Mathematics Interdisciplinary Research **7** (2022) 61 – 76

Original Scientific Paper

Hyperbolic Ricci-Bourguignon-Harmonic Flow

Shahroud Azami ⋆

Abstract

In this paper, we consider hyperbolic Ricci-Bourguignon flow on a compact Riemannian manifold M coupled with the harmonic map flow between M and a fixed manifold N. At the first, we prove the unique short-time existence to solution of this system. Then, we find the second variational of some geometric structure of M along this system such as, curvature tensors. In addition, for emphasize the importance of hyperbolic Ricci-Bourguignon flow, we give some examples of this flow on Riemannian manifolds.

Keywords: Ricci flow, hyperbolic equation, harmonic map, strictly hyperbolicity.

2020 Mathematics Subject Classification: 58J45, 58J47.

How to cite this article
S. Azami, Hyperbolic Ricci-Bourguignon-harmonic flow, *Math. Interdisc. Res.* **7** (2022) 61 – 76.

1. Introduction

Let (M^m, g) be a smooth compact Riemannian manifold. Geometric flows on M are evolution of a geometric structures under differential equations with some functionals on M. Geometric flows play important role in mathematics, because in differential geometry by some geometric flows we can obtain canonical metrics on Riemannian manifolds. A fundamental study of geometric flows began when Hamilton [8] introduced the Ricci flow on compact m-dimensional Riemannain

⋆Corresponding author (E-mail: azami@sci.ikiu.ac.ir)
Academic Editor: Seyfollah Mosazadeh
Received 27 January 2021, Accepted 18 September 2021
DOI: 10.22052/mir.2021.240451.1272

manifold (M, g_0) as follows

$$\frac{\partial}{\partial t} g = -2Ric, \qquad g(0) = g_0, \tag{1}$$

and normalized Ricci flow as follows

$$\frac{\partial}{\partial t} g = -2Ric + \frac{2r}{m} g, \qquad g(0) = g_0.$$

Here Ric is the Ricci curvature tensor of $g(t)$, $r = \frac{\int_M R\, d\mu}{\int_M d\mu}$, R denotes the scalar curvature of $g(t)$ and $d\mu$ denotes the volume form of $g(t)$. The short-time existence and uniqueness for solution of (1) was first shown by R. S. Hamilton (see [8]), using the Nash-Moser theorem and harmonic maps and then by D. DeTurck (see [6]) on a compact Riemannian manifold. The Ricci flow played a techniqual role in the study of topology and geometry of a manifold, for instance the Ricci flow is used in the proof of the Poincaré conjecture [13] and the sphere theorem [3]. Also, the harmonic map between Riemannian manifolds introduced for first time by Eells and Sampson [7].

A generalization of the Ricci flow is as follows

$$\frac{\partial}{\partial t} g = -2Ric + 2\rho R g = -2(Ric - \rho R g), \qquad g(0) = g_0, \tag{2}$$

for some real constant ρ where R denotes the scalar curvature. This flow is called Ricci-Bourguignon flow. Catino et'al [4] shown that if $\rho < \frac{1}{2(m-1)}$ then the flow (2) has a unique short-time solution.

Another, generalization of the Ricci flow is the harmonic-Ricci flow, is defined as follows

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2Ric + 2\eta(t) \nabla\phi \otimes \nabla\phi, & g(0) = g_0, \\ \frac{\partial}{\partial t} \phi = \tau_g \phi, & \phi(0) = \phi_0, \end{cases}$$

where $\eta(t) > 0$ only dependent to t and m, $\phi(t) : (M, g(t)) \to (N, h)$ is a smooth maps from $(M, g(t))$ to a target compact Riemannian manifold (N, h), and $\tau_g \phi$ is the tension field of the map ϕ given by the evolving metric $g(t)$. The harmonic-Ricci flow studied by Müller [11] and when $(N, h) = (\mathbb{R}, dr^2)$, this flow become the Bernhard List's flow [10],

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2Ric + 2\eta(t) \nabla\phi \otimes \nabla\phi, & g(0) = g_0, \\ \frac{\partial}{\partial t} \phi = \Delta_g \phi, & \phi(0) = \phi_0. \end{cases}$$

Then author [1], extended harmonic-Ricci flow as follows

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2Ric + 2\rho R g + 2\eta \nabla\phi \otimes \nabla\phi, & g(0) = g_0, \\ \frac{\partial}{\partial t} \phi = \tau_g \phi, & \phi(0) = \phi_0, \end{cases}$$

and provn that if $\rho < \frac{1}{2(m-1)}$ then this flow has a unique short-time solution for a positive time interval on any smooth compact Riemannian manifold (M^m, g_0) with intial condition (g_0, ϕ_0).

The geometric flows which introduced in above, were of the first variation of metric and similar to heat equation. Now, we introduce some of geometric flows which are family of the second order nonlinear partial differential equations analogous to wave equation flow metrics. Hyperbolic geometric flow is similar to Einstein equation

$$\frac{\partial^2}{\partial t^2} g_{ij} = -2R_{ij} - \frac{1}{2} g^{pq} \frac{\partial g_{ij}}{\partial t} \frac{\partial g_{pq}}{\partial t} + g^{pq} \frac{\partial g_{ip}}{\partial t} \frac{\partial g_{jq}}{\partial t},$$

and is defined as

$$\frac{\partial^2}{\partial t^2} g = -2Ric, \qquad g(0) = g_0, \qquad \frac{\partial g}{\partial t}(0) = k_0, \tag{3}$$

where k_0 is a symmetric tensor on M. The unique short-time existences of (3) investigated in [5] on compact Riemannian manifold. Author [2] extended hyperbolic geometric flow to harmonic-hyperbolic geometric flow which is defined as follows

$$\begin{cases} \frac{\partial^2 g}{\partial t^2} = -2Ric + 2\eta \nabla\phi \otimes \nabla\phi, & g(0) = g_0(x), \qquad \frac{\partial g}{\partial t}(0) = k_0, \\ \frac{\partial}{\partial t}\phi = \tau_g \phi, & \phi(0) = \phi_0, \end{cases} \tag{4}$$

and shown that system (4) has a unique short-time solution and funded evolution equations of curvature tensors of manifold under flow (4).

In present paper, we introduce a generalized of the harmonic-hyperbolic geometric flow on smooth compact Riemannian manifold (M^m, g_0) which we call it HRBH flow and we investigate the short-time existence and uniqueness for solution to the HRBH flow. We obtain the evolution equation of some geometric structures of M under the HRBH flow. Finally, we present some examples of the HRBH flow.

