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On the Riemann-Stieltjes Integral

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Abstract

This study contributes to the theory of Riemann-Stieltjes integral. We prove that if all continuous piecewise linear functions are Riemann-Stieltjes integrable with respect to a bounded integrator $\alpha : [a, b] \to R$, then α must be of bounded variation on $[a, b]$. We also provide some other consequences.

Keywords: Cantor's intersection theorem, function of bounded variation, Riemann-Stieltjes integral

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1. Introduction

Although the theory of Riemann-Stieltjes (simply R-S) integral has been established for a century, it remains as the subject of many current mathematical researches. Some of the studies in R-S integration in the last four decades are given in its own theory. For instance, in [11], Ross has proposed an alternative definition for the R-S integral that has some advantages over the classical definition.

As is well known, Riemann integrability in Euclidean space is characterized by continuity almost everywhere, with respect to Lebesgue measure. The main purpose of Ter Horst in [15], is to generalize this classical theorem to Stieltjes integral in Euclidean space.

In [9], Lukkarinen and Pakkanen has considered the question "Whether an R-S integral of a positive continuous function with respect to a nonnegative function of bounded variation is positive?" before providing an answer to it. They deduced

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that, the general answer to the question is no. An affirmative answer to this question under a slightly different set of assumptions, is given by Satyanarayana in [13].

In [6], Chen, Leskelä, and Viitasaari studied the existence of pathwise Stieltjes integrals of the form $\int f(X_t)dY_t$ for nonrandom, possibly discontinuous, evaluation functions f, and Hölder continuous random processes X and Y. In [16], Yaskov derived the results of [6] under weaker conditions.
Some of the recent studies in this area, similar to [6] and [16] are related to the approximation theory, or finding the bounds of functionals. For instance, in [5], Cerone and Dragomir gave lower and upper bounds of the Čebyšev functional for the R-S integral. In [1], Alomari proved several bounds for the difference between two R-S integral means, under various assumptions.

Many papers have been espically devoted to study the analogies of the R-S integration known for different types of partial differential equations, oscillation theory, fractional differential equations, and integro-differential equations as well as their applications to analysis of their solutions [7, 17].

Finally, some of the studies in this area, in the last two decades, were realated to the modelling in life situations such as dynamics of multi-body systems, time scales, applications in complex analysis, and probability theory. Some of the references in this area are [4, 8, 10].

The present paper is also a study on the theory of R-S integral. Our approach, as some of the previous references, is in its own theory. We turn to a basic question in the theory of R-S integration that seems to be less well-known. In fact, as a theorem in mathematical analysis, every continuous function is R-S integrable with respect to a monotonic (hence a bounded variation) integrator $\alpha : [a, b] \to R$ [2, 3, 12]. Here, we provide a converse for this theorem. We show that if all continuous piecewise linear functions are R-S integrable with respect to a bounded integrator $\alpha : [a, b] \to R$, then α must be of bounded variation on $[a, b]$.

2. Preliminary Notes

Let's start with recalling some definitions and the most essential concepts needed in our study. A partition of $[a, b]$ is a set of points $P = \{x_0, x_1, \dots, x_l\}$, $(l \in N)$ such that $a = x_0 < x_1 < x_2 < \cdots < x_l = b$. The set of all partitions of $[a, b]$ is denoted by $\mathbb{P}[a, b]$. A function $\alpha : [a, b] \to R$ is said to be of bounded variation on $[a, b]$, if there exists a positive constant R such that $\Sigma(P) = \Sigma_{i=1}^{i=l} |\alpha(x_i) - \alpha(x_{i-1})| \leq R$ for all possible partitions $P \in \mathbb{P}[a, b]$. If α is of bounded variation on $[a, b]$, then the number $V_\alpha[a, b] = sup\{\Sigma(P) | P \in \mathbb{P}[a, b]\}$ is called the total variation of α on $[a, b]$. In this case, the function $V : [a, b] \to R$ defined by $V(x) = V_\alpha[a, x]$ for $x \in [a, b]$, is called the function of total variation of α [2]. Total variation of α on $[a, b]$ is defined by $V(b)$. A real valued function defined on $[a, b]$ is of bounded variation, if and only if its total variation on $[a, b]$ is bounded [2].

Suppose that f and α are real-valued functions defined on $[a, b]$. Let P be a partition of $[a, b]$ and $S(P, f, \alpha) = \Sigma_{i=1}^{i=l} f(t_i)(\alpha(x_i) - \alpha(x_{i-1}))$, where t_i is an arbitrary point in $[x_{i-1}, x_i]$. The function f is said to be R-S integrable with respect to α, if there exists a real number I that satisfies the following condition: For any $\epsilon > 0$ there exists a partition $P_\epsilon \in \mathbb{P}[a, b]$ such that for every partition $P \in \mathbb{P}[a, b]$ refines P_ϵ, the inequality $|S(P, f, \alpha) - I| < \epsilon$ holds [2, 3]. Since I is unique, it is called the R-S integral of f with respect to α, and is denoted by $I = \int_a^b f d\alpha$. Clearly, if f is R-S integrable with respect to α, then there exists a partition $P_0 \in \mathbb{P}[a, b]$ such that the set $\{S(P, f, \alpha) | P \in \mathbb{P}[a, b], P_0 \subseteq P\}$ is bounded.

3. Main Results

We provide some results about the integrators of bounded variation. Regarding the continution of the investigation, we start with some theorems on the functions of bounded variation.

Theorem 3.1. ([2], [3]) Let the function $\alpha : [a, b] \to R$ be of bounded variation on $[a, b]$ and $p \in [a, b]$. Then α is of bounded variation