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Existence Solution of a Biharmonic-Type Kirchhoff-Schrödinger-Maxwell System

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Abstract

This article addresses the following biharmonic type of the Kirchhoff-Schrödinger-Maxwell system;

$$\begin{cases} \Delta^2 w - (a_1 + b_1 \int\limits_{\mathbb{R}^N} |\nabla w|^2)\Delta w + \eta\psi w = q(w) & \text{in } \mathbb{R}^N, \\ -\Delta\psi = \eta w^2 & \text{in } \mathbb{R}^N, \end{cases} \qquad \text{(bKSM)}$$

in which a_1, b_1 and η are fixed positive numbers and q is a continuous real valued function in $\mathbb{R}$. We are going to prove the existence solution for this system via variational methods, delicate cut-off technique and Pohozaev identity.

Keywords: Biharmonic equations, Kirchhoff-Schrödinger-Maxwell, cut-off function, Pohozaev identity.

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1. Introduction

This note deals with the following system of nonlinear Kirchhoff-Schrödinger-Maxwell equations of biharmonic type

$$\begin{cases} \Delta^2 w - (a_1 + b_1 \int\limits_{\mathbb{R}^N} |\nabla w|^2)\Delta w + \eta\psi w = q(w) & \text{in } \mathbb{R}^N, \\ -\Delta\psi = \eta w^2 & \text{in } \mathbb{R}^N, \end{cases} \tag{bKSM}$$

where a_1, b_1 and η are fixed positive real numbers, $N \geq 5$ and the operator $\Delta^2 w = \Delta(\Delta w)$ is biharmonic operator. We take that into account the nonlinear term q has the following properties:

(q_1) q is a real valued continuous map on $\mathbb{R}$.

(q_2) $-\infty < \lim\limits_{s\to 0^+} \inf \frac{q(s)}{s} \leq \lim\limits_{s\to 0^+} \sup \frac{q(s)}{s} = -m < 0.$

(q_3) But at infinity $-\infty \leq \lim\limits_{s\to +\infty} \sup \frac{q(s)}{s^{2^*-1}} \leq 0.$

(q_4) For a suitable $\xi > 0$, $Q(\xi) := \int_{\cdot}^{\xi} q(s)ds > 0.$

Note that this type of properties for data q during the study of a nonlinear scalar field was first introduced by Berestyci and Lions [9]. using these properties and minimizing arguments, they showed the existence of a ground state solution $w \in H^1(\mathbb{R}^N)$ for

$$-\Delta w = q(w),$$

where $N \geq 3$.

Regardless of the first term and when $\eta = 0$ and if $\mathbb{R}^N$ is changes to a smooth bounded domain Ω, then the system of equations (bKSM) would reduced to the following equation of Kirchhoff type

$$-(a_1 + b_1 \int\limits_{\Omega} |\nabla w|^2)\Delta w = q(w), \quad x \in \Omega.$$

This equation is relevant to the changeless analogue of

$$\rho \frac{\partial^2 w}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial w}{\partial x} \right|^2 \mathrm{d}x \right) \frac{\partial^2 w}{\partial x^2} = 0,$$

which was studied by Kirchhoff in [17], for the case that ρ, ρ_0, h, E and L are real constants, which it expands the classical D'Alembert's wave equation for free vibrations of elastic strings. In this model, it is considered the changes in length of the string, where h is area of cross-section, E is the Young modules of the material, ρ and ρ_0 are represented the mass density and the primary traction respectively.

For example, from the application of Kirchhoff law, we can refer to the celebrated paper by C. Tunç and e.al [14] which represented the discrete nonlinear transmission is given through the modified Zakharov-Kuznetsov equation that is expressed by Duan when he implemented the Kirchhoff law. Biharmonic equations have numerous applications in solid and fluid mechanics, but they are hard to solve due to the presence of fourth-order derivative terms, especially in complicated geometries. Generally biharmonic equations are some kind of fourth-order partial differential equations which arises in several branches of sciences for example in the study of travelling wave, Stokes flows, static deflection of an elastic plate in a fluid, vertical wavy wall, in suspension bridge and electrical engineering. Moreover, it is a mathematical model of some thin structures that react elastically to external forces(see [1, 3]). C. Tunç and et al. [13] studied about flow features and heat transport inside enclosure which its principal objective is to examine the impacts of Grashof number, aspect ratio, waviness on the thermal variation, and flow stream of free convection inside a wavy compartment covered with Newtonian fluid. F. Wang and et al. [20] studied the following biharmonic elliptic equations:

$$\begin{cases} \Delta^2 w - \lambda(a + b\int\limits_{\Omega} |\nabla w|^2)\Delta w = f(x,w) & \text{in } \Omega, \\ w = 0, \quad \Delta w = 0, & \text{on } \partial\Omega, \end{cases}$$

where λ is a positive parameter, a, b are positive constants, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and the function f has the properties of a locally Lipschitz continuous. They showed existence of a non-trivial solution for this system via the mountain pass techniques and trunction method.

Regardless of the first sentence and the case that $a_1 = 1$, $b_1 = 0$ and $N = 3$, (bKSM) reduced to the following known as the nonlinear electrostatic Schrödinger-Poisson system

$$\begin{cases} -\Delta u + q\psi w = g(u) & \text{in } \mathbb{R}^3, \\ -\Delta\psi = qu^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{SM}$$

where

$$g(u) = -u + |u|^{p-1}u.$$

(see [2, 5, 7]). Moreover, the linear case and asymptotic linear form of (SM) have been investigated in [11, 12] but for a bounded domain and Neumann conditions.

A. Azzollini and e.al studied (SM) but for the same general hypothesis introduced by Berestycki and Lions [6]. Under motivated by Azzollini celebrated article we would show the existence of at least a nontrivial solution for (bKSM) by applying Pohozaev manifold combined with the monotone trick of L. Jeanjean [15] (see Section 4).

Main result of this article under review and writing is as follows:

Theorem 1.1. *If the typotheses $(q_1) - (q_4)$ are correct, then there is a constant $\eta_0 > 0$ such that for any $0 < \eta < \eta_0$, (bKSM) possesses a non-trivial positive radial solution (w, ψ) in $H^2(\mathbb{R}^N) \times D^{2,2}(\mathbb{R}^N)$.*

In order to reach our conclusion, we have to overcome two important problems. First one is that we do not propose some usual conditions, namely $\frac{q(s)}{s^{2^*-1}}$ is increasing in $(0, +\infty)$ and existence of $\nu > 2$ such that $0 < \nu Q(s) \leq q(s)s$ for all $s \in \mathbb{R}$ (Ambrosetti-Rabinowitz type condition) which would lead us in the discussion of geometric assumptions to standard mountain pass arguments for its corresponding energy functional and improving the boundedness of its Palais-Smale sequence ((PS) for short). Second one is that the effect of the competition of the two nonlocal terms with the nonlinear item q makes some difficulties, for instance, nonlocal expressions $(\int\limits_{\mathbb{R}^N} |\nabla w|^2)^2$ and $\int\limits_{\mathbb{R}^N} \psi_w w^2$ are too obstacles for using variational method, so by using the cut-off function we can achieve the goal, where cut-off function is a technique inspired by L. Jeanjean [15] for controlling the nonlocal nonlinear term $\int\limits_{\mathbb{R}^N} \psi_w w^2$.

The organization of the contents of this article is as follows. In Section 3.1, we mention some of the concepts that are used to prove the main result. In Section , some functional frameworks by a variational approach are given. Section 2 deal with a sequence the modified functional that provide us situation to use the mountain pass theorem. In Section 3 we will prove the existence of radial solutions for problem (bKSM). Section 4 contains proof of Pohozaev identity as a powerful and fundamental tool in our arguments. In Section 5 we will prove main Theorem 1.1. Finally, Section 6 is devoted to the conclusion.

