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Incomplete and Interior Inverse Problem for a Discontinuous Differential Pencil with the Spectral Boundary Condition on the Half-Line

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Abstract

Differential pencils on the half-line with a spectral boundary condition having a discontinuity in an interior point are investigated. We prove two uniqueness theorems: (i) knowing β_1, β_0 and potentials p, q on $(0, a)$, only eigenvalues suffice to determine the boundary value problem B. (ii) some information on eigenfunctions at $x = a$ and eigenvalues establish the boundary value problem B.

Keywords: inverse problem, differential pencil, spectral boundary condition, discontinuity

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1. Introduction

Let us consider the quadratic pencil of Sturm-Liouville operator

$$y''(x) + (\rho^2 + i\rho\, p(x) + q(x))y(x) = 0, \quad x \geq 0, \tag{1}$$

with the spectral boundary condition

$$U(y) := y'(0) + (\beta_1\rho + \beta_0)y(0) = 0, \tag{2}$$

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and discontinuous conditions

$$y(a+0,\rho)=a_1 y(a-0,\rho), \quad y'(a+0,\rho)=a_1^{-1}y'(a-0,\rho)+a_2 y(a-0,\rho), \quad (3)$$

at the point $x=a>0$. It is assumed that the numbers β_0, $\beta_1(\beta_1 \neq \pm i)$, $a_1(a_1 \neq \pm 1, \pm i)$ and a_2 are complex and the parameter ρ is spectral in the boundary value problem (1)-(3). The functions $q(x)$ and $p(x)$ (the absolutely continuous function $p(x)$) are complex and satisfy the condition $p(x), p'(x), q(x) \in L(0,\infty)$. The throughout of the paper, the boundary value problem (1)-(3) is denoted by $B := B(q,p,\beta_0,\beta_1)$.

Inverse spectral problems can show the phenomena in the physics and other natural sciences. Inverse problems are reconstructing of the problem from its spectral characteristics and cab be seen applications in sciences and engineering (see [1, 8, 13, 22, 25]). For example, Jaulent and Jean have also investigated inverse problems for differential pencils in the physical science [10].

In the general case, two spectra produced from two different boundary value problems are requirement to recover $q(x)$ completely [4, 14, 20]. It was seen in some works that providing $q(x)$ on the half of the interval a priori, this function can be uniquely given on the whole interval by one spectrum [12, 21, 28]. In [3, 22], incomplete inverse problem was also used for the Sturm-Liouville problems to recover $q(x)$ taking some of the spectrum assuming that $q(x)$ is prescribed on some of the segment. Also the interior inverse problem is a technique which has been firstly applied by Mochizuki and Trooshin to study the inverse problem [16]. They showed that one spectrum and some of values of eigenfunctions at some interior point are enough to find $q(x)$. This method has been taken in many articles in order to study the inverse Sturm-Liouville problems [18, 23, 24, 26]. For example, the research in [27] has been considered the inverse problem for the interior spectral data and has been shown that one spectra and some information on eigenfunctions at some interior point can establish the potential function. We are motivated to show two uniqueness theorems for the discontinuous diffusion operator with the boundary condition dependent on the spectrum on the half line that have not been considered yet. This problem has been studied by the spectral mappings method in [19]. In this article, firstly we investigate the incomplete inverse problem of B and recover it using eigenvalues assuming that potential functions are determined on some of the segment a priori. In addition, we prove the interior inverse problem for B and show that some information of eigenfunctions at $x=a$ and eigenvalues are sufficient to reconstruct the problem B. The technique employed in this paper is an extension of Hochstadt-Lieberman's and Mochizuki-Trooshin's theorems [9, 16].

The paper is organized as follows. In Section 2, some properties of spectral characteristics are given. In Section 3, taking Hochstadt-Lieberman's technique, a uniqueness theorem of the incomplete inverse problem is proved. Developing Mochizuki-Trooshin's method, we prove the interior inverse problem for B in Section 4.

2. Preliminaries

Let $\tau = \Im\rho$. Denote $\Pi_+ := \{\rho : \tau > 0\}$ and $\Pi := \overline{\Pi_+} \setminus \{0\}$. With helping the well known method [6, 15, 29]), we have the Jost-type solution of Equation (1) for uniformly in $x > a$ and sufficiently large $\rho \in \Pi$ with the following formula for $m = 0, 1$,

$$S^{(m)}(x,\rho) = (i\rho)^m \exp(i\rho x - A(x))\lceil 1\rfloor,$$

in which $A(x) = \frac{1}{2}\int_0^x p(t)dt$ and $\lceil 1\rfloor := 1 + O\left(\rho^{-1}\right)$.

