Lemma 5.1.1 of [6], $\vartheta \in T_{\Psi,y}$ and we conclude that $h \in S(y) \subset S(K)$ and therefore $S(K)$ is compact. If $y \in D$, the proof is similar.

Now, we show that S is cyclic contraction. Let $y \in C, z \in D$ and $h \in S(y)$, then there exists $\vartheta \in T_{\Psi,y}$ such that for each $t \in I$,

$$h(t) = u_2 + \int_{t_0}^{t} \vartheta(r)dr.$$

From condition $(\mathcal{A}_2)$,

$$H(\Phi(t, z(t)), \Psi(t, y(t))) \leq l(t)(\|y - z\| - \|u_1 - u_2\|) \qquad t \in I.$$

Since, Φ is compact valued, then there is $\varphi(t) \in \Phi(t, z(t))$ such that

$$\|\vartheta(t) - \varphi(t)\| \leq l(t)(\|y - z\| - \|u_1 - u_2\|) \quad t \in I.$$

Now we define the multivalued map $U : I \rightrightarrows H$ as

$$U(t) := \{\varphi \in H \mid \|\vartheta(t) - \varphi\| \leq l(t)(\|y - z\| - \|u_1 - u_2\|)\}.$$

Since ϑ is measurable, then U is measurable, hence, by Proposition III.4 of [2], $F(t) = \Phi(t, z(t)) \cap U(t)$ is measurable. Therefore, Kuratowski Ryll-Nardzewski Theorem of [8] conclude that F has a measurable selection ω. Thus, $\omega(t) \in \Phi(t, z(t))$ and

$$\|\vartheta(t) - \omega(t)\| \leq l(t)(\|y - z\| - \|u_1 - u_2\|) \ \ for\ each\ t \in I.$$

For each $t \in I$, we define

$$g(t) = u_1 + \int_{t_0}^{t} \omega(r)dr,$$

therefore

$$\begin{aligned} \|h(t) - g(t)\| &\leq \|u_2 - u_1\| + \int_{t_0}^{t} |\vartheta(r) - \omega(r)|dr \\ &\leq \|u_2 - u_1\| + \int_{t_0}^{t} l(r)(\|y - z\| - \|u_2 - u_1\|)dr \\ &\leq \|u_2 - u_1\| + \int_{t_0}^{t} l(r)(e^{\tau L(r)}e^{-\tau L(r)}\|y - z\| - e^{\tau L(r)}\|u_2 - u_1\|dr \\ &\leq \|u_2 - u_1\| + \frac{1}{\tau}(\|y - z\|_1 - \|u_2 - u_1\|)\int_{t_0}^{t} (e^{\tau L(r)})'dr \\ &\leq \|u_2 - u_1\| + \frac{1}{\tau}e^{\tau L(t)}(\|y - z\|_1 - \|u_2 - u_1\|). \end{aligned}$$

where $L(t) = \int_{t_0}^{t} l(r)dr, t \in I$, and

$$\|z\|_1 = \sup\{e^{-\tau L(t)}\|z(t)\| \mid t \in I\} \qquad \tau > 1.$$

Then

$$\|h - g\|_1 \leq \frac{1}{\tau}\|y - z\|_1 - (1 - \frac{1}{\tau})\|u_2 - u_1\|.$$

Therefore, by the same argument as in the above, we deduce that

$$H(S(y), S(z)) \leq \gamma\|y - z\|_1 - (1 - \gamma)\|u_2 - u_1\|,$$

where $\gamma = \frac{1}{\tau} < 1$. Also, $d_1(C, D) = \|u_2 - u_1\|_1 = \|u_2 - u_1\| = d(C, D)$, where $d_1(C, D) = \inf\{\|y - z\|_1 : y \in C, z \in D\}$. Therefore, the map S is cyclic contraction.

Then by Theorems 2.10 of [4], S has a best proximity point y in C, so $d(y, S(y)) = d(C, D)$. Since S is compact valued, there exists $z \in S(y)$ such that $\|y - z\| = d(C, D)$, then z is a best proximity point in D, and $d(z, S(z)) = d(C, D)$. Since the pair (C, D) satisfies the property UC, we conclude that $y \in S(z)$.

Now we show that y, z are solution of system of differential inclusion given in (3) and (4), respectively.

First notice that since $\|y - z\| = d(C, D) = \|u_1 - u_2\|$, then by condition $(\mathcal{A}_2)$, we deduce that $\Psi(t, y(t)) = \Phi(t, z(t))$ for each $t \in I$. We have $y \in S(z)$ and $z \in S(y)$, so there exist $\vartheta, \omega \in L^1(I, H)$ such that $\vartheta(t) \in \Psi(t, y(t)) = \Phi(t, z(t))$ and $\omega(t) \in \Phi(t, z(t)) = \Psi(t, y(t))$ for a.e. $t \in I$ and

$$z(t) = u_2 + \int_{t_0}^{t} \vartheta(r)dr,$$

$$y(t) = u_1 + \int_{t_0}^{t} \omega(r)dr.$$

Then,

$$\begin{aligned} z'(t) &\in \Phi(t, z(t)) \qquad a.e.\ t \in I \qquad and \qquad z(t_0) = u_2, \\ y'(t) &\in \Psi(t, y(t)) \qquad a.e.\ t \in I \qquad and \qquad y(t_0) = u_1. \end{aligned}$$

Therefore, the pair (y, z) is an optimal solution for the system of differential inclusion given in (3) and (4). □

Now we present the following system of differential inclusion

$$\begin{aligned} &(5) \qquad y' \in By(t) + \Phi(t, y(t)) \qquad t \in I = [0, d] \qquad y(0) = u_2, \\ &(6) \qquad y' \in By(t) + \Psi(t, y(t)) \qquad t \in I = [0, d] \qquad y(0) = u_1. \end{aligned}$$

in a Hilbert spaces in which $\Phi, \Psi : I \times H \rightrightarrows H$ are multivalued maps, $\{B(t)\}_{t\in[0,d]}$ is a family of linear operators in H and $u_1, u_2 \in H$ are distinct.