2. Preliminaries

Assume that (M^m, g) and (N^n, h) are smooth closed Riemannian manifolds. By Nash's embedding theorem [12], let N be isometrically embedded into Euclidean space by $e_N : (N^n, \gamma) \hookrightarrow \mathbb{R}^d$ for d large enough. As in [11], we identify map $\phi : M \to N$ with the map $e_N \circ \phi : M \hookrightarrow \mathbb{R}^d$, then we can write $\phi = (\phi^\lambda),\ 1 \leq \lambda \leq d$. We often delete the summation indices for ϕ when there is no ambiguity and we denote the tension field of the map ϕ by $\tau_g \phi = \nabla_p \nabla_p \phi$ for the covariant derivative ∇ on $T^*M \otimes \phi^* TN$.

Let M^m be a soomth compact Riemannian manifold. In present work, we assume that the metric of M evolves as

$$\begin{cases} \frac{\partial^2 g}{\partial t^2} = -2Ric + 2\rho R g + 2\eta \nabla\phi \otimes \nabla\phi, & g(0) = g_0(x), \qquad \frac{\partial g}{\partial t}(0) = k_0, \\ \frac{\partial}{\partial t}\phi = \tau_g \phi, & \phi(0) = \phi_0, \end{cases} \tag{5}$$

where Ric denotes the Ricci curvature tensor of M, ρ denotes a real constant, R denotes the scalar curvature, η denotes a positive coupling constant, k_0 denotes a symmetric tensor on M, and $\phi : M \to N$ is a map beteween of M and a fixed compact Riemannian manifold N. The flow (5) reduce to the hyperbolic geometric flow when $\rho = \eta = 0$ and it reduce to the harmonic-hyperbolic geometric flow when $\rho = 0$. We say that this flow is hyperbolic Ricci-Bourguignon-harmonic flow or shortly the HRBH flow.

3. Existence and Uniqueness for the HRBH Flow

In the following, similar to process of reviewing the existence and uniqueness of geometric flows such as Ricci-Bourguignon flow, hyperbolic geometric flow, harmonic-Ricci flow, we prove the existence and uniqueness for the solution to th HRBH flow in short time on a compact Riemannian manifold (M^m, g_0).

Theorem 3.1. *Let (M^m, g_0) and (N^n, h) be two compact Riemannian manifolds, k_0 be a symmetric tensor on M and $\rho < \frac{1}{2(m-1)}$. Then there is a constant $T > 0$ such that the HRBH flow* (5) *has a unique smooth solution $(g(t), \phi(t))$ on $[0, T)$.*

Proof. The HRBH flow is a family of nonlinear weakly hyperbolic second order partial differential equations. Therefore our method of proof will be applying the gauge fixing idea and the push-forward of a solution of (5) we can obtain a family of nonlinear strictly-hyperbolic second order partial differential equations and then the short-time existence and uniqueness for solution on a compact Riemannain manifold, implies that the existence and uniqueness forthe solution of this system and in finally the pull-back of this solution complete the proof of theorem. For this end, suppose that $(g(t), \phi(t))_{t\in[0,T)}$ is a solution to the HRBH flow with initial condition $(g(0), \phi(0)) = (g_0, \phi_0)$, $\frac{\partial g_{ij}}{\partial t}(0) = k_{ij}(0)$. Let $\psi_t : (M, \hat{g}(t)) \to (M, g_0)$ be solution of $\frac{\partial}{\partial t}\psi = \tau_g \psi$, with $\psi(0) = id_M$. Let

$$\hat{g}_{ij}(t) = \psi_* g_{ij}, \quad \hat{\phi}(t) = \psi_* \phi(t),$$

be the push-forward of g_{ij} and ϕ respectively. We now begin by calculating the evolution of $(\hat{g}_{ij}(t), \hat{\phi}(t))$. We suppose in local coordinate ψ_t is given by $y(x, t) = \psi_t(x) = (y^1(x, t), \ldots, y^n(x, t))$. Then we have

$$\hat{g}_{ij}(x, t) = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} g_{\alpha\beta}(y, t), \tag{6}$$

therefore, we arrive at the following:

$$
\begin{aligned}
\frac{\partial^2 \hat{g}_{ij}}{\partial t^2}(x,t) = & \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{d^2 g_{\alpha\beta}}{dt^2}(y(x,t),t) + \frac{\partial}{\partial x^i}(\frac{\partial^2 y^\alpha}{\partial t^2})\frac{\partial y^\beta}{\partial x^j} g_{\alpha\beta} \\
& + \frac{\partial}{\partial x^j}(\frac{\partial^2 y^\beta}{\partial t^2})\frac{\partial y^\alpha}{\partial x^i} g_{\alpha\beta} + 2\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial y^\beta}{\partial x^j}\frac{d g_{\alpha\beta}}{dt} \\
& + 2\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t}) g_{\alpha\beta} + 2\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t})\frac{\partial y^\alpha}{\partial x^i}\frac{d g_{\alpha\beta}}{dt}.
\end{aligned}
$$

Also,

$$
\frac{d g_{\alpha\beta}}{dt}(y(x,t),t) = \frac{\partial g_{\alpha\beta}}{\partial t} + \frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t},
$$

and

$$
\frac{d^2 g_{\alpha\beta}}{dt^2}(y(x,t),t) = \frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial y^\lambda}\frac{\partial y^\gamma}{\partial t}\frac{\partial y^\lambda}{\partial t} + 2\frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial t}\frac{\partial y^\gamma}{\partial t} + \frac{\partial^2 g_{\alpha\beta}}{\partial t^2} + \frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial^2 y^\gamma}{\partial t^2}.
$$

Hence

$$
\begin{aligned}
\frac{\partial^2 \hat{g}_{ij}}{\partial t^2}(x,t) = & \frac{\partial^2 g_{\alpha\beta}}{\partial t^2}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} + \frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial y^\lambda}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t}\frac{\partial y^\lambda}{\partial t} \\
& + 2\frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial t}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t} + \frac{\partial}{\partial x^j}(g_{\alpha\beta}\frac{\partial y^\beta}{\partial x^i}\frac{\partial^2 y^\alpha}{\partial t^2}) \\
& + \frac{\partial}{\partial x^i}(g_{\alpha\beta}\frac{\partial y^\beta}{\partial x^j}\frac{\partial^2 y^\alpha}{\partial t^2}) \\
& + \left[\frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} - \frac{\partial}{\partial x^i}(g_{\beta\gamma}\frac{\partial y^\beta}{\partial x^j}) - \frac{\partial}{\partial x^j}(g_{\beta\gamma}\frac{\partial y^\beta}{\partial x^i})\right]\frac{\partial^2 y^\gamma}{\partial t^2} \\
& + 2\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial y^\beta}{\partial x^j}(\frac{\partial g_{\alpha\beta}}{\partial t} + \frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t}) \\
& + 2 g_{\alpha\beta}\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t}) \\
& + 2\frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t})(\frac{\partial g_{\alpha\beta}}{\partial t} + \frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t}).
\end{aligned}
$$