The solutions $S^{(m)}(x,\rho)$, $m = 0, 1$, are analytic for $\rho \in \Pi_+$ and continuous for $\rho \in \overline{\Pi_+}$ as each fixed $x > a$. Also these solutions are continuously differentiable for $\rho \in \Pi$. Moreover, the function $S(x,\rho)$ satisfies the property

$$\lim_{x\to\infty} S(x,\rho) = 0, \quad \rho \in \Pi. \tag{4}$$

The properties of $S(x,\rho)$ remain true on $[0, a)$. By taking the Birkhoff-type solutions

$$y_k^{(m)}(x,\rho) = ((-1)^{k-1} i\rho)^m \exp((-1)^{k-1}(i\rho x - A(x)))\lceil 1\rfloor, \ \rho \in \Pi, \ \ x \geq 0, \tag{5}$$

as $k = 1, 2$ (see [17]), we will have for $x \in [0, a)$,

$$\begin{aligned} S^{(m)}(x,\rho) = {} & \frac{(i\rho)^m}{2a_1} \exp(i\rho a - A(a)) \\ & \times \Big((a_1^2 + 1) \exp\big(i\rho(x-a) - (A(x) - A(a))\big)\lceil 1\rfloor \\ & + (-1)^m (a_1^2 - 1) \exp\big(-i\rho(x-a) + (A(x) - A(a))\big)\lceil 1\rfloor \Big) \end{aligned} \tag{6}$$

(see [11] for more details).

The integral form of this solution is as follows

$$S(x,\rho) = \exp(i\rho x - A(x)) + \int_x^\infty \mathcal{H}_1(x,t)\exp(i\rho t)dt, \quad x \geq a, \tag{7}$$

and

$$\begin{aligned} S(x,\rho) = {} & \frac{1}{2a_1} \exp(i\rho a - A(a)) \\ & \times \Big((a_1^2 + 1) \exp\big(i\rho(x-a) - (A(x) - A(a))\big) \\ & + (a_1^2 - 1) \exp\big(-i\rho(x-a) + (A(x) - A(a))\big) \Big) \\ & + \int_0^x \mathcal{H}_2(x,t)\exp(i\rho t)dt + \int_0^x \mathcal{H}_3(x,t)\exp(i\rho(2a-t))dt, \quad x \in [0,a], \end{aligned} \tag{8}$$

where the kernels $\mathcal{H}_j(x,t)$, $j=1,2,3$ are bounded [2, 26, 27].

Let us consider the characteristic function of B as

$$\Delta(\rho) = S'(0,\rho) + (\beta_1\rho + \beta_0)S(0,\rho),$$

which is entire function in $\rho \in \Pi_+$ of exponential type. Therefore by using (6), we can give for sufficiently large $\rho \in \Pi$,

$$\Delta(\rho) = \frac{\rho}{2a_1}\exp(i\rho a - A(a))\Big((a_1^2+1)(\beta_1+i)\exp(-i\rho a + A(a))\lceil 1\rfloor \\ + (a_1^2-1)(\beta_1-i)\exp(i\rho a - A(a))\lceil 1\rfloor\Big). \tag{9}$$

Assume that $\delta > 0$ be fixed and $C > 0$ be a constant. Put $G_\delta := \{\rho \in \Pi_+; | \rho - \rho_n | \geq \delta,\ \forall n\}$. Taking (9) and the known technique [7], one gets

$$|\Delta(\rho)| \geq C|\rho|, \quad \rho \in G_\delta. \tag{10}$$

The eigenvalues ρ_n of B coincide with the roots of $\Delta(\rho)$ for $\rho \in \Pi_+$. By the Rouche's theorem [5] and the known technique [7, 29], one can give that the roots have the asymptotics

$$\rho_n = \frac{1}{a}\left(n\pi - iA(a) + \kappa_1 + \kappa_2\right) + O\left(n^{-1}\right), \tag{11}$$

for large enough n, wherein

$$\kappa_1 = \frac{1}{2i}ln\frac{a_1^2+1}{a_1^2-1}, \quad \kappa_2 = \frac{1}{2i}ln\frac{i+\beta_1}{i-\beta_1}.$$

Now here we express the following inverse problems.

Inverse Problem 1. Given the eigenvalues of $B := B(p,q,\beta_1,\beta_0)$, find p,q, assume that the parameters β_1, β_0 and the potentials p,q on $(0,a)$ are prescribed a priori.