Definition 2.5. The two parameter family $\{E(t, s)\}_{(t,s)\in\triangle}$ that

$$\triangle = \{(t, s) \in [0, d] \times [0, d] : \ 0 \le s \le t \le d\},$$

is said to be an evolution contraction system if $E(t, s) : H \to H$ for each $(t, s) \in \triangle$, is a bounded linear operator and satisfies in the following conditions:

(*i*) $E(s, s) = \mathcal{I}, s \in [0, d]; \quad E(t, r)E(r, s) = E(t, s) \ \ 0 \le s \le r \le t \le d,$

(*ii*) for each $0 \le s \le t \le d$, map $(t, s) \to E(t, s)$ is strongly continuous.

(*ii*) $\|E(s, t)\| \le 1$ for each $0 \le s \le t \le d$,

Now we prove an existence optimal solution for the system of differential inclusion given in (5) and (6).
First we define $C = \{y \in C(I, H) : \ y(0) = u_1\}$ and $D = \{y \in C(I, H) : \ y(0) = u_2\}$. We have $d(C, D) = \|u_1 - u_2\|$. We define the multivalued map $S : C \cup D \rightrightarrows C(I, H)$ as

$$S(y) := \{h \in C(I, H) \mid h(t) = E(t, 0)u_2 + \int_0^t E(t, r)\vartheta(r)dr, \ \ t \in I\}, \qquad y \in C,$$

where

$$\vartheta \in T_{\Psi,y} = \{\vartheta \in L^1(I, H) \mid \vartheta(t) \in \Psi(t, y(t)) \quad for\ a.e.\ t \in I\}.$$

and

$$S(z) := \{g \in C(I, H) \mid g(t) = E(t, 0)u_1 + \int_0^t E(t, r)\omega(r)dr, \ \ t \in I\}, \qquad z \in D,$$

where

$$\omega \in T_{\Phi,z} = \{\omega \in L^1(I, H) \mid \omega(t) \in \Phi(t, z(t)) \quad for\ a.e.\ t \in I\}.$$

Theorem 2.2. *Let H be a real separable Hilbert space. Let S is defined as above. Suppose that $\Phi, \Psi : I \times H \rightrightarrows H$ are compact convex valued satisfy the conditions* $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ *and* $(\mathcal{A}_3)$ *of Theorem* 2.1. *We assume the family of linear operators* $\{B(t)\}_{t\in[0,d]}$ *satisfy the following hypotheses:*

$(\mathcal{A}_4)$ $B(t) : Dom(B) \subseteq H \to H$, *where* $Dom(B)$ *is not depending on* $t \in [0, d]$ *and dense in* H *and* $\{B(t)\}_{t\in[0,d]}$ *generates an evolution contraction system* $\{E(t, s)\}_{(t,s)\in\triangle}$.

Then there exists $y \in C \cup D$ *such that* $d(y, Sy) = d(C, D)$. *Therefore, the pair* (y, z) *such that* $z \in Sy$ *and* $d(y, z) = d(C, D)$, *is an optimal solution for the system of differential inclusion given in* (5) *and* (6).

Proof. First, notice that since H is separable and Φ satisfies condition $(\mathcal{A}_1)$, then by Theorem 2.2, $S(y)$ is nonempty and S is well-defined. It is easy to check that $S(C) \subset D$ and $S(D) \subset C$. By some minor modifications in the proof of Theorem 2.1, we obtain the conclusion for the system of differential inclusion given in (5) and (6). □

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] J. P. Aubin and A. Cellina, *Differential Inclusions: Set-Valued Maps and Viability Theory*, Springer, Berlin Heidelberg, New York, Tokyo, 1984.

[2] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions,* Lecture Notes in Mathematics 580, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[3] S. Djebali, L. Górniewicz and A. Ouahab, *Solution Sets for Differential Equations and Inclusions,* De Gruyter, Berlin, 2013.

[4] M. Fakhar, Z. Soltani and J. Zafarani, Existence of best proximity points for set-valued cyclic Mier–Keeler contraction, *Fixed Point Theory* **19** (1) (2018) 211 − 218.

[5] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis*, Vol. I: Theory, Kluwer Academic Publishers, Dordrecht, 1997.

[6] M. Kamenskii, V. Obukhovskii and P. Zecca, *Condensing multivalued maps and semilinear differential inclusions in Banach spaces,* De Gruyter Series in Nonlinear Analysis and Applications. **7**, Walter de Gruyter Co., Berlin, 2001.

[7] E. .N. Mahmudov, *Approximation and Optimization of Discrete and Differential Inclusions*, Elsevier, Boston, USA, 2011.

[8] M. Srebrny, Measurable Selectors of PCA Multifunctions with Applications, *Mem. Amer. Math. Soc.* **52** (311) (1984) pp. 50.

[9] T. Suzuki, M. Kikkawa and C. Vetro, The existence of best proximity points in metric spaces with the property UC, *Nonlinear Anal* **71** (2009) 2918–2926.

[10] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*,Vol. 3 : Variational methods and optimization, Springer, New York, 1985.

[11] P. Veeramani and S. Rajesh, *Best Proximity Points,*In: Ansari Q. (eds) Nonlinear Analysis. Trends in Mathematics. Birkhüser, New Delhi, 2014.

[12] D. Wagner, Survey of measurable selection theorems, *SIAM J. Control Optim* **15** (1977) 859–903.

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