2. Preliminaries

We briefly describe the natural framework to treat solution for the problem. In the paper it is understood that all functions, unless otherwise stated, are real-valued, but for simplicity we write $L^s(\mathbb{R}^N)$, $H^2(\mathbb{R}^N)$, ..., and for

- Given any $1 \leq s \leq +\infty$, $\|\cdot\|_s$ is notation of norm for the Lebesgue space $L^s(\mathbb{R}^N)$;
- $H^2(\mathbb{R}^N)$ is stand for usual Sobolev space endowed with the following norm:
$$\|w\|^2 := \int\limits_{\mathbb{R}^N} |\Delta w|^2 + a_1 \int\limits_{\mathbb{R}^N} |\nabla w|^2 + \int\limits_{\mathbb{R}^N} w^2.$$
- $D^{2,2}(\mathbb{R}^N)$ is completion of $C_0^\infty(\mathbb{R}^N)$ by the following norm
$$\|w\|^2_{D^{2,2}(\mathbb{R}^N)} := \int\limits_{\mathbb{R}^N} |\Delta w|^2 + a_1 \int\limits_{\mathbb{R}^N} |\nabla w|^2.$$

It is easy to understand that system (bKSM) can be changed to a single equation. In fact, for all w in $H^2(\mathbb{R}^N)$, considering on $D^{2,2}(\mathbb{R}^N)$ by
$$L_w(\upsilon) := \int\limits_{\mathbb{R}^N} w^2 \upsilon.$$

Then by applying the well- known Hölder inequality and the applied theorem of Sobolev embedding, we deduce the following relation

$$|L_w(\upsilon)| \leq \left(\int\limits_{\mathbb{R}^N} (w^2)^{\frac{2^*}{2^*-1}} \right)^{\frac{2^*-1}{2^*}} \left(\int\limits_{\mathbb{R}^N} |\upsilon|^{2^*} \right)^{\frac{1}{2^*}} \leq C|w|^2_{\frac{22^*}{2^*-1}} \|\upsilon\|_D,$$

for all $\upsilon \in D^{2,2}(\mathbb{R}^{\mathbb{N}})$.

Thanks to the Lax-Milgram theorem, there is only one ψ_w in $D^{2,2}(\mathbb{R}^N)$ such that

$$\int\limits_{\mathbb{R}^N} \nabla\psi_w \nabla\upsilon = \eta \int\limits_{\mathbb{R}^N} w^2\upsilon, \quad \text{for any } \upsilon \text{ in } D^{2,2}(\mathbb{R}^N).$$

In fact, ψ_w satisfies as a solution in the Maxwell equation

$$-\Delta\psi = \eta w^2 \ \text{ in } \mathbb{R}^N, \tag{1}$$

in a weak sense. On the other hand, (1) has the following form:

$$\psi_w(x) = \frac{\eta}{4\pi} \int\limits_{\mathbb{R}^N} \frac{w^2(y)}{|x-y|} dy.$$

- Throught this note we introduced B_R as the open ball with radius R and centered at origin and also ∂B_R denotes it's boundary,
- For simplicity, we set $\alpha := \frac{22^*}{2^*-1}$.

3. Functional Setting

Before starting the details about proving the Theorem 1.1, we state the following famous facts that will be essential later to achieve the main result (see, for instance [7, 8, 4, 18]).

Lemma 3.1. *Given any w belongs to $H^2(\mathbb{R}^N)$, The following relationships are hold:*

(i) $\psi_w \geq 0$ *and* $\|\psi_w\|^2_{D^{2,2}(\mathbb{R}^N)} = \eta \int\limits_{\mathbb{R}^N} \psi_w w^2$.

(ii) *For any $\theta > 0$, $\psi_{w\theta}(x) = \theta^2\psi_w(\frac{x}{\theta})$, where $w_\theta(x) = w(\frac{x}{\theta})$.*

(iii) *There exist $c, c' > 0$ independent of any choice $w \in H^2(\mathbb{R}^N)$ such that*

$$\|\psi_w\|_{D^{2,2}(\mathbb{R}^N)} \leq c\eta\|w\|^2_\alpha,$$

$$\text{and} \int_{\mathbb{R}^N} \psi_w w^2 \leq c'\eta \|w\|_\alpha^4.$$

(iv) *ψ_w is a radial function if w is radial.*

(v) *Converging weakly a sequence (w_n) to w in $H_r^2(\mathbb{R}^N)$ implies that converging of ψ_{w_n} to ψ_w in $D^{2,2}(\mathbb{R}^N)$ and converging of $\int_{\mathbb{R}^N} \psi_{w_n} w_n^2$ to $\int_{\mathbb{R}^N} \psi_w w^2$.*

Inspired by [9], for given ξ, set $s_0 := +\infty$ if $q(s) \neq 0$ and if $s \geq \xi$ then $s_0 := \min\{s \in [\xi, +\infty) | q(s) = 0\}$. Considering $\tilde{q} : \mathbb{R} \to \mathbb{R}$ in which

$$\tilde{q}(s) = \begin{cases} q(s) & [0, s_0], \\ 0 & \mathbb{R}_+ \backslash [0, s_0], \\ (q(-s) - ms)^+ - q(-s) & \mathbb{R}_-, \end{cases}$$

where m is a constant given by hypothesis (q_2). It is direct to verify that $\tilde{q}$ holds too in conditions mentioned for q.

Remark 1. Consider non-trivial solution $w \in H^2(\mathbb{R}^N)$ of (bKSM) but with $\tilde{q}$ instead of q. From the maximum principle and Lemma 3.1(i) we may assume that $w > 0$ and so is an upper bound for w, which means precisely that w is in fact a solution for (bKSM) but for q.

As already observed in Remark 1, we may assume $\tilde{q}$ instead of q but for simplicity we denote again by q. This substitution implies that q satisfies the stronger condition

$$\lim_{s \longrightarrow \pm\infty} \frac{q(s)}{s^{2^*-1}} = 0. \tag{2}$$

Furthermore, for $s \geq 0$, define

$$q_1(s) := \begin{cases} 0 & \text{if } s < 0, \\ (q(s) + ms)^+ & \text{if } s \geq 0. \end{cases}$$

$$q_2(s) := q_1(s) - q(s), \qquad \text{if } s \in \mathbb{R}.$$

From the following limits

$$\lim_{s \longrightarrow 0} \frac{q_1(s)}{s} = 0,$$
$$\lim_{s \longrightarrow \pm\infty} \frac{q_1(s)}{s^{2^*-1}} = 0, \tag{3}$$

and since

$$q_2(s) \geq ms, \quad \forall\, s \geq 0, \tag{4}$$

for any $\epsilon > 0$ one can find $c_\epsilon > 0$ in which

$$q_1(s) \leq c_\epsilon s^{2^*-1} + \epsilon q_2(s), \quad \forall\, s \geq 0. \tag{5}$$

Set

$$Q_i(t) := \int_0^t q_i(s)ds, \quad i = 1, 2\ ,$$

then by (4) and (5),

$$Q_2(s) \geq \frac{m}{2}s^2, \quad \forall\, s \in \mathbb{R}, \tag{6}$$

and

$$Q_1(s) \leq \frac{c_\epsilon}{2^*}s^{2^*} + \epsilon Q_2(s), \quad \forall\, s \in \mathbb{R}. \tag{7}$$

It is clear that the solutions $(w, \psi) \in H^2(\mathbb{R}^N) \times D^{2,2}(\mathbb{R}^N)$ of (bKSM) are the critical points for the following functional $\varepsilon_\eta : H^2(\mathbb{R}^N) \times D^{2,2}(\mathbb{R}^N) \to \mathbb{R}$, defined as

$$\begin{aligned}\varepsilon_\eta(w, \psi) := &\frac{1}{2}\int_{\mathbb{R}^N} |\Delta w|^2 + \frac{a_1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 \\ &+ \frac{b_1}{4}(\int_{\mathbb{R}^N} |\nabla w|^2)^2 - \frac{1}{4}\int_{\mathbb{R}^N} |\nabla \psi|^2 + \frac{\eta}{2}\int_{\mathbb{R}^N} \psi w^2 - \int_{\mathbb{R}^N} Q(w).\end{aligned}$$

It is clear to see that ε_η is both from below and from above unbounded on infinite dimensional subspace. We can change this indefiniteness behavior by using the reduction method [7, 8] which it lead us to one variable functional.

Multiplying equation $-\Delta\psi_w = \eta w^2$ by ψ_w and using integration by parts we deduce

$$\int_{\mathbb{R}^N} |\nabla \psi_w|^2 dx = \eta \int_{\mathbb{R}^N} \psi_w w^2 dx.$$

Then, changes to the following reduced form

$$I_\eta(w) = \frac{1}{2}\int_{\mathbb{R}^N} |\Delta w|^2 + \frac{a_1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 + \frac{b_1}{4}(\int_{\mathbb{R}^N} |\nabla w|^2)^2 + \frac{\eta}{4}\int_{\mathbb{R}^N} \psi_w w^2 - \int_{\mathbb{R}^N} Q(w).$$

Therefore, it suffices to show that w is a solution of the desired function I_η on $H_r^2(\mathbb{R}^N)$, that is a critical point that leads us to the intended destination.