Inverse Problem 2. Given the eigenfunctions at some interior point $x = a$ and eigenvalues of $B := B(p,q,\beta_1,\beta_0)$, find p,q and β_1.

To show the uniqueness theorems in the next sections, alongside $B := B(p,q,\beta_1,\beta_0)$, a boundary value problem $\widetilde{B} := B(\widetilde{p},\widetilde{q},\widetilde{\beta}_1,\beta_0)$ like (1)-(3) is taken. We accept that if α denotes an item relevant to B, then $\widetilde{\alpha}$ will denote an analogous item related to $\widetilde{B}$.

The eigenvalues and eigenfunctions of B is considered by $\lambda_n = \rho_n^2$ and $y_n(x) = y(x,\rho_n)$ as $\rho_n \in \Lambda_+ = \{\rho \in \Pi_+; \Delta(\rho) = 0\}$, respectively.

3. Incomplete Inverse Problem

In this section, we investigate the so-called incomplete inverse problem for B which consists in recovering the operator and the coefficients of the boundary condition from its eigenvalues provided that the boundary condition and the potentials p, q on $(0, a)$ are determined a priori. We will show the uniqueness theorem by the Hochstadt and Lieberman's method that is based on ideas in [9].

Theorem 3.1. *Let* $\beta_1 = \widetilde{\beta}_1,\ \beta_0 = \widetilde{\beta}_0$, $p(x) = \widetilde{p}(x)$ *and* $q(x) = \widetilde{q}(x)$ *a.e. on* $(0, a)$. *Then the set of the eigenvalues uniquely establishes* $p(x)$ *and* $q(x)$ *a.e. on* $x \geq 0$.

Proof. We assume that $S(x, \rho)$ be the solution of the following pencil

$$y''(x) + (\rho^2 + i\rho\ p(x) + q(x))y(x) = 0, \quad x \geq 0, \tag{12}$$

and $\widetilde{S}(x, \rho)$ be the solution of the secondary pencil

$$\widetilde{y}''(x) + (\rho^2 + i\rho\ \widetilde{p}(x) + \widetilde{q}(x))\widetilde{y}(x) = 0, \quad x \geq 0. \tag{13}$$

Multiplying (12) by $\widetilde{y}(x, \rho)$ and (13) by $y(x, \rho)$ and subtracting, we can write

$$\big(q(x) - \widetilde{q}(x) + i\rho(p(x) - \widetilde{p}(x))\big)y(x)\widetilde{y}(x) = y'(x)\widetilde{y}(x) - y(x)\widetilde{y}'(x). \tag{14}$$

Now by integrating the above equation on $[0, \infty)$, one gets

$$\int_0^\infty \big(Q(x) + i\rho P(x)\big)y(x)\widetilde{y}(x)dx = (y'(x)\widetilde{y}(x) - y(x)\widetilde{y}'(x))|_0^a + |_a^\infty,$$

where $Q(x) = q(x) - \widetilde{q}(x)$ and $P(x) = p(x) - \widetilde{p}(x)$. Taking the hypothesis of the theorem and (4), we infer that

$$H_a(\rho) := \int_a^\infty \big(Q(x) + i\rho P(x)\big)y(x)\widetilde{y}(x)dx = y(0)\widetilde{y}'(0) - y'(0)\widetilde{y}(0). \tag{15}$$

Therefore

$$H_a(\rho) = y(0)\widetilde{y}'(0) - y'(0)\widetilde{y}(0).$$

Considering the hypothesis of the theorem again, one gets $H_a(\rho_n) = 0$ as $\rho_n \in \Lambda_+$. Thus we shall prove $H_a(\rho) = 0$ as $\rho \in \Pi_+ \setminus \Lambda_+$.

From (7), knowing

$$\begin{aligned} S(x, \rho)\widetilde{S}(x, \rho) &= \exp(2i\rho x - A_+(x)) \\ &\quad + \int_x^\infty \mathbf{H}_1(x, t)\exp(2i\rho t - A_+(t))dt, \qquad x \geq a, \end{aligned} \tag{16}$$

where $A_+(x) = A(x) + \widetilde{A}(x)$, one gets

$$|S(x, \rho)\widetilde{S}(x, \rho)| \leq M_1 \exp(-2|\Im\rho|x), \quad x \geq a,$$

for some constant $M_1 > 0$. So, together with (15), this yields that

$$|H_a(\rho)| \leq \mathbf{M}|\rho|\exp(-2\Im\rho a). \tag{17}$$

Putting

$$G(\rho)=\frac{H_a(\rho)}{\Delta(\rho)},$$

that is holomorphic in $\Pi_+\setminus\Lambda_+$ and applying (10) and (17), one gets $G(\rho)=0$ and therefore $H_a(\rho)=0$.