We consider a fixed point $p \in M$. Using the normal coordinates $\{x^i\}$ around p, we get $\frac{\partial g_{ij}}{\partial x^k}(p) = 0$ and

$$
\frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} - \frac{\partial}{\partial x^i}(g_{\beta\gamma}\frac{\partial y^\beta}{\partial x^j}) - \frac{\partial}{\partial x^j}(g_{\beta\gamma}\frac{\partial y^\beta}{\partial x^i}) = 0, \quad \forall i, j, \gamma = 1, 2, \ldots, n.
$$

Thus we get

$$\begin{aligned}\frac{\partial^2\hat{g}_{ij}}{\partial t^2}(x,t) &= \frac{\partial^2 g_{\alpha\beta}}{\partial t^2}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} + \frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial y^\lambda}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t}\frac{\partial y^\lambda}{\partial t}\\ &+ 2\frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial t}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t} + \frac{\partial}{\partial x^i}(g_{\alpha\beta}\frac{\partial y^\beta}{\partial x^j}\frac{\partial^2 y^\alpha}{\partial t^2})\\ &+ \frac{\partial}{\partial x^j}(g_{\alpha\beta}\frac{\partial y^\beta}{\partial x^i}\frac{\partial^2 y^\alpha}{\partial t^2}) + 2g_{\alpha\beta}\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t})\\ &+ 2\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial y^\beta}{\partial x^j}(\frac{\partial g_{\alpha\beta}}{\partial t} + \frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t})\\ &+ 2\frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t})(\frac{\partial g_{\alpha\beta}}{\partial t} + \frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t}).\end{aligned} \tag{7}$$

Now we assume that $y(x,t)$ is a solution to the following problem

$$\begin{cases}\frac{\partial^2 y^\alpha}{\partial t^2} = \frac{\partial y^\alpha}{\partial x^k} g^{il}(\hat{\Gamma}^k_{jl} - \mathring{\Gamma}^k_{jl}),\\ y^\alpha(x,0) = x^\alpha, \quad \frac{\partial}{\partial t} y^\alpha(x,0) = y^\alpha_1(x).\end{cases} \tag{8}$$

We consider the vector field

$$V_i = g_{ik} g^{jl}(\hat{\Gamma}^k_{jl} - \mathring{\Gamma}^k_{jl}),$$

here $\hat{\Gamma}^k_{jl}$ and $\mathring{\Gamma}^k_{jl}$ are the Christoffel symboles of the metrics $\hat{g}_{ij}(t)$ and $g_{ij}(0)$, respectively, $y^\alpha_1(x) \in C^\infty(M)$ for $1 \le \alpha \le m$. Since

$$\frac{\partial^2}{\partial t^2} g_{ij} = -2R_{ij} + 2\rho R g_{ij} + 2\eta \nabla_i \phi \nabla_j \phi,$$

therefore plugging last equation into (7) we obtain the evolution equation for $\hat{g}_{ij}$ as follows

$$\frac{\partial^2}{\partial t^2}\hat{g}_{ij} = -2\hat{R}_{ij} + 2\rho \hat{R}\hat{g}_{ij} + 2\eta \nabla_i \hat{\phi} \nabla_j \hat{\phi} + \hat{\nabla}_i V_j + \hat{\nabla}_j V_i + F(Dy, D_t D_x y),$$

where

$$Dy = (\frac{\partial y^\alpha}{\partial t}, \frac{\partial y^\alpha}{\partial x^i}), \qquad D_t D_x y = (\frac{\partial^2 y^\alpha}{\partial x^i \partial t}), \qquad \alpha, i = 1, 2, \ldots, n.$$

The relation

$$\hat{\Gamma}^k_{jl} = \frac{\partial y^\alpha}{\partial x^j}\frac{\partial y^\beta}{\partial x^i}\frac{\partial x^k}{\partial y^\gamma}\Gamma^\gamma_{\alpha\beta} + \frac{\partial x^k}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^j \partial x^i},$$

results that

$$\frac{\partial^2 y^\alpha}{\partial t^2} = g^{jl}(\frac{\partial^2 y^\alpha}{\partial x^j \partial x^i} - \mathring{\Gamma}^k_{jl}\frac{\partial y^\alpha}{\partial x^j} + \Gamma^\gamma_{\alpha\beta}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^i}),$$

and

$$\begin{aligned}\frac{\partial^2 \hat{g}_{ij}}{\partial t^2}(x,t) = &\ \hat{g}^{kl}\frac{\partial^2 \hat{g}_{ij}}{\partial x^k \partial x^l}(x,t) - 2\rho \hat{g}_{ij}\hat{g}^{pq}\hat{g}^{kl}\frac{\partial^2 \hat{g}_{kl}}{\partial x^p \partial x^q}(x,t)\\ &+ 2\rho \hat{g}_{ij}\hat{g}^{pq}\hat{g}^{kl}\frac{\partial^2 \hat{g}_{ql}}{\partial x^p \partial x^k}(x,t) + 2\eta \nabla_i \hat{\phi}\nabla_j \hat{\phi}\\ &+ G(\hat{g}, D_x \hat{g}) + F(Dy, D_t D_x y).\end{aligned} \tag{9}$$

Here $\hat{g} = (\hat{g}_{ij})$ and $D_x\hat{g} = (\frac{\partial \hat{g}_{ij}}{\partial x^k})$ for $i, j, k = 1, 2, \ldots, m$. Now we show that the system (9) is strictly hyperbolic partial differential equation. Upon doing this, we consider only the highest order terms of (9), then we compute its principal symbol and we show that is elliptic. Let the linear differential operator $L_{\hat{g}}$ defined as

$$L_{\hat{g}}(h_{ij}) = \hat{g}^{kl}\frac{\partial^2 h_{ij}}{\partial x^k \partial x^l} - 2\rho \hat{g}_{ij}\hat{g}^{pq}\hat{g}^{kl}\frac{\partial^2 h_{kl}}{\partial x^p \partial x^q} + 2\rho \hat{g}_{ij}\hat{g}^{pq}\hat{g}^{kl}\frac{\partial^2 h_{ql}}{\partial x^p \partial x^k}.$$