4. The Perturbed Functional

As previously mentioned in the beginning, because of the existence of two expressions $(\int_{\mathbb{R}^N} |\nabla w|^2)^2$ and $\int_{\mathbb{R}^N} \psi_w w^2$ in I_η, in order to employ the mountain-pass geometry and the boundedness of (PS) sequences for the functional I_η, following [16] we consider the cut-off function χ in $C^\infty(\mathbb{R}_+, \mathbb{R})$ which satisfies in the following conditions:

$$\begin{cases} \chi(s) = 1, & \text{for } s \in [0,1], \\ 0 \le \chi(s) \le 1, & \text{for } s \in (1,2), \\ \chi(s) = 0, & \text{for } s \in [2,+\infty), \\ \|\chi'\|_\infty \le 2. \end{cases} \tag{8}$$

Define

$$K_T(w) := \chi\left(\frac{\|w\|_\alpha^\alpha}{T^\alpha}\right),$$

where $T > 0$ and α was introduced in previous section. Here, we consider the following modified form of I_η as functional $I_\eta^T : H_r^2(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I_\eta^T(w) = \frac{1}{2}\int\limits_{\mathbb{R}^N} |\Delta w|^2 + \frac{a_1}{2}\int\limits_{\mathbb{R}^N} |\nabla w|^2 + \frac{b_1}{4}(\int\limits_{\mathbb{R}^N} |\nabla w|^2)^2 + \frac{\eta}{4}K_T(w)\int\limits_{\mathbb{R}^N} \psi_w w^2 - \int\limits_{\mathbb{R}^N} Q(w).$$

It is direct to see that any critical point w of I_η^T in which $\|w\|_\alpha \le T$ is a critical point of I_η too. Since, q does not satisfy in the (AR) type condition, so we may not able to obtain directly that the (PS) sequences are bounded. To preponderate of this difficultly, we recall and use the following result [15] of L. Jeanjean.

Theorem 4.1. *Assume that X is a Banach space equipped with a norm $\|\cdot\|_X$ and $J \subset \mathbb{R}^+$ is an given interval. And $\{I_\tau\}_{\tau\in J}$ is a family of C^1-functionals on X of the form*

$$I_\tau(w) = A(w) - \tau B(w), \quad \forall\, \tau \in J,$$

in which $B(w)$ is nonnegative for all w in X, $I_\tau(0) = 0$ and either $A(w) \to +\infty$ or $B(w) \to +\infty$ when $\|w\|_X \to \infty$. For any $\tau \in J$, set

$$\Sigma_\tau := \{\sigma \in C([0,1], X) | I_\tau(\sigma(1)) < 0, \sigma(0) = 0\}. \tag{9}$$

If for any τ belongs to J, Σ_τ is a nonempty set and

$$c_\tau := \inf_{\sigma\in\Sigma_\tau} \max_{t\in[0,1]} I_\tau(\sigma(t)),$$

is positive, then for almost every $\tau \in J$ there exists $\{\upsilon_n\}_n \subset X$ satisfying in the following:

(i) *$\{\upsilon_n\}$ is a bounded sequence*;

(ii) *$I_\tau(\upsilon_n)$ approaches to c_τ as $n \to \infty$;*

(iii) *$I_\tau'(\upsilon_n)$ approaches to 0 in X' the dual of X as $n \to \infty$.*

For space $X := H_r^2(\mathbb{R}^N)$, we consider the following functionals:

$$A(w) :=\frac{1}{2}\int\limits_{\mathbb{R}^N} |\Delta w|^2 + \frac{a_1}{2}\int\limits_{\mathbb{R}^N} |\nabla w|^2 + \frac{b_1}{4}(\int\limits_{\mathbb{R}^N} |\nabla w|^2)^2 + \frac{\eta}{4}K_T(w)\int\limits_{\mathbb{R}^N} \psi_w w^2 + \int\limits_{\mathbb{R}^N} Q_2(w),$$

$$B(w) := \int\limits_{\mathbb{R}^N} Q_1(w).$$

One can directly see that, $A(w)$ approaches to $+\infty$ as $\|w\| \to +\infty$ and $B(w) \geq 0$, for any $w \in H_r^2(\mathbb{R}^N)$.

The perturbed functional is defined as follows

$$I_{\eta,\tau}^T(w) = \frac{1}{2}\int\limits_{\mathbb{R}^N} |\Delta w|^2 + \frac{a_1}{2}\int\limits_{\mathbb{R}^N} |\nabla w|^2 + \frac{b_1}{4}(\int\limits_{\mathbb{R}^N} |\nabla w|^2)^2 + \frac{\eta}{4}K_T(w)\int\limits_{\mathbb{R}^N} \psi_w w^2 + \int\limits_{\mathbb{R}^N} Q_2(w) - \tau\int\limits_{\mathbb{R}^N} Q_1(w).$$

It can be seen easily and without difficulty $I_{\eta,\tau}^T \in C^1(H^2(\mathbb{R}^N), \mathbb{R})$ and for any $w, \upsilon \in H^2(\mathbb{R}^N)$, by the above functional and using Gateaux derivative, i.e., by replacing $w + t\upsilon$ instead of w and by taking the derivative of t and putting $t = 0$, we obtain

$$\langle (I_{\eta,\tau}^T)'(w), \upsilon\rangle = \int\limits_{\mathbb{R}^N} \Delta w.\Delta \upsilon + a_1\int\limits_{\mathbb{R}^N} \nabla w.\nabla \upsilon + b_1\int\limits_{\mathbb{R}^N} |\nabla w|^2\int\limits_{\mathbb{R}^N} \nabla w.\nabla \upsilon + \eta K_T(w)\int\limits_{\mathbb{R}^N} \psi_w w\upsilon + \frac{\eta\alpha}{4T^\alpha}\chi'\left(\frac{\|w\|_\alpha^\alpha}{T^\alpha}\right)\int\limits_{\mathbb{R}^N} \psi_w w^2\int\limits_{\mathbb{R}^N} |w|^{\alpha-2}w\upsilon + \int\limits_{\mathbb{R}^N} q_2(w)\upsilon - \tau\int\limits_{\mathbb{R}^N} q_1(w)\upsilon.$$

In order to exploit Theorem 1.1, we require an appropriate interval J, in fact $\Sigma_\tau \neq \psi$, for any $\tau \in J$ and (9) holds.

Based on [9], and (q_4), there exists a function z in $H_r^2(\mathbb{R}^N)$ such that the following relation

$$\int\limits_{\mathbb{R}^N} Q_1(z) - \int\limits_{\mathbb{R}^N} Q_2(z) = \int\limits_{\mathbb{R}^N} Q(z) > 0,$$

holds.

Then there is $0 < \bar{\delta} < 1$ in which

$$\bar{\delta}\int\limits_{\mathbb{R}^N} Q_1(z) - \int\limits_{\mathbb{R}^N} Q_2(z) > 0. \tag{10}$$

We consider the interval $J := [\bar{\delta}, 1]$.

The following Lemma ensures that $I^T_{\eta,\tau}$ satisfies in conditions of the mountain-pass geometry.

Lemma 4.2. *Suppose that conditions $(q_1) - (q_4)$ hold. Then for any $\tau \in \left[\bar{\delta}, 1\right]$,*

(i) *There are two constants $\rho, \tilde{c} > 0$ in which $I^T_{\eta,\tau}(w) \geq \tilde{c} > 0$ for all w in $H^2_r(\mathbb{R}^N)$ with $\|w\| = \rho$;*

(ii) *There is e in $H^2_r(\mathbb{R}^N)\backslash\{0\}$ such that $I^T_{\eta,\tau}(e) < 0$;*

(iii) *There is a positive constant $\tilde{c} > 0$ in which*

$$c_\tau \geq \tilde{c} \geq \max\left\{I^T_{\eta,\tau}(0), I^T_{\eta,\tau}(e)\right\},$$

where

$$c_\tau := \inf_{\sigma\in\Sigma} \max_{t\in[0,1]} I^T_{\eta,\tau}(\sigma(t)),$$

and

$$\Sigma := \left\{\sigma \in C\left([0,1], H^2_r(\mathbb{R}^N)\right) : \sigma(1) = e, \sigma(0) = 0\right\}.$$

Proof. **(i)** For any $w \in H^2_r(\mathbb{R}^N)$ and $\tau \in [\bar{\delta}, 1]$, using (6) and (7) for $\epsilon < 1$,

$$\begin{aligned} I^T_{\eta,\tau}(w) \geq& I^T_{\eta,1}(w) \\ =& \frac{1}{2}\int_{\mathbb{R}^N} |\Delta w|^2 + \frac{a_1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 + \frac{b_1}{4}(\int_{\mathbb{R}^N} |\nabla w|^2)^2 + \frac{\eta}{4}K_T(w)\int_{\mathbb{R}^N} \psi_w w^2 \\ &+ \int_{\mathbb{R}^N} Q_2(w) - \int_{\mathbb{R}^N} Q_1(w) \\ \geq& \frac{1}{2}(\int_{\mathbb{R}^N} |\Delta w|^2 + a_1\int_{\mathbb{R}^N} |\nabla w|^2) + (1-\epsilon)\eta\int_{\mathbb{R}^N} w^2 - \frac{c_\epsilon}{2^*}|w|^{2^*}_{2^*} \\ \geq& \beta_0\|w\|^2 - \beta_1\|w\|^{2^*}. \end{aligned}$$

This is positive for $\|w\|$ small enough.