With respect to this result, we obtain

$$\int_a^\infty \big(Q(x)+i\rho P(x)\big)y(x)\widetilde{y}(x)dx=0. \tag{18}$$

Substituting (16) in (18) and then rewriting it, we have for $t>a$,

$$\int_a^\infty \exp(2i\rho t-A_+(t))\left(Q(t)+\int_a^t Q(x)\mathbf{H}_1(x,t)dx\right)dt$$
$$+i\rho\int_a^\infty \exp(2i\rho t-A_+(t))\left(P(t)+\int_a^t P(x)\mathbf{H}_1(x,t)dx\right)dt=0.$$

Applying Riemann-Lebesgue lemma, one gives that for sufficiently large ρ,

$$\int_a^\infty \exp(2i\rho t-A_+(t))\left(Q(t)+\int_a^t Q(x)\mathbf{H}_1(x,t)dx\right)dt=0,$$
$$\int_a^\infty \exp(2i\rho t-A_+(t))\left(P(t)+\int_a^t P(x)\mathbf{H}_1(x,t)dx\right)dt=0.$$

From the completeness of "exp" [7, 28], we can write that

$$Q(t)+\int_a^t Q(x)\mathbf{H}_1(x,t)dx=0,$$
$$P(t)+\int_a^t P(x)\mathbf{H}_1(x,t)dx=0,$$

which are homogeneous integral equations of Volterra type with trivial solutions, i.e., $Q(x)=P(x)=0$, for $x>a$. Thus we result that $p(x)=\widetilde{p}(x)$ and $q(x)=\widetilde{q}(x)$ a.e. in $x>a$. The proof is completed. □

4. Interior Inverse Problem

In this section, we give the interior inverse problem for B which recovers the boundary value problem B from virtues of eigenfunctions on $x=a$ and eigenvalues. We will show the uniqueness theorem by the method of Mochizuki and Trooshin that is based on ideas in [16].

Theorem 4.1. *Suppose that* $\lambda_n = \widetilde{\lambda}_n$ *and for any* n*, the Wronskian* $< y_n, \widetilde{y}_n >$ *is zero in* $x = a - 0$*. Thus the potentials* $p(x), q(x)$ *a.e. on* $x \geq 0$ *and the coefficient* β_1 *are uniquely determined.*

Remark 1. Consider $\theta(x)$ and $\Theta(x)$ as two solutions of B. Due to the equality of the Wronskian at the neighborhood of $x = a$, Theorem 4.1 can be expressed by the condition $< \theta, \Theta >= 0$ in $x = a + 0$.

Proof. Taking (14) and integrating on $[0, a)$, one gives

$$\int_0^a \big(Q(x) + i\rho P(x)\big)y(x)\widetilde{y}(x)dx = \big(y'(x)\widetilde{y}(x) - y(x)\widetilde{y}'(x)\big)|_0^a.$$

So

$$\begin{aligned} H_0(\rho) := &\int_0^a \big(Q(x) + i\rho P(x)\big)y(x)\widetilde{y}(x)dx \\ &- i\rho(a_1^2 - a_1^{-2})\sinh A_-(a)\exp(2i\rho a - A_+(a)) \\ &= (y'(x)\widetilde{y}(x) - y(x)\widetilde{y}'(x))_{|x=a}, \end{aligned} \tag{19}$$

wherein $A_-(x) = A(x) - \widetilde{A}(x)$. On the base of the hypothesis of the theorem, we result that

$$H_0(\rho_n) = y_n'(a)\widetilde{y}_n(a) - y_n(a)\widetilde{y}_n'(a) = 0, \quad \rho_n \in \Lambda_+.$$

Now we have to prove that $H_0(\rho) = 0$ as other $\rho \in \Pi_+$.

From (8), having

$$\begin{aligned} S(x,\rho)\widetilde{S}(x,\rho) = &\frac{1}{4a_1^2}\exp(2i\rho a - A_+(a)) \\ &\times \Big((a_1^2+1)^2\exp(2i\rho(x-a) - (A_+(x) - A_+(a))) \\ &+ (a_1^2-1)^2\exp(-2i\rho(x-a) + (A_+(x) - A_+(a))) \\ &+ 2(a_1^4 - 1)\cosh(A_-(x) - A_-(a))\Big) \\ &+ \int_0^x \mathbf{H}_2(x,t)\exp(2i\rho t - A_+(t))dt \\ &+ \int_0^x \mathbf{H}_3(x,t)\exp(2i\rho(2a-t) - (2A_+(a) - A_+(t)))dt, \ x \in [0,a], \end{aligned} \tag{20}$$

can be obtained

$$|S(x,\rho)\widetilde{S}(x,\rho)| \leq M_2\exp(-2|\Im\rho|x), \quad x \in [0,a],$$

as a constant $M_2 > 0$.