The principal symbol of differential operator $L_{\hat{g}}$ in direction $\zeta = (\zeta_1, \ldots, \zeta_n)$ is

$$\sigma L_{\hat{g}}(\zeta)h_{ij} = \hat{g}^{kl}\zeta_k\zeta_l h_{ij} - 2\rho \hat{g}_{ij}\hat{g}^{pq}\hat{g}^{kl}\zeta_p\zeta_q h_{kl} + 2\rho \hat{g}_{ij}\hat{g}^{pq}\hat{g}^{kl}\zeta_p\zeta_k h_{ql}, \tag{10}$$

where in last equation we replace $\frac{\partial}{\partial x^i}$ by the Fourier transform variable ζ_i. Since the principal symbol is homogeneous, then we can always assume ζ has length 1 and we perform all the computing in an orthonormal basis $\{e_i\}_{i=1}^n$ of T_pM such that $\zeta = g(e_1, .)$ that is $\zeta_i = 0$ for $i \neq 1$, then

$$\begin{cases}\hat{g}_{ij} = \delta_{ij},\\ \zeta = (1, 0, \ldots, 0).\end{cases}$$

Let O be zero matrix and $A[n-1]$ be the $(n-1)\times(n-1)$ matrix as

$$A[n-1] = \begin{pmatrix} 1-2\rho & -2\rho & \cdots & -2\rho\\ -2\rho & 1-2\rho & \cdots & -2\rho\\ \vdots & \vdots & \ddots & \vdots\\ -2\rho & -2\rho & \cdots & 1-2\rho\end{pmatrix}.$$

Thus (10) becomes

$$\sigma L_{\hat{g}}(\zeta)h_{ij} = h_{ij} - 2\rho\delta_{ij}\delta_{kl}h_{kl} + 2\rho\delta_{ij}h_{11},$$

which for any $h \in \Gamma(S^2M)$, in the coordinate system

$$(h_{11}, h_{22}, \ldots, h_{nn}, h_{12}, \ldots, h_{1n}, h_{23}, h_{24}, \ldots, h_{n-1,n})$$

the operator $\sigma L_{\hat{g}}$ can be represented as

$$\sigma L_{\hat{g}} = \left(\begin{array}{cccc|c|c} 1 & -2\rho & \cdots & -2\rho & & \\ \vdots & & A[n-1] & & O & O \\ 0 & & & & & \\ \hline & & O & & I_{(n-1)} & O \\ \hline & & O & & O & I_{\frac{(n-1)(n-2)}{2}} \end{array} \right).$$

Because, the matrix σL_g has 1 eigenvalue equal to $1-2(n-1)\rho$ and $\frac{1}{2}n(n+1)-1$ eigenvalues equal to 1 (see [4]), hence the operator $L_{\hat{g}}$ is elliptic for $\rho < \frac{1}{2(n-1)}$ and then the Equations (9) are strictly hyperbolic partial differential equation for $\rho < \frac{1}{2(n-1)}$. Also, we have

$$\frac{\partial \hat{\phi}}{\partial t} = \psi_*(\frac{\partial \phi}{\partial t}) + L_V \hat{\phi} = \tau_{\hat{g}} \hat{\phi} + < \nabla \hat{\phi}, V >= \tau_{\hat{g}} \hat{\phi} + d\hat{\phi}(V).$$

Via normal coordinates on (N, γ) at the base point we get ${}^N\Gamma^{\lambda}_{\mu\nu} = 0$. Hence $\tau_{\hat{g}} \hat{\phi} = \Delta_{\hat{g}} \hat{\phi}$ which this implies that

$$\begin{aligned} \frac{\partial \hat{\phi}}{\partial t} &= \Delta_{\hat{g}} \hat{\phi} + d\hat{\phi}(V) = \hat{g}^{kl}(\frac{\partial^2}{\partial x^k \partial x^l} \hat{\phi}^{\lambda} - \hat{\Gamma}^j_{kl} \nabla_j \hat{\phi}^{\lambda}) + \nabla_j \hat{\phi}^{\lambda} \hat{g}^{kl}(\hat{\Gamma}^j_{kl} - \mathring{\Gamma}^k_{jl}) \\ &= \hat{g}^{kl}(\partial_k \partial_l \hat{\phi}^{\lambda} - \mathring{\Gamma}^k_{jl} \nabla_j \hat{\phi}^{\lambda}), \end{aligned} \tag{11}$$

and it is strictly hyperbolic equation. Using the standard theory of hyperbolic equations [9] on compact manifold M we conclude the system (9) has a unique short-time smooth solution. The solution to (9) yields to a solution of the HRBH flow (5) from (6) and (8). Hence the proof of existence to solution of (5) complete. Using the fact that

$$\hat{\Gamma}^k_{jl} = \frac{\partial y^{\alpha}}{\partial x^j} \frac{\partial y^{\beta}}{\partial x^i} \frac{\partial x^k}{\partial y^{\gamma}} \Gamma^{\gamma}_{\alpha\beta} + \frac{\partial x^k}{\partial y^{\alpha}} \frac{\partial^2 y^{\alpha}}{\partial x^j \partial x^i},$$

we can rewritten the initial value problem (8) as follows:

$$\begin{cases} \frac{\partial^2 y^{\alpha}}{\partial t^2} = g^{jl} \left(\frac{\partial^2 y^{\alpha}}{\partial x^j \partial x^l} - \mathring{\Gamma}^k_{jl} \frac{\partial y^{\alpha}}{\partial x^k} + \Gamma^{\alpha}_{\beta\gamma} \frac{\partial y^{\beta}}{\partial x^j} \frac{\partial y^{\gamma}}{\partial x^i} \right), \\ y^{\alpha}(x, 0) = x^{\alpha}, \quad \frac{\partial}{\partial t} y^{\alpha}(x, 0) = y^{\alpha}_1(x). \end{cases} \tag{12}$$

Manifold M is compact and the Equations (12) strictly hyperbolic PDE, therefore (12) has a unique smooth solution $\psi_t(x) = (y^1(x,t), \ldots, y^n(x,t))$ for short time. Suppose that $(g^{(1)}_{ij}(t), \phi^1(t))$ and $(g^{(2)}_{ij}(t), \phi^2(t))$ are two solutions to the HRBH flow

(5) with the same initial data. Let $\psi_t^{(1)}$ and $\psi_t^{(2)}$ be two solutions of the initial value problem (12) corresponding to $g^{(1)}$ and $g^{(2)}$, respectively. Then the metrics $\hat{g}_{ij}^{(1)}(t) = (\psi_t^{(1)})_* g_{ij}^{(1)}(t)$ and $\hat{g}_{ij}^{(2)}(t) = (\psi_t^{(2)})_* g_{ij}^{(2)}(x,t)$ are two solutions to the modified evolution Equation (9) wich share the same initial data. Also, $\hat{\phi}^{(1)}(t) = (\psi_t^{(1)})_* \phi^{(1)}(t)$ and $\hat{\phi}^{(2)}(t) = (\psi_t^{(2)})_* \phi^{(2)}(x,t)$ are two solutions to the modified evolution Equation (11) which share the same initial data. Since the Equation (9) only have one solution with the same initial data, we get that $\hat{g}_{ij}^{(1)}(x,t) = \hat{g}_{ij}^{(2)}(x,t)$. Thus using the standard uniqueness result of PDE system for system (12) we deduce the corresponding solutions $\psi_t^{(1)}$ and $\psi_t^{(2)}$ of (6) must agree. Therefore, the metrics $g_{ij}^{(1)}(x,t)$ and $g_{ij}^{(2)}(x,t)$ must agree. Since Equation (11) only have one solution with the same initial value then $\hat{\phi}_t^{(1)} = \hat{\phi}_t^{(1)}$, then $\phi_t^{(1)} = \phi_t^{(1)}$. Then we complete the proof of the uniqueness for solution to the HRBH flow (5). □