Moreover, thanks to Sobolev embedding, one can find $\rho, \tilde{c} > 0$ such that $I^T_{\eta,\tau}(w) \geq \tilde{c} > 0$ for $\|w\| = \rho$ and for small enough $\tau \in [\bar{\delta}, 1]$.

(ii) Fix $\tau \in J$ and $\bar{\theta} > 0$ and $\bar{z} := z(\frac{\cdot}{\bar{\theta}})$. We define $\sigma : [0,1] \to H^2_r(\mathbb{R}^N)$ as the following

$$\sigma(t) := \begin{cases} 0, & \text{if } t = 0, \\ \bar{z}(\frac{\cdot}{t}), & \text{if } 0 < t \leq 1. \end{cases}$$

This map is a continuous pass from 0 to $\bar{z}$. Since $\tau > \bar{\delta}$, by simple computation we have

$$\begin{aligned} I^T_{\eta,\tau}(\sigma(1)) &\leq I^T_{\eta,\bar{\delta}}(\sigma(1)) \\ &= \frac{\bar{\theta}^{N-4}}{2}\int\limits_{\mathbb{R}^N}|\Delta z|^2 + \frac{a_1\bar{\theta}^{N-2}}{2}\int\limits_{\mathbb{R}^N}|\nabla z|^2 + \frac{b_1\bar{\theta}^{2N-4}}{4}(\int\limits_{\mathbb{R}^N}|\nabla z|^2)^2 \\ &+ \frac{\eta}{4}\bar{\theta}^{N+2}\chi\left(\frac{\bar{\theta}^N\|z\|^\alpha_\alpha}{T^\alpha}\right)\int\limits_{\mathbb{R}^N}\psi_z z^2 + \bar{\theta}^N(\int\limits_{\mathbb{R}^N}Q_2(z) - \bar{\delta}\int\limits_{\mathbb{R}^N}Q_1(z)). \end{aligned}$$

(8) and (10) imply that $I^T_{\eta,\tau}(\sigma(1)) < 0$, provided we choose $\bar{\theta}$ sufficiently large. Therefore, (ii) can be deduced for $e = \sigma(1)$.

(iii) Fix $\tau \in [\bar{\delta}, 1]$ and $\sigma \in \Sigma$. Then $I^T_{\eta,\tau}(\sigma(1)) < 0$ and $\sigma(0) = 0$, so $\|\sigma(1)\| > \rho$. Set $q(t) := \|\sigma(t)\| : [0,1] \to \mathbb{R}$, then q is a continuous map and

$$\begin{aligned} q(0) &= \|\sigma(0)\| = 0, \\ q(1) &= \|\sigma(1)\| > \rho. \end{aligned}$$

Mean value theorem implies that the existence of a constant $t_\sigma \in (0,1)$ in which $q(t_\sigma) = \rho$. It follows that $\|\sigma(t_\sigma)\| = \rho$. Therefore, by (ii)

$$c_\tau \geq \inf_{\sigma\in\Sigma} I^T_{\eta,\tau}(\sigma(t_\sigma)) \geq \tilde{c} > 0 = \max\left\{I^T_{\eta,\tau}(0), I^T_{\eta,\tau}(\sigma(1))\right\},$$

for any $\tau \in [\bar{\delta}, 1]$.

□

Remark 2. proof of (ii) in Lemma 4.2 implies the existence $\sigma \in C\left([0,1], H^2_r(\mathbb{R}^N)\right)$ in which $\sigma(0) = 0$ and $I^T_{\eta,\tau}(\sigma(1)) < 0$, i.e., $\sigma \in \Sigma$ and so $\Sigma \neq \psi$.

It is remarkable that Theorem 4.1 and Lemma 4.2 for almost any $\tau \in [\bar{\delta}, 1]$ imply the existence of boundedness a $(PS)_{c_{\eta,\tau}}$-sequences for $I^T_{\eta,\tau}$, i.e., there exists a sequence $\{u^\tau_n\} \subset H^2_r(\mathbb{R}^N)$ such that, $I^T_{\eta,\tau}(w^\tau_n) \to c_{\eta,\tau}$ and $\left(I^T_{\eta,\tau}\right)'(w^\tau_n) \to 0$.

In what follows, we recall a compactness result due to Strauss which will be apply Lemma 5.1.

Lemma 4.3. [19] *Suppose two continuous function P, $P' : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\lim_{t\to\infty}\frac{P(t)}{P'(t)} = 0,$$

$(\upsilon_n)_n$, υ and w_1 measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ in which

$$\sup_n \int\limits_{\mathbb{R}^N}|P'(\upsilon_n(x))w_1|dx < +\infty,$$

and $P(\upsilon_n(x)) \to \upsilon(x)$ a.e. in $\mathbb{R}^N$. Then $(P(\upsilon_n(x)) - \upsilon)\, w_1 \to 0$ in $L^1(B)$, for any bounded Borel set B.
Moreover, if

$$\lim_{s\to 0} \frac{P(s)}{P'(s)} = 0,$$

and

$$\lim_{x\to\infty} \sup_n |\upsilon_n(x)| = 0,$$

then $(P(\upsilon_n) - \upsilon)\, w_1 \to 0$, in $L^1(\mathbb{R}^N)$.

5. Existence Radial Solution

In this section we are going to prove that problem (bKSM) has a radial solution with asymptotic behavior with respect to the parameter τ.

Lemma 5.1. *Assume that $(q_1) - (q_4)$ hold. Let $\{w_n\} \subset H_r^2(\mathbb{R}^N)$ is a bounded sequence in which*

$$I_{\eta,\tau}^T(w_n) \le c \qquad \text{and} \qquad \left(I_{\eta,\tau}^T\right)'(w_n) \to 0. \tag{11}$$

Then for any $\tau \in [\bar{\delta}, 1]$, $\{w_n\}$ has a convergent subsequence(strongly).

Proof. Since $\{w_n\}$ is a bounded sequence, we may assume that, up to subsequence, it is a weak convergence sequence w in $H_r^2(\mathbb{R}^N)$ such that

$$\begin{cases} w_n \rightharpoonup w & \text{in } H_r^2(\mathbb{R}^N), \quad (12)\\ w_n \to w & \text{a.e in } \mathbb{R}^N, \quad (13)\\ w_n \to w & \text{in } L^p(\mathbb{R}^N), \quad 2 < p < 2^*. \quad (14)\end{cases}$$

It is direct to see that $\int \Delta w_n \,.\, \Delta w_1 \to \int \Delta w \,.\, \Delta w_1$ for any $w_1 \in C_0^\infty(\mathbb{R}^N)$.
Applying Lemma 4.3 for $P(s) := q_i(s),\ i = 1, 2$ and $P'(s) := |s|^{2^*-1}$, $\upsilon_n = w_n,\ \upsilon := q_i(w),\ i = 1, 2$ and from (2), (3), (14) yields

$$\int_{\mathbb{R}^N} q_i(w_n) w_1 \to \int_{\mathbb{R}^N} q_i(w) w_1. \quad i = 1, 2 \ \text{ as } n \to +\infty$$

(13) and [[18], Lemma 2.1] imply that

$$K_T(w_n) \int_{\mathbb{R}^N} \psi_{w_n}.w_n w_1 \to K_T(w) \int_{\mathbb{R}^N} \psi_w.w w_1,$$

and

$$\chi'\left(\frac{\|w_n\|_\alpha^\alpha}{T^\alpha}\right) \int_{\mathbb{R}^N} \psi_{w_n} w_n^2 \int_{\mathbb{R}^N} |w_n|^{\frac{2}{2^*-1}} w_n w_1 \to \chi'\left(\frac{\|w\|_\alpha^\alpha}{T^\alpha}\right) \int_{\mathbb{R}^N} \psi_w w^2 \int_{\mathbb{R}^N} |w|^{\frac{2}{2^*-1}} w w_1,$$

as $n \to +\infty$.