Repeating the discussions done in the proof of Theorem 3.1, one gives that $H_0(\rho) = 0$ for all $\rho \in \Pi_+$.

Now taking this result and (19), we have

$$\int_0^a \big(Q(x) + i\rho P(x)\big)y(x)\widetilde{y}(x)dx - i\rho(a_1^2 - a_1^{-2})\sinh A_-(a)\exp(2i\rho a - A_+(a)) = 0. \tag{21}$$

Substituting (20) in (21) and then rewriting it, we infer that

$$\begin{aligned}
&i\rho(a_1^{-2} - a_1^2)\sinh A_-(a)\exp(2i\rho a - A_+(a)) \\
&+ \int_0^a \frac{2(a_1^4 - 1)}{4a_1^2}\exp(2i\rho a - A_+(a))\cosh(A_-(t) - A_-(a))(Q(t) + i\rho P(t))dt \\
&+ \int_0^a \frac{(a_1^2 + 1)^2}{4a_1^2}\exp(2i\rho t - A_+(t))(\mathcal{Q}_2(t) + i\rho\mathcal{P}_2(t))dt \\
&+ \int_0^a \frac{(a_1^2 - 1)^2}{4a_1^2}\exp(2i\rho(2a - t) - (2A_+(a) - A_+(t)))(\mathcal{Q}_3(t) + i\rho\mathcal{P}_3(t))dt = 0,
\end{aligned}$$

where $\mathcal{Q}_2 = Q(t) + \int_t^a Q(x)\mathbf{H}_2(x,t)dx$, $\mathcal{P}_2 = P(t) + \int_t^a P(x)\mathbf{H}_2(x,t)dx$, $\mathcal{Q}_3 = Q(t) + \int_t^a Q(x)\mathbf{H}_3(x,t)dx$, and $\mathcal{P}_3 = P(t) + \int_t^a P(x)\mathbf{H}_3(x,t)dx$. Using Riemann-Lebesgue lemma for continuous functions, we can get as large enough ρ,

$$\begin{aligned}
&\int_0^a \exp(2i\rho t - A_+(t))\mathcal{Q}_2(t)dt = 0, \\
&\int_0^a \exp(2i\rho t - A_+(t))\mathcal{P}_2(t)dt = 0, \\
&\int_0^a \exp(2i\rho(2a - t) - (2A_+(a) - A_+(t)))\mathcal{Q}_3(t)dt = 0, \\
&\int_0^a \exp(2i\rho(2a - t) - (2A_+(a) - A_+(t)))\mathcal{P}_3(t)dt = 0.
\end{aligned}$$

From the completeness of "exp" [7, 28], we can write for $0 \le t < a$,

$$\mathcal{Q}_\nu(t) = \mathcal{P}_\nu(t) = 0, \quad \nu = 2, 3.$$

So the homogeneous integral equations of Volterra type

$$Q(t) + \int_t^a Q(x)\mathbf{H}_\nu(x,t)dx = 0, \qquad P(t) + \int_t^a P(x)\mathbf{H}_\nu(x,t)dx = 0,$$

have only trivial solutions. Therefore $Q(x) = P(x) = 0$ in $[0, a]$. Thus $p(x) = \widetilde{p}(x)$ and $q(x) = \widetilde{q}(x)$ a.e. in $[0, a]$.

By regarding to Remark 1, we can show that for $\rho_n \in \Lambda_+$,

$$H_a(\rho_n) = y_n(a)\widetilde{y}_n'(a) - y_n'(a)\widetilde{y}_n(a) = 0.$$

Similar to the pervious section, one can show that $H_a(\rho) = 0$ as $\rho \in \Pi_+ \setminus \Lambda_+$. So again, like the previous section, we will have $q(x) = \widetilde{q}(x)$ and $p(x) = \widetilde{p}(x)$ a.e. in $[a, \infty)$.

Therefore $q(x) = \widetilde{q}(x)$ and $p(x) = \widetilde{p}(x)$ a.e. in $x \geq 0$. A direct computation and (11) give that $\beta_1 = \widetilde{\beta}_1$. The proof is completed. □

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