4. Evolution Formulas for Curvature Tensor under the HRBH Flow

The evolution for the metric implies some the nonlinear evolution of geometric quantities and the HRBH flow shows that the metric deforms by an evolution equation. In the following, we apply the method used to obtain the evolution formulas of geometric quantities along the harmonic-Ricic flow and the hyperbolic geometric flow (see [5], [11]). We compute the evolution formulas for curvature tensors of (M, g) along the HRBH flow.

Lemma 4.1. *Along the HRBH flow the quantities g^{ij} evolves as follows:*

$$\frac{\partial^2}{\partial t^2} g^{ij} = 2R^{ij} - 2\rho R g^{ij} - 2\eta \nabla^i \phi \nabla^j \phi + 2 g^{ip} g^{rq} g^{sj} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{rs}}{\partial t}. \tag{13}$$

Proof. Since $g^{ij} g_{jk} = \delta_k^i$, we get $\frac{\partial}{\partial t} g^{ij} = -g^{ip} g^{jq} \frac{\partial g_{pq}}{\partial t}$. Therefore by direct computation we can obtain

$$\frac{\partial^2}{\partial t^2} g^{ij} = -g^{ip} g^{jq} \frac{\partial^2 g_{pq}}{\partial t^2} + 2 g^{ip} g^{rq} g^{sj} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{rs}}{\partial t}. \tag{14}$$

Plugging (5) into (14) completes the proof. □

Theorem 4.2. *Along the HRBH flow the tensor R_{ijkl} of M satisfies*

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ijkl} - B_{iljk} - B_{ijlk} + B_{ikjl}) \\ &\quad - g^{pq}(R_{ipkl}R_{qj} + R_{pjkl}R_{qi} + R_{ijkp}R_{ql} + R_{ijpl}R_{qk}) \\ &\quad + 2g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}.\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}.\frac{\partial}{\partial t}\Gamma^q_{ik}\Big) \\ &\quad - \rho\left[\nabla_i\nabla_k R g_{jl} - \nabla_i\nabla_l R g_{jk} - \nabla_j\nabla_k R g_{il} + \nabla_j\nabla_l R g_{ik}\right] + 2\rho R R_{ijkl} \\ &\quad + \eta\left[\frac{\partial^2(\nabla_k\phi\nabla_j\phi)}{\partial x^i\partial x^l} - \frac{\partial^2(\nabla_j\phi\nabla_l\phi)}{\partial x^i\partial x^k} - \frac{\partial^2(\nabla_k\phi\nabla_i\phi)}{\partial x^j\partial x^l} - \frac{\partial^2(\nabla_i\phi\nabla_l\phi)}{\partial x^j\partial x^k}\right],\end{aligned} \tag{15}$$

where Δ is the Beltrami-Laplace operator with respect to the metric g and $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$.

Proof. By direct computations the Christoffel symbol of metric g, $\Gamma^h_{jl} = \frac{1}{2}g^{hm}\big(\frac{\partial g_{mj}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^j} - \frac{\partial g_{jl}}{\partial x^m}\big)$, satisfies the evolution equation

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\Gamma^h_{jl} &= \frac{1}{2}\frac{\partial^2 g^{hm}}{\partial t^2}\Big(\frac{\partial g_{mj}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^j} - \frac{\partial g_{jl}}{\partial x^m}\Big) + \frac{\partial g^{hm}}{\partial t}\Big(\frac{\partial^2 g_{mj}}{\partial x^l\partial t} + \frac{\partial^2 g_{ml}}{\partial x^j\partial t} - \frac{\partial^2 g_{jl}}{\partial x^m\partial t}\Big) \\ &\quad + \frac{1}{2}g^{hm}\left[\frac{\partial}{\partial x^l}(\frac{\partial^2 g_{mj}}{\partial t^2}) + \frac{\partial}{\partial x^j}(\frac{\partial^2 g_{ml}}{\partial t^2}) - \frac{\partial}{\partial x^m}(\frac{\partial^2 g_{jl}}{\partial t^2})\right].\end{aligned}$$

Since $R_{ijkl} = g_{hk}R^h_{ijl}$ where $R^h_{ijl} = \frac{\partial\Gamma^h_{jl}}{\partial x^i} - \frac{\partial\Gamma^h_{il}}{\partial x^j} + \Gamma^h_{ip}\Gamma^p_{jl} - \Gamma^h_{jp}\Gamma^p_{il}$, with a double differentiation of the above equation respect to time, we have

$$\frac{\partial^2}{\partial t^2}R^h_{ijl} = \frac{\partial}{\partial x^i}(\frac{\partial^2}{\partial t^2}\Gamma^h_{jl}) - \frac{\partial}{\partial x^j}(\frac{\partial^2}{\partial t^2}\Gamma^h_{il}) + \frac{\partial^2}{\partial t^2}(\Gamma^h_{ip}\Gamma^p_{jl} - \Gamma^h_{jp}\Gamma^p_{il}),$$

and

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{ijkl} &= g_{hk}\left[\frac{\partial}{\partial x^i}(\frac{\partial^2\Gamma^h_{jl}}{\partial t^2}) - \frac{\partial}{\partial x^j}(\frac{\partial^2\Gamma^h_{il}}{\partial t^2}) + \frac{\partial^2}{\partial t^2}(\Gamma^h_{ip}\Gamma^p_{jl} - \Gamma^h_{jp}\Gamma^p_{il})\right] \\ &\quad + 2\frac{\partial g_{hk}}{\partial t}\left[\frac{\partial}{\partial x^i}(\frac{\partial\Gamma^h_{jl}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial\Gamma^h_{il}}{\partial t}) + \frac{\partial}{\partial t}(\Gamma^h_{ip}\Gamma^p_{jl} - \Gamma^h_{jp}\Gamma^p_{il})\right] \\ &\quad + R^h_{ijl}\frac{\partial^2 g_{hk}}{\partial t^2}.\end{aligned} \tag{16}$$

We consider a fixed point $p \in M$ and assume that $\{x^1, \ldots, x^n\}$ is the normal coordinates around p. Then in point p we have $\frac{\partial g_{ij}}{\partial x^k}(p) = 0$, $\Gamma^k_{ij}(p) = 0$, and

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} R_{ijkl} = &\frac{1}{2}\left[\frac{\partial^2}{\partial x^i \partial x^j}(\frac{\partial^2 g_{kl}}{\partial t^2}) + \frac{\partial^2}{\partial x^i \partial x^l}(\frac{\partial^2 g_{kj}}{\partial t^2}) - \frac{\partial^2}{\partial x^i \partial x^k}(\frac{\partial^2 g_{jl}}{\partial t^2})\right] \\
&- \frac{1}{2}\left[\frac{\partial^2}{\partial x^j \partial x^i}(\frac{\partial^2 g_{kl}}{\partial t^2}) + \frac{\partial^2}{\partial x^j \partial x^l}(\frac{\partial^2 g_{kj}}{\partial t^2}) - \frac{\partial^2}{\partial x^j \partial x^k}(\frac{\partial^2 g_{il}}{\partial t^2})\right] \\
&- g^{pm}\frac{\partial^2 g_{kp}}{\partial x^i \partial t}\Big(\frac{\partial^2 g_{ml}}{\partial x^j \partial t} + \frac{\partial^2 g_{mj}}{\partial x^l \partial t} - \frac{\partial^2 g_{jl}}{\partial x^m \partial t}\Big) \\
&+ g^{pm}\frac{\partial^2 g_{kp}}{\partial x^j \partial t}\Big(\frac{\partial^2 g_{ml}}{\partial x^i \partial t} + \frac{\partial^2 g_{mi}}{\partial x^l \partial t} - \frac{\partial^2 g_{il}}{\partial x^m \partial t}\Big) \\
&+ 2g_{hk}\Big(\frac{\partial}{\partial t}\Gamma^h_{ip}.\frac{\partial}{\partial t}\Gamma^p_{jl} - \frac{\partial}{\partial t}\Gamma^h_{jp}.\frac{\partial}{\partial t}\Gamma^p_{il}\Big).
\end{aligned}$$

Since $\frac{\partial^2}{\partial t^2} g = -2Ric + 2\rho R g + 2\eta \nabla\phi \otimes \nabla\phi$, then (16) yields

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} R_{ijkl} = &\frac{1}{2}\frac{\partial^2}{\partial x^i \partial x^l}\big(- 2R_{kj} + 2\rho R g_{kj} + 2\eta \nabla_k\phi \nabla_j \phi\big) \\
&- \frac{1}{2}\frac{\partial^2}{\partial x^i \partial x^k}\big(- 2R_{jl} + 2\rho R g_{jl} + 2\eta \nabla_j\phi \nabla_l \phi\big) \\
&- \frac{1}{2}\frac{\partial^2}{\partial x^j \partial x^l}\big(- 2R_{ki} + 2\rho R g_{ki} + 2\eta \nabla_k\phi \nabla_i \phi\big) \\
&+ \frac{1}{2}\frac{\partial^2}{\partial x^j \partial x^k}\big(- 2R_{il} + 2\rho R g_{il} + 2\eta \nabla_i\phi \nabla_l \phi\big) + \\
&+ \frac{1}{2}\frac{\partial^2}{\partial x^i \partial x^j}\big(- 2R_{kl} + 2\rho R g_{kl} + 2\eta \nabla_k\phi \nabla_l \phi\big) \\
&- \frac{1}{2}\frac{\partial^2}{\partial x^j \partial x^i}\big(- 2R_{kl} + 2\rho R g_{kl} + 2\eta \nabla_k\phi \nabla_l \phi\big) \\
&- g^{pm}\frac{\partial^2 g_{kp}}{\partial x^i \partial t}\Big(\frac{\partial^2 g_{ml}}{\partial x^j \partial t} + \frac{\partial^2 g_{mj}}{\partial x^l \partial t} - \frac{\partial^2 g_{jl}}{\partial x^m \partial t}\Big) \\
&+ g^{pm}\frac{\partial^2 g_{kp}}{\partial x^j \partial t}\Big(\frac{\partial^2 g_{ml}}{\partial x^i \partial t} + \frac{\partial^2 g_{mi}}{\partial x^l \partial t} - \frac{\partial^2 g_{il}}{\partial x^m \partial t}\Big) \\
&+ 2g_{hk}\Big(\frac{\partial}{\partial t}\Gamma^h_{ip}.\frac{\partial}{\partial t}\Gamma^p_{jl} - \frac{\partial}{\partial t}\Gamma^h_{jp}.\frac{\partial}{\partial t}\Gamma^p_{il}\Big). \qquad (17)
\end{aligned}$$

The following identities hold

$$\frac{\partial^2}{\partial x^i \partial x^l} R_{jk} = \nabla_i \nabla_l R_{jk} - R_{jp}\nabla_i \Gamma^p_{lk} - R_{kp}\nabla_i \Gamma^p_{lj}, \qquad (18)$$

and

$$\begin{aligned}
&-g^{pm}\frac{\partial^2 g_{kp}}{\partial x^i \partial t}\left(\frac{\partial^2 g_{ml}}{\partial x^j \partial t} + \frac{\partial^2 g_{mj}}{\partial x^l \partial t} - \frac{\partial^2 g_{jl}}{\partial x^m \partial t}\right) \\
&+g^{pm}\frac{\partial^2 g_{kp}}{\partial x^j \partial t}\left(\frac{\partial^2 g_{ml}}{\partial x^i \partial t} + \frac{\partial^2 g_{mi}}{\partial x^l \partial t} - \frac{\partial^2 g_{il}}{\partial x^m \partial t}\right)
\end{aligned}$$

$$+\ 2g_{hk}\Big(\tfrac{\partial}{\partial t}\Gamma^h_{ip}.\tfrac{\partial}{\partial t}\Gamma^p_{jl} - \tfrac{\partial}{\partial t}\Gamma^h_{jp}.\tfrac{\partial}{\partial t}\Gamma^p_{il}\Big)$$
$$= 2g_{pq}\Big(\tfrac{\partial}{\partial t}\Gamma^p_{il}.\tfrac{\partial}{\partial t}\Gamma^q_{jk} - \tfrac{\partial}{\partial t}\Gamma^p_{jl}.\tfrac{\partial}{\partial t}\Gamma^q_{ik}\Big). \tag{19}$$