From (11) and (12), we have

$$\left(I_{\eta,\tau}^T\right)'(w) = 0,$$

Hence,

$$\int_{\mathbb{R}^N} |\Delta w|^2 + a_1 \int_{\mathbb{R}^N} |\nabla w|^2 + b_1 (\int_{\mathbb{R}^N} |\nabla w|^2)^2 + \eta K_T(w) \int_{\mathbb{R}^N} \psi_w w^2$$
$$+ \frac{\eta\alpha}{4T^\alpha} \chi' \left(\frac{\|w\|_\alpha^\alpha}{T^\alpha} \right) \|w\|_\alpha^\alpha \int_{\mathbb{R}^N} \psi_w w^2 + \int_{\mathbb{R}^N} w q_2(w) = \tau \int_{\mathbb{R}^N} w q_1(w). \tag{15}$$

Weak lower semicontinuity of norms imply that

$$\int_{\mathbb{R}^N} |\Delta w|^2 + a_1 \int_{\mathbb{R}^N} |\nabla w|^2 + b_1 (\int_{\mathbb{R}^N} |\nabla w|^2)^2$$
$$\leq \liminf_n \left(\int_{\mathbb{R}^N} |\Delta w_n|^2 + a_1 \int_{\mathbb{R}^N} |\nabla w_n|^2 + b_1 (\int_{\mathbb{R}^N} |\nabla w_n|^2)^2 \right). \tag{16}$$

Thanks to (13) and Lemma 3.1(v), we get that

$$K_T(w_n) \int_{\mathbb{R}^N} \psi_{w_n} w_n^2 \to K_T(w) \int_{\mathbb{R}^N} \psi_w w^2, \tag{17}$$

and

$$\chi' \left(\frac{\|w_n\|_\alpha^\alpha}{T^\alpha} \right) \|w_n\|_\alpha^\alpha \int_{\mathbb{R}^N} \psi_{w_n} w_n^2 \to \chi' \left(\frac{\|w\|_\alpha^\alpha}{T^\alpha} \right) \|w\|_\alpha^\alpha \int_{\mathbb{R}^N} \psi_w w^2. \tag{18}$$

If we applying Lemma 4.3 to $P(s) = q_1(s)s$, $P'(s) = s^2 + s^{2^*}$ (where $s \geq 0$), $\upsilon_n = w_n$, $\upsilon = wq_1(w)$, and $w_1 \equiv 1$, then according to (2), (3) and (14), we deduce that

$$\int_{\mathbb{R}^N} w_n q_1(w_n) \to \int_{\mathbb{R}^N} w q_1(w). \tag{19}$$

By applying Fatou's Lemma and (14) we deduce the following relation

$$\int_{\mathbb{R}^N} w q_2(w) \leq \lim_{n\to\infty} \inf \int_{\mathbb{R}^N} w_n q_2(w_n). \tag{20}$$

On the other hand however, the fact that $\langle I'_\tau(w_n), w_n\rangle$ converges to zero as $n \to +\infty$ together with (15), (17), (18), (19) and (20) imply that

$$\begin{aligned}
&\limsup_n \left[\int_{\mathbb{R}^N} |\Delta w_n|^2 + a_1 \int_{\mathbb{R}^N} |\nabla w_n|^2 + b_1\Big(\int_{\mathbb{R}^N} |\nabla w_n|^2\Big)^2 \right] \\
&= \limsup_n \left[\tau \int_{\mathbb{R}^N} q_1(w_n) w_n - \int_{\mathbb{R}^N} w_n q_2(w_n) - \eta K_T(w_n) \int_{\mathbb{R}^N} \psi_{w_n} w_n^2 \right. \\
&\left. - \frac{\eta\alpha}{4T^\alpha} \chi'\left(\frac{\|w_n\|_\alpha^\alpha}{T^\alpha}\right) \|w_n\|_\alpha^\alpha \int_{\mathbb{R}^N} \psi_{w_n} w_n^2 \right] \\
&\leq \tau \int_{\mathbb{R}^N} w q_1(w) - \int_{\mathbb{R}^N} w q_2(w) - \eta K_T(w) \int_{\mathbb{R}^N} \psi_w w^2 \\
&- \frac{\eta\alpha}{4T^\alpha} \chi'\left(\frac{\|w\|_\alpha^\alpha}{T^\alpha}\right) \|w\|_\alpha^\alpha \int_{\mathbb{R}^N} \psi_w w^2 \\
&= \int_{\mathbb{R}^N} |\Delta w|^2 + a_1 \int_{\mathbb{R}^N} |\nabla w|^2 + b_1\Big(\int_{\mathbb{R}^N} |\nabla w|^2\Big)^2. \qquad (21)
\end{aligned}$$

From (16) and (21)

$$\begin{aligned}
\lim_n \left[\int_{\mathbb{R}^N} |\Delta w_n|^2 + a_1 \int_{\mathbb{R}^N} |\nabla w_n|^2 + b_1\Big(\int_{\mathbb{R}^N} |\nabla w_n|^2\Big)^2 \right] &= \int_{\mathbb{R}^N} |\Delta w|^2 + a_1 \int_{\mathbb{R}^N} |\nabla w|^2 \\
&+ b_1\Big(\int_{\mathbb{R}^N} |\nabla w|^2\Big)^2. \qquad (22)
\end{aligned}$$

Hence,

$$\lim_n \int_{\mathbb{R}^N} w_n q_2(w_n) = \int_{\mathbb{R}^N} w q_2(w). \qquad (23)$$

Since $q_2(s)s = ms^2 + h(s)$, where h is a continuous and positive function, so Fatou's Lemma implies that

$$\int_{\mathbb{R}^N} h(w) \leq \liminf_n \int_{\mathbb{R}^N} h(w_n),$$

and

$$\int_{\mathbb{R}^N} w^2 \leq \liminf_n \int_{\mathbb{R}^N} w_n^2.$$

These two inequalities and (23) imply that going if necessary to a subsequence, we may assume that

$$\int_{\mathbb{R}^N} w^2 = \lim_n \int_{\mathbb{R}^N} w_n^2,$$

and by (22), strong convergence of $w_n \to w$ in $H_r^2(\mathbb{R}^N)$ is concluded. □

Lemma 5.2. *Assume that* $(q_1)-(q_4)$ *hold. Then for almost every* $\tau \in [\bar{\delta}, 1]$, *there exists* w^τ *in* $H_r^2(\mathbb{R}^N)\backslash\{0\}$ *such that*

$$I_{\eta,\tau}^T(w^\tau) = c_\tau, \quad \left(I_{\eta,\tau}^T\right)'(w^\tau) = 0. \tag{24}$$

Proof. By the Theorem 4.1, for almost every $\tau \in [\bar{\delta}, 1]$, there is a bounded sequence $(w_n^\tau)_n$ in $H_r^2(\mathbb{R}^N)$ such that

$$I_{\eta,\tau}^T(w_n^\tau) \to c_\tau; \tag{25}$$

$$\left(I_{\eta,\tau}^T\right)'(w_n^\tau) \to 0 \text{ in } \left(H_r^2(\mathbb{R}^N)\right)'. \tag{26}$$

From Lemma 5.1, up to a subsequence, we may assume that there exists w^τ in $H_r^2(\mathbb{R}^N)$ such that w_n^τ converges to w^τ in $H_r^2(\mathbb{R}^N)$. Due to the continuity and uniquness of the limits $I_{\eta,\tau}^T$ and $\left(I_{\eta,\tau}^T\right)'$ and from (25) and (26) we conclude (24). □

In order to prove non-triviality of w^τ, on the contrary, suppose that $w^\tau = 0$. Then $w_n^\tau \to 0$ in $H_r^2(\mathbb{R}^N)$, implies that $I_{\eta,\tau}^T(w_n^\tau) \to 0$ which contradicts $I_{\eta,\tau}^T(w_n^\tau) \to c_\tau \geq \tilde{c} > 0$ (by Lemma 4.2 (iii)). Therefore, $w^\tau \neq 0$.
Given $(\tau_n)_n \subset J$, there exists $(w_n)_n \subset H_r^2(\mathbb{R}^N)$ such that

$$I_{\eta,\tau_n}^T(w_n) = c_{\tau_n}, \qquad \left(I_{\eta,\tau_n}^T\right)'(w_n) = 0. \tag{27}$$

6. Application of Pohozaev Type Identity

This section is intended to show that $\sup_n \|w_n\| \leq T$. To reach this goal, we need the following result called Pohozaev identity which as a catalyst it plays a vital role in proving the main theorem.

Lemma 6.1. *Fix $\tau \in [\bar{\delta}, 1]$. If $w, \psi \in H^2_{loc}(\mathbb{R}^N)$ solve*

$$\begin{cases} \Delta^2 w - (a_1 + b_1 \int\limits_{\mathbb{R}^N} |\nabla w|^2)\Delta w \\ +\eta K_T(w)\psi w + \eta \frac{\alpha}{T^\alpha}\chi'\left(\frac{\|w\|_\alpha^\alpha}{T^\alpha}\right)|w|^{\frac{2}{2^*-1}} w \int\limits_{\mathbb{R}^N} \psi w^2 = q(w) & \text{in } \mathbb{R}^N, \\ -\Delta\psi = \eta w^2 & \text{in } \mathbb{R}^N, \end{cases} \tag{28}$$

then the following Pohozaev type identity hold.

$$\begin{aligned} &\frac{N-4}{2}\int\limits_{\mathbb{R}^N} |\Delta w|^2 + \frac{a_1(N-2)}{2}\int\limits_{\mathbb{R}^N} |\nabla w|^2 + \frac{b_1(N-2)}{2}(\int\limits_{\mathbb{R}^N} |\nabla w|^2)^2 \\ &+ \frac{(N+2)\eta}{4}K_T(w)\int\limits_{\mathbb{R}^N} \psi w^2 + \frac{Nk}{T^\alpha}\chi'\left(\frac{\|w\|_\alpha^\alpha}{T^\alpha}\right)\|w\|_\alpha^\alpha \int\limits_{\mathbb{R}^N} \psi w^2 = N\int\limits_{\mathbb{R}^N} Q(w). \end{aligned} \tag{29}$$

Proof. According to the proof of [10], by multiplying the first equation of (28) by $x.\nabla w$ and doing integral by using integral parts by means of the Green's formula and simplifying it, we get

$$\begin{aligned} \int\limits_{B_R} (\Delta^2 w)(x.\nabla w)dx &= -\left(\frac{N-4}{2}\right)\int\limits_{B_R} |\Delta w|^2 dx - \frac{1}{2}\int\limits_{\partial B_R} |\Delta w|^2 (x.\upsilon)d\sigma \\ &= -\left(\frac{N-4}{2}\right)\int\limits_{B_R} |\Delta w|^2 dx - \frac{R}{2}\int\limits_{\partial B_R} |\Delta w|^2 d\sigma. \end{aligned} \tag{30}$$

Again, multiplying the second equation of (28) by $x.\nabla w$, and integrating by parts by means of the Green's formula, we have

$$\begin{aligned} \int\limits_{B_R} (a_1 + b_1 \int\limits_{B_R} |\nabla w|^2)\Delta w(x.\nabla w) =& \frac{a_1(N-2)}{2}\int\limits_{B_R} |\nabla w|^2 + \frac{b_1(N-2)}{2}(\int\limits_{B_R} |\nabla w|^2)^2 \\ &+ \frac{a_1}{R}\int\limits_{\partial B_R} |x.\nabla w|^2 - \frac{a_1 R}{2}\int\limits_{\partial B_R} |\nabla w|^2 \\ &+ \frac{b_1}{R}\int\limits_{B_R} |\nabla w|^2 \int\limits_{\partial B_R} |x.\nabla w|^2 \\ &- \frac{b_1 R}{2}\int\limits_{B_R} |\nabla w|^2 \int\limits_{\partial B_R} |\nabla w|^2. \end{aligned} \tag{31}$$

And also for third sentence of equation (28), as in the previous method with multiply by $x.\nabla w$ and integrating by parts by means of the Green's formula, we have

$$\begin{aligned}\int\limits_{B_R} \psi w(x.\nabla w) =&\frac{1}{2}\int\limits_{B_R} \psi\nabla(w^2).x\\ =&\frac{1}{2}\int\limits_{B_R} \nabla(w^2).\psi x\\ =&\frac{1}{2}\int\limits_{B_R} \nabla(w^2).\psi\nabla(\frac{1}{2}|x|^2)\\ =&-\frac{1}{2}\int\limits_{B_R} w^2\nabla(\psi\nabla(\frac{1}{2}|x|^2))+\frac{1}{2}\int\limits_{\partial B_R} w^2\psi x.\upsilon d\sigma\\ =&-\frac{1}{2}\int\limits_{B_R} w^2(\nabla\psi\cdot x)-\frac{N}{2}\int\limits_{B_R} w^2\psi+\frac{R}{2}\int\limits_{\partial B_R} w^2\psi. \end{aligned}\tag{32}$$

In the same way as before the treatment of the following sentence is much easier, indeed we have

$$\begin{aligned}\int\limits_{B_R} q(w)(x.\nabla w) =&\int\limits_{B_R} \nabla\left(Q(w)\right).x\\ =&\int\limits_{B_R} \nabla\left(Q(w)\right).\nabla\left(\frac{1}{2}|x|^2\right)\\ =&-\int\limits_{B_R} NQ(w)+\int\limits_{\partial B_R} Q(w)\frac{\partial}{\partial\upsilon}\left(\frac{1}{2}|x|^2\right)d\sigma\\ =&-\int\limits_{B_R} NQ(w)+\int\limits_{\partial B_R} Q(w)(x.\upsilon)d\sigma\\ =&-N\int\limits_{B_R} Q(w)+R\int\limits_{\partial B_R} Q(w). \end{aligned}\tag{33}$$

And finally for the last sentence, we have

$$\begin{aligned}\int\limits_{B_R} |w|^{\frac{2}{2^*-1}}w\left(x\cdot\nabla w\right) =&\frac{2^*-1}{2(2^*)}\int\limits_{B_R} \nabla\left(|w|^{2\left(\frac{2^*}{2^*-1}\right)}\right)\cdot x\\ =&\frac{1}{\alpha}\int\limits_{B_R} \nabla\left(|w|^{\alpha}\right)\cdot\nabla\left(\frac{1}{2}|x|^2\right)\\ =&-\frac{N}{\alpha}\int\limits_{B_R} |w|^{\alpha}+\frac{1}{\alpha}\int\limits_{\partial B_R} |w|^{\alpha}(\upsilon\cdot x)d\sigma \end{aligned}$$

$$= -\frac{N}{\alpha}\int\limits_{B_R}|w|^{\alpha}+\frac{R}{\alpha}\int\limits_{\partial B_R}|w|^{\alpha} \tag{34}$$

where as usual $B_R=\{x\in\mathbb{R}^N,|x|<\mathbb{R}\}$ denotes the open ball in $\mathbb{R}^N$ with radius R, centered at origin . According to (30), (31), (32), (33) and (34)

$$\begin{aligned}
&-\left(\frac{N-4}{2}\right)\int\limits_{B_R}|\Delta w|^2-\frac{R}{2}\int\limits_{\partial B_R}|\Delta w|^2-\left(\frac{a_1(N-2)}{2}\right)\int\limits_{B_R}|\nabla w|^2\\
&-\left(\frac{b_1(N-2)}{2}\right)(\int\limits_{B_R}|\nabla w|^2)^2-\frac{a_1}{R}\int\limits_{\partial B_R}|x\cdot\nabla w|^2\\
&+\frac{a_1R}{2}\int\limits_{\partial B_R}|\nabla w|^2-\frac{b_1}{R}\int\limits_{B_R}|\nabla w|^2\int\limits_{\partial B_R}|x.\nabla w|^2+\frac{b_1R}{2}\int\limits_{B_R}|\nabla w|^2\int\limits_{\partial B_R}|\nabla w|^2\\
&-\frac{\eta}{2}K_T(w)\int\limits_{B_R}w^2\left(x\cdot\nabla\psi\right)-\frac{N\eta}{2}K_T(w)\int\limits_{B_R}\psi w^2+\frac{R\eta}{2}K_T(w)\int\limits_{\partial B_R}\psi w^2\\
&-\frac{N\eta}{T^{\alpha}}\chi'\left(\frac{\|w\|_{\alpha}^{\alpha}}{T^{\alpha}}\right)\int\limits_{B_R}\psi w^2\int\limits_{B_R}|w|^{\alpha}+\frac{R\eta}{T^{\alpha}}\chi'\left(\frac{\|w\|_{\alpha}^{\alpha}}{T^{\alpha}}\right)\int\limits_{B_R}\psi w^2\int\limits_{\partial B_R}|w|^{\alpha}\\
&=-N\int\limits_{B_R}Q(w)+R\int\limits_{\partial B_R}Q(w).
\end{aligned}\tag{35}$$

Multiplying by $x\cdot\nabla\psi$ the second equation and integrating on B_R,

$$\eta\int\limits_{B_R}(x\cdot\nabla\psi)w^2=-\frac{N-2}{2}\int\limits_{B_R}|\nabla\psi|^2+\frac{R}{2}\int\limits_{\partial B_R}|\nabla\psi|^2-\frac{1}{R}\int\limits_{\partial B_R}|x\cdot\nabla\psi|^2. \tag{36}$$