Therefore, replacing (18) and (19) in (17) lead to

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{ijkl} &= -\nabla_i\nabla_l R_{jk} + \nabla_j\nabla_l R_{ki} + \nabla_i\nabla_k R_{jl} - \nabla_j\nabla_k R_{il}\\
&- g^{pq}(-R_{ijdk}R_{ql} + R_{ijql}R_{kp} + 2R_{ilqj}R_{kp})\\
&+ 2g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}.\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}.\frac{\partial}{\partial t}\Gamma^q_{ik}\Big)\\
&+ \rho\left[\frac{\partial^2 R}{\partial x^i\partial x^l}g_{kj} - \frac{\partial^2 R}{\partial x^j\partial x^l}g_{ki} - \frac{\partial^2 R}{\partial x^i\partial x^k}g_{jl} + \frac{\partial^2 R}{\partial x^j\partial x^k}g_{il}\right]\\
&+ \rho\left[\frac{\partial^2 g_{kj}}{\partial x^i\partial x^l}R - \frac{\partial^2 g_{ki}}{\partial x^j\partial x^l}R - \frac{\partial^2 g_{jl}}{\partial x^i\partial x^k}R + \frac{\partial^2 g_{il}}{\partial x^j\partial x^k}R\right]\\
&+ \eta\left[\frac{\partial^2(\nabla_k\phi\nabla_j\phi)}{\partial x^i\partial x^l} - \frac{\partial^2(\nabla_j\phi\nabla_l\phi)}{\partial x^i\partial x^k} - \frac{\partial^2(\nabla_k\phi\nabla_i\phi)}{\partial x^j\partial x^l} - \frac{\partial^2(\nabla_i\phi\nabla_l\phi)}{\partial x^j\partial x^k}\right],\end{aligned}$$

and equivalently

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ikjl} - B_{ijlk} - B_{iljk} + B_{ijkl})\\
&- g^{pq}(R_{ipkl}R_{qj} + R_{pjkl}R_{qi} + R_{ijkp}R_{ql} + R_{ijpl}R_{qk} + 2R_{ilqj}R_{kp})\\
&+ 2g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}.\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}.\frac{\partial}{\partial t}\Gamma^q_{ik}\Big)\\
&- \rho\left[\nabla_i\nabla_k R g_{jl} - \nabla_i\nabla_l R g_{jk} - \nabla_j\nabla_k R g_{il} + \nabla_j\nabla_l R g_{ik}\right] + 2\rho R R_{ijkl}\\
&+ \eta\left[\frac{\partial^2(\nabla_k\phi\nabla_j\phi)}{\partial x^i\partial x^l} - \frac{\partial^2(\nabla_j\phi\nabla_l\phi)}{\partial x^i\partial x^k} - \frac{\partial^2(\nabla_k\phi\nabla_i\phi)}{\partial x^j\partial x^l} - \frac{\partial^2(\nabla_i\phi\nabla_l\phi)}{\partial x^j\partial x^k}\right],\end{aligned}$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$, so the proof of the theorem is complete. □

Theorem 4.3. *Along the HRBH flow Ricci curvature tensor is satisfies*

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{ij} &= \Delta R_{ij} - (n-2)\rho\nabla_i\nabla_j R - \rho\Delta R g_{ij} + 2g^{pr}g^{qs}R_{piqj}R_{rs} - 2g^{pq}R_{pi}R_{qj}\\
&+ 2g^{kl}g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}.\frac{\partial}{\partial t}\Gamma^q_{kj} - \frac{\partial}{\partial t}\Gamma^p_{kl}.\frac{\partial}{\partial t}\Gamma^q_{ij}\Big) - 2g^{kp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} \qquad (20)\\
&+ \eta g^{kl}\left[\frac{\partial^2(\nabla_k\phi\nabla_j\phi)}{\partial x^i\partial x^l} - \frac{\partial^2(\nabla_j\phi\nabla_l\phi)}{\partial x^i\partial x^k} - \frac{\partial^2(\nabla_k\phi\nabla_i\phi)}{\partial x^j\partial x^l} - \frac{\partial^2(\nabla_i\phi\nabla_l\phi)}{\partial x^j\partial x^k}\right]\\
&+ 2g^{kp}g^{rq}g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}R_{ikjl} - 2\eta g^{kp}g^{lq}\nabla_p\phi\nabla_q\phi R_{ikjl}.\end{aligned}$$

Proof. We choose a fixed point $p \in M$ and the normal coordinates $\{x^1, \ldots, x^n\}$ around p. At point p, we have

$$\frac{\partial^2}{\partial t^2} R_{ij} = \frac{\partial^2}{\partial t^2}(g^{kl} R_{ikjl}) = g^{kl} \frac{\partial^2}{\partial t^2} R_{ikjl} + 2\frac{\partial g^{kl}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} + R_{ikjl}\frac{\partial^2 g^{kl}}{\partial t^2}.$$

Since $\frac{\partial g^{kl}}{\partial t} = -g^{kp} g^{lq} \frac{\partial g_{pq}}{\partial t}$ and $\frac{\partial^2 g^{kl}}{\partial t^2} = -g^{kp} g^{lq} \frac{\partial^2 g_{pq}}{\partial t^2} + 2g^{kp} g^{rq} g^{sl} \frac{\partial g_{pq}}{\partial t} \frac{\partial g_{rs}}{\partial t}$, then we get

$$\begin{aligned}\frac{\partial^2}{\partial t^2} R_{ij} = g^{kl} \frac{\partial^2}{\partial t^2} R_{ikjl} - 2g^{kp} g^{lq} \frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} - g^{kp} g^{lq} \frac{\partial^2 g_{pq}}{\partial t^2} R_{ikjl} \\ + 2g^{kp} g^{rq} g^{sl} \frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} R_{ikjl}. \qquad (21)\end{aligned}$$

Replacing (15) and (5) in (21) we have

$$\begin{aligned}\frac{\partial^2}{\partial t^2} R_{ij} = {} & \Delta R_{ij} + 2g^{kl}(B_{ikjl} - B_{iljk} - B_{ijlk} + B_{ijkl}) \\ & - g^{kl} g^{pq}(R_{ipkl} R_{qj} + R_{pjkl} R_{qi} + R_{ijkp} R_{ql} + R_{ijpl} R_{qk} + 2R_{ilqj} R_{kp}) \\ & + 2g^{kl} g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}.\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}.\frac{\partial}{\partial t}\Gamma^q_{ik}\Big) \qquad (22) \\ & - (n-2)\rho \nabla_i \nabla_j R - \rho \Delta R g_{ij} - 2g^{kp} g^{lq} \frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} \\ & - g^{kp} g^{lq}(-2R_{pq} + 2\rho R g_{pq}) R_{ikjl} + 2g^{kp} g^{rq} g^{sl} \frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} R_{ikjl} \\ & + \eta g^{kl}\left[\frac{\partial^2(\nabla_k\phi\nabla_j\phi)}{\partial x^i \partial x^l} - \frac{\partial^2(\nabla_j\phi\nabla_l\phi)}{\partial x^i \partial x^k} - \frac{\partial^2(\nabla_k\phi\nabla_i\phi)}{\partial x^j \partial x^l} - \frac{\partial^2(\nabla_i\phi\nabla_l\phi)}{\partial x^j \partial x^k}\right] \\ & - 2\eta g^{kp} g^{lq} \nabla_p\phi \nabla_q\phi R_{ikjl},\end{aligned}$$

where

$$2g^{kl}(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) = 2g^{kl}(B_{ikjl} - 2B_{iklj}) + 2g^{pr} g^{qs} R_{piqj} R_{rs},$$

and

$$\begin{aligned}& g^{kl} g^{pq}(R_{pjkl} R_{qi} + R_{ipkl} R_{qj} + R_{ijpl} R_{qk} + R_{ijkp} R_{ql} + 2R_{ilqj} R_{kp}) \\ & = 2g^{pq} R_{pi} R_{qj} + 2g^{pr} g^{qs} R_{piqj} R_{rs},\end{aligned}$$

but $g^{kl}(B_{ikjl} - 2B_{iklj}) = 0$, thus the result of theorem obtain by replace above Equations in (22). □

Since $R = g^{ij} R_{ij}$ we have

$$\frac{\partial^2}{\partial t^2} R = \frac{\partial^2}{\partial t^2}(g^{ij} R_{ij}) = g^{ij} \frac{\partial^2 R_{ij}}{\partial t^2} + 2\frac{\partial g^{ij}}{\partial t}\frac{\partial R_{ij}}{\partial t} + R_{ij}\frac{\partial^2 g^{ij}}{\partial t^2}.$$

Plugging (13) and (20) in last equation we get the following corollary.

Corollary 4.4. *Along the HRBH flow, the scalar curvature satisfies*

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}R &= (1-2(n-1)\rho)\Delta R + 2|Ric|^2 - 2\rho R^2 \\
&+ 2g^{ij}g^{kl}g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}\cdot\frac{\partial}{\partial t}\Gamma^q_{kj} - \frac{\partial}{\partial t}\Gamma^p_{kl}\cdot\frac{\partial}{\partial t}\Gamma^q_{ij}\Big) - 4\eta g^{ij}g^{kp}g^{lq}\nabla_p\phi\nabla_q\phi R_{ikjl} \\
&- 2g^{ij}g^{kp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} + 4g^{kp}g^{rq}g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}R_{kl} - 2g^{ip}g^{jq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ij}}{\partial t} \\
&+ \eta g^{ij}g^{kl}\left[\frac{\partial^2(\nabla_k\phi\nabla_j\phi)}{\partial x^i\partial x^l} - \frac{\partial^2(\nabla_j\phi\nabla_l\phi)}{\partial x^i\partial x^k} - \frac{\partial^2(\nabla_k\phi\nabla_i\phi)}{\partial x^j\partial x^l} - \frac{\partial^2(\nabla_i\phi\nabla_l\phi)}{\partial x^j\partial x^k}\right].
\end{aligned}$$

5. Examples

In following, we precent some examples of HRBH flow.

Example 5.1. Let $(M, g(0))$ be a round two-sphere of constant scalar curvature 2. Let $(N, h) = (M, g(0))$ and $\phi(0)$ be the identity map. Consider the HRBH flow and suppose $g(t) = c(t)g(0)$, $c(0) = 1$, $c'(0) = 0$ is a solution for it. For all $g(t)$, the map $\phi(t) = \phi(0)$ will be harmonic map. Therefore the HRBH flow on M becomes

$$\frac{\partial^2 c(t)}{\partial t^2} = -2 + 2\rho c(t) + 2\eta. \tag{23}$$

When $\rho = 0$, the Equation (23) has solution $c(t) = (-1+\eta)t^2 + 1$. For $\eta < 1$, $c(t)$ tends to zero in finite time. For $\eta = 1$ we have $c(t) = 1$ and for $\eta > 1$, $c(t)$ is increasing. Now, when $\rho > 0$ then the Equation (23) has solution

$$c(t) = \big(\frac{1}{2} - \frac{1}{2\rho}(1-\eta)\big)\big(e^{\sqrt{2\rho}t} + e^{-\sqrt{2\rho}t}\big) + \frac{1}{\rho}(1-\eta).$$

For $\rho < 0$ the solution to the Equation (23) is

$$c(t) = \big(1 - \frac{1}{\rho}(1-\eta)\big)\cos\sqrt{-2\rho}t + \frac{1}{\rho}(1-\eta).$$

Example 5.2. Suppose that $(M, g(0))$ is a compact Riemannian manifold, (N, h) $= (M, g(0))$ and $\phi(0)$ is the identity map. Let the initial metric $g_{ij}(x, 0)$ be Ricci flat, i.e. $R_{ij}(x, 0) = 0$, then $g_{ij}(x, t) = (\eta t^2 + t + 1)g_{ij}(x, 0)$ will be a solution to the HRBH flow with $\frac{\partial g}{\partial t}(x, 0) = g(x, 0)$. Thus any Ricci flat metric is a solution of the HRBH flow.

Example 5.3. Suppose that $(M, g(0))$ is a compact Einstein Riemannian manifold. Let $R_{ij}(0) = ag_{ij}(0)$ for some constant a, $(N, h) = (M, g(0))$, and $\phi(0)$ be the identity map. Consider the HRBH flow and suppose $g(t) = c(t)g(0)$, $c(0) =$

1, $c'(0) = 0$ is a solution for it, then $\phi(t) = \phi(0)$ is harmonic map for all $g(t)$. Definition of the Ricci tensor implies that

$$R_{ij}(t) = R_{ij}(0) = ag_{ij}(0), \quad R_{g(t)} = \frac{ma}{c}.$$

Hence, the HRBH flow on $(M, g(0))$ yields

$$\frac{\partial^2 c(t) g_{ij}(0))}{\partial t^2} = -2ag_{ij}(0) + 2\rho m a g_{ij}(0) + 2\eta g_{ij}(0),$$

this gives $\frac{d^2 c(t)}{\partial t^2} = -2a + 2\rho m a + 2\eta$. The solution of this ODE is as follows

$$\rho(t) = (-2\lambda + 2\rho m\lambda + 2\eta)t^2 + \nu t + 1.$$

Then the Einstein metrics preserves by the HRBH flow.

Confilcts of Interest. The author declares that there are no conflicts of interest regarding the publication of this article.

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Shahroud Azami
Department of Pure Mathematics,
Faculty of Science,
Imam Khomeini International University,
Qazvin, Iran
e-mail: azami@sci.ikiu.ac.ir