By placing the relation (36) in (35)

$$\begin{aligned}
&-\left(\frac{N-4}{2}\right)\int\limits_{B_R}|\Delta w|^2\\
&-\left(\frac{a_1(N-2)}{2}\right)\int\limits_{B_R}|\nabla w|^2-\left(\frac{b_1(N-2)}{2}\right)(\int\limits_{B_R}|\nabla w|^2)^2-\frac{N\eta}{2}K_T(w)\int\limits_{B_R}\psi w^2\\
&+\frac{N-2}{4}K_T(w)\int\limits_{B_R}|\nabla\psi|^2-\frac{N\eta}{T^{\alpha}}\chi'\left(\frac{\|w\|_{\alpha}^{\alpha}}{T^{\alpha}}\right)\int\limits_{B_R}\psi w^2\int\limits_{B_R}|w|^{\alpha}+N\int\limits_{B_R}Q(w)\\
&=\frac{R}{2}\int\limits_{\partial B_R}|\Delta w|^2+\frac{a_1}{R}\int\limits_{\partial B_R}|x\cdot\nabla w|^2-\frac{a_1R}{2}\int\limits_{\partial B_R}|\nabla w|^2-\frac{1}{2R}K_T(w)\int\limits_{\partial B_R}|x\cdot\nabla\psi|^2\\
&-\frac{R}{4}K_T(w)\int\limits_{\partial B_R}|\nabla\psi|^2-\frac{R\eta}{2}K_T(w)\int\limits_{\partial B_R}\psi w^2
\end{aligned}$$

$$+ R \int\limits_{\partial B_R} Q(w) - \frac{R\eta}{T^\alpha}\chi'\left(\frac{\|w\|_\alpha^\alpha}{T^\alpha}\right) \int\limits_{\partial B_R} \psi w^2 \int\limits_{\partial B_R} |w|^\alpha$$
$$+ \frac{b_1}{R} \int\limits_{B_R} |\nabla w|^2 \int\limits_{\partial B_R} |x \cdot \nabla w|^2 - \frac{b_1 R}{2} \int\limits_{B_R} |\nabla w|^2 \int\limits_{\partial B_R} |\nabla w|^2.$$

As in [9], the right-hand side tends to zero for at least one suitably chosen sequence $R = R_n \to +\infty$ as $n \to \infty$ and so

$$\begin{aligned} &- \frac{N-4}{2} \int\limits_{\mathbb{R}^N} |\Delta w|^2 - \frac{a_1(N-2)}{2} \int\limits_{B_R} |\nabla w|^2 \\ &- \frac{b_1(N-2)}{2} (\int\limits_{B_R} |\nabla w|^2)^2 - \frac{N\eta}{2} K_T(w) \int\limits_{\mathbb{R}^N} \psi w^2 \\ &+ \frac{N-2}{4} K_T(w) \int\limits_{\mathbb{R}^N} |\nabla \psi|^2 - \frac{N\eta}{T^\alpha}\chi'\left(\frac{\|w\|_\alpha^\alpha}{T^\alpha}\right) \int\limits_{\mathbb{R}^N} \psi w^2 \int\limits_{\mathbb{R}^N} |w|^\alpha \\ &+ N \int\limits_{\mathbb{R}^N} Q(w) = 0. \end{aligned}$$

The standard regularity results show that if $(w, \psi_w) \in H^2(\mathbb{R}^N) \times D^{2,2}(\mathbb{R}^N)$ is a solution of (28) then $w, \psi_w \in H^2_{Loc}(\mathbb{R}^N)$ and by (i) of Lemma 3.1, we deduce (29). $\square$

It what follows, we shall show $\|w_n\|_\alpha \leq T$, which is a critical key in the proof of existence of solutions to (bKSM).

Lemma 6.2. *Suppose that q satisfies $(q_1)-(q_4)$ and w_n is a critical point (for any n) for I^T_{η,τ_n} at level c_{τ_n} corresponding to (27). Then for large enough $T > 0$, there exists $\eta_0 = \eta_0(T)$ such that, up to a subsequence, for any $0 < \eta < \eta_0$, $\|w_n\|_\alpha \leq T$, for any $n \geq 1$.*

Proof. Since $\left(I^T_{\eta,\tau_n}\right)'(w_n) = 0$, so w_n satisfies the following Pohozaev type identity

$$\begin{aligned} &\frac{N-4}{2} \int\limits_{\mathbb{R}^N} |\Delta w_n|^2 + \frac{a_1(N-2)}{2} \int\limits_{\mathbb{R}^N} |\nabla w_n|^2 + \frac{b_1(N-2)}{2} (\int\limits_{\mathbb{R}^N} |\nabla w_n|^2)^2 \\ &+ \frac{(N+2)\eta}{4} K_T(w_n) \int\limits_{\mathbb{R}^N} \psi_{w_n} w_n^2 + \frac{N\eta}{T^\alpha}\chi'\left(\frac{\|w_n\|_\alpha^\alpha}{T^\alpha}\right) \|w_n\|_\alpha^\alpha \int\limits_{\mathbb{R}^N} \psi_{w_n} w_n^2 \\ &= N\tau_n \int\limits_{\mathbb{R}^N} Q_1(w_n) - N \int\limits_{\mathbb{R}^N} Q_2(w_n). \end{aligned} \tag{37}$$

Multiplying the first of statement of (27) by N and subtracting from the (37), we obtain

$$2\int\limits_{\mathbb{R}^N}|\Delta w_n|^2+a_1\int\limits_{\mathbb{R}^N}|\nabla w_n|^2+\frac{(4-N)b_1}{4}(\int\limits_{\mathbb{R}^N}|\nabla w_n|^2)^2-\frac{\eta}{2}K_T(w_n)\int\limits_{\mathbb{R}^N}\psi_{w_n}w_n^2$$
$$-\frac{N\eta}{T^\alpha}\chi'\left(\frac{\|w_n\|_\alpha^\alpha}{T^\alpha}\right)\|w_n\|_\alpha^\alpha\int\limits_{\mathbb{R}^N}\psi_{w_n}w_n^2=Nc_{\tau_n}.$$

Lemma 3.1(iii) implies that

$$\begin{aligned}&2\int\limits_{\mathbb{R}^N}|\Delta w_n|^2+a_1\int\limits_{\mathbb{R}^N}|\nabla w_n|^2+\frac{(4-N)b_1}{4}(\int\limits_{\mathbb{R}^N}|\nabla w_n|^2)^2\\ =&Nc_{\tau_n}+\frac{\eta}{2}K_T(w_n)\int\limits_{\mathbb{R}^N}\psi_{w_n}w_n^2+\frac{N\eta}{T^\alpha}\chi'\left(\frac{\|w_n\|_\alpha^\alpha}{T^\alpha}\right)\|w_n\|_\alpha^\alpha\int\limits_{\mathbb{R}^N}\psi_{w_n}w_n^2\\ \le&Nc_{\tau_n}+c_1\eta^2K_T(w_n)\|w_n\|_\alpha^4+c_2\frac{\eta^2}{T^\alpha}\chi'\left(\frac{\|w_n\|_\alpha^\alpha}{T^\alpha}\right)\|w_n\|_\alpha^{4+\alpha}.\end{aligned}\tag{38}$$

We intend to estimate the right-hand side of the inequality (38).

About mountain-pass level, we have

$$\begin{aligned}c_{\tau_n}\le&\max_{\bar\theta}I^T_{\eta,\tau_n}\left(z(\frac{\cdot}{\bar\theta})\right)\\ \le&\max_{\bar\theta}\{\frac{\bar\theta^{N-4}}{2}\int\limits_{\mathbb{R}^N}|\Delta z|^2+\frac{a_1\bar\theta^{N-2}}{2}\int\limits_{\mathbb{R}^N}|\nabla z|^2+\frac{b_1\bar\theta^{2N-4}}{4}(\int\limits_{\mathbb{R}^N}|\nabla z|^2)^2\\ &+\bar\theta^N\left(\int\limits_{\mathbb{R}^N}Q_2(z)-\bar\delta\int\limits_{\mathbb{R}^N}Q_1(z)\right)\}+\max_{\bar\theta}\left\{\frac{\eta}{4}\bar\theta^{N+2}\chi\left(\frac{\bar\theta^N\|z\|_\alpha^\alpha}{T^\alpha}\right)\int\limits_{\mathbb{R}^N}\psi_z z^2\right\}\\ =&A_1+A_2(T).\end{aligned}\tag{39}$$

If $\bar\theta^N\ge\frac{2T^\alpha}{\|z\|_\alpha^\alpha}$ then according to the relationship (8), $A_2(T)=0$. Otherwise $\bar\theta^N<\frac{2T^\alpha}{\|z\|_\alpha^\alpha}$, so by Lemma 3.1(iii) and notice that $\alpha=\frac{22^*}{2^*-1}$,

$$A_2(T)\le\frac{\eta}{4}\left(\frac{2T^\alpha}{\|z\|_\alpha^\alpha}\right)^{\frac{N+2}{N}}\int\limits_{\mathbb{R}^N}\psi_z z^2=c_3\eta^2T^4.\tag{40}$$

Similarly, we can also obtain that

$$c_1\eta^2K_T(w_n)\|w_n\|_\alpha^4\le c_4\eta^2T^4,\tag{41}$$

and

$$c_2\chi'\left(\frac{\|w_n\|_\alpha^\alpha}{T^\alpha}\right)\frac{\eta^2}{T^\alpha}\|w_n\|_\alpha^{4+\alpha} \leq c_5\eta^2T^4, \tag{42}$$

By (38)-(42) we would have the following

$$2\int\limits_{\mathbb{R}^N}|\Delta w_n|^2 + a_1\int\limits_{\mathbb{R}^N}|\nabla w_n|^2 + \frac{(4-N)b_1}{4}(\int\limits_{\mathbb{R}^N}|\nabla w_n|^2)^2 \leq NA_1 + c_6\eta^2T^4. \tag{43}$$

In the other words, since $\langle \left(I^T_{\eta,\tau_n}\right)'(w_n), w_n\rangle = 0$, by (5)

$$\begin{aligned}&\int\limits_{\mathbb{R}^N}|\Delta w_n|^2 + a_1\int\limits_{\mathbb{R}^N}|\nabla w_n|^2 + b_1(\int\limits_{\mathbb{R}^N}|\nabla w_n|^2)^2 + \eta K_T(w_n)\int\limits_{\mathbb{R}^N}\psi_{w_n}w_n^2\\ &+\frac{\eta\alpha}{4T^\alpha}\chi'\left(\frac{\|w_n\|_\alpha^\alpha}{T^\alpha}\right)\|w_n\|_\alpha^\alpha\int\limits_{\mathbb{R}^N}\psi_{w_n}w_n^2 + \int\limits_{\mathbb{R}^N}w_nq_2(w_n) = \tau_n\int\limits_{\mathbb{R}^N}w_nq_1(w_n)\\ &\leq c_3\int\limits_{\mathbb{R}^N}|w_n|^{2^*} + \epsilon\int\limits_{\mathbb{R}^N}w_nq_2(w_n).\end{aligned} \tag{44}$$

From (4), (43) and (44)

$$\begin{aligned}m(1-\epsilon)\int\limits_{\mathbb{R}^N}w_n^2 \leq&(1-\epsilon)\int\limits_{\mathbb{R}^N}w_nq_2(w_n)\\ \leq&c_\epsilon\int\limits_{\mathbb{R}^N}|w_n|^{2^*} - \frac{\eta\alpha}{4T^\alpha}\chi'\left(\frac{\|w_n\|_\alpha^\alpha}{T^\alpha}\right)\|w_n\|_\alpha^\alpha\int\limits_{\mathbb{R}^N}\psi_{w_n}w_n^2\\ \leq&c\left(\int\limits_{\mathbb{R}^N}|\Delta w_n|^2 + a_1\int\limits_{\mathbb{R}^N}|\nabla w_n|^2 + \frac{(4-N)b_1}{4}(\int\limits_{\mathbb{R}^N}|\nabla w_n|^2)^2\right)^{\frac{2^*}{2}}\\ &+\bar{c}\eta^2T^4\\ \leq&c\left(NA_1 + c_6\eta^2T^4\right)^{\frac{2^*}{2}} + \bar{c}\eta^2T^4.\end{aligned} \tag{45}$$

Therefore, $(w_n)_n$ is bounded in $L^2(\mathbb{R}^N)$. Now we will show that up to a subsequence $\|w_n\|_\alpha \leq T$. If not, there is no subsequence uniformly bounded by T of $(w_n)_n$ by the α-norm. Then for a certain $\bar{n}$

$$\|w_n\|_\alpha > T, \quad \forall n \geq \bar{n}. \tag{46}$$

Without losing of totality, we may assume that (46) is true for any w_n. (43) and (45) imply that

$$T^2 < \|w_n\|_\alpha^2 \leq c_7 + c_8\eta^2\left(T^2\right)^{2^*},$$

which is not true for T large enough and η small enough: in fact there are $T_0 > 0$ such that $T_0^2 > c_7 + 1$ and $\eta_0 = \eta_0(T_0)$ such that $c_8\eta^2\left(T_0^2\right)^{2^*} < 1$, for any $\eta < \eta_0$, which leads to a antagonism. □

7. Proof of Theorem 1.1

By the process of the proof of Lemma 6.2, now it's time to prove Theorem 1.1.

Proof of Theorem 1.1. Fix T, η_0 as in Lemma 6.2 and consider any $0 < \eta < \eta_0$. By (27) we can take a sequence $\{\tau_n\} \subset [\bar{\delta}, 1]$ such that $\tau_n \nearrow 1$ and a sequence $\{w_n\} \subset H_r^2(\mathbb{R}^N)\backslash\{0\}$ in which

$$I^T_{\eta,\tau_n}(w_n) = c_{\tau_n} \quad \text{and} \quad \left(I^T_{\eta,\tau_n}\right)'(w_n) = 0.$$

According to Lemma 6.2 we may assume that $\|w_n\|_\alpha \leq T$. Then by (8)

$$\begin{aligned} I^T_{\eta,\tau_n}(w_n) =& \frac{1}{2}\int_{\mathbb{R}^N} |\Delta w_n|^2 + \frac{a_1}{2}\int_{\mathbb{R}^N} |\nabla w_n|^2 + \frac{b_1}{4}(\int_{\mathbb{R}^N} |\nabla w_n|^2)^2 + \frac{\eta}{4}\int_{\mathbb{R}^N} \psi_w w^2 \\ &+ \int_{\mathbb{R}^N} Q_2(w_n) - \tau_n \int_{\mathbb{R}^N} Q_1(w_n). \end{aligned} \tag{47}$$

Moreover, according to the arguments of relationship (43) and (45), it shows the boundedness of $\{w_n\}$ in $H_r^2(\mathbb{R}^N)$.

As a result, it remains to be seen that $\{w_n\}$ is a (PS) sequence for I_η at the level $c_{\eta,1}$. Indeed, the boundedness of $\{w_n\}$ implies that $\{I_\eta(w_n)\}$ is bounded. By (47), $\tau_n \nearrow 1$, then for any $\psi \in C_0^\infty(\mathbb{R}^N)$,

$$\langle I'_\eta(w_n), \psi\rangle = \langle \left(I^T_{\eta,\tau_n}\right)'(w_n), \psi\rangle + (\tau_n - 1)\int_{\mathbb{R}^N} q_1(w_n)\psi \to 0 \quad \text{as } n \to \infty.$$

Hence, $I'_\eta(w_n) \to 0$ as $n \to \infty$. Moreover, left continuity of $\tau \to c_{\eta,\tau}$ and Lemma 4.2(iii) imply that

$$\lim_{n\to\infty} I_\eta(w_n) = \lim_n \left(I^T_{\eta,\tau_n}(w_n) + (\tau_n - 1)\int_{\mathbb{R}^N} Q_1(w_n)\right) = \lim_{n\to\infty} c_{\eta,\tau_n} = c_{\eta,1}.$$

Therefore, $\{w_n\}$ is a bounded (PS) sequence of I_η. Then by using Lemma 5.1, $\{w_n\}$ has a convergent subsequence, $w_n \to w_\eta$ strongly in $H_r^2(\mathbb{R}^N)$. Consequently, due to the continuity of the functional I_η and I'_η and the uniqueness of the limits we deduce, $I'_\eta(w_\eta) = 0$ and $I_\eta(w_\eta) = c_{\eta,1}$. By Lemma 4.2(iii) $c_{\eta,1} \geq \tilde{c} > 0$, which implies that $w_\eta \neq 0$. Hence, w_η is a non-trivial positive solution by Remark 1. □

8. Conclusion

The current paper introduced a fundamental relation to find nontrivial solution in such a way that under some condition.Variable and topological methods are powerful tools in solving nonlinear concrete boundary value problems that appear in many disciplines where classical methods have failed. The ideas presented here use inspirational analysis in the geometry of a mountain pass. The ridge theorem is the result of great intuitive appeal as well as practical importance in determining functional critical points, especially those occurring in the theory of ordinary differential equations. The concept of Palais-Smale is introduced, which is the analog compression in the variable account. both mountain pass and Palais-Smale use in order to find critical points. The concept of Pohozaev identity has been proposed and has many applications in fields such as fractional equations and semi-linear equations, proving the existence and absence of nontrivial star-shaped solutions for supercritical nonlinearities and elliptical PDEs. Pohozaev identity surrenders to uniform formulas, unique continuation features, radial symmetry of solutions, and unique results. It is also used in other fields such as hyperbolic equations, harmonic maps, control theory and geometry. In this article, we have examined all of them. We try to use these results in future studies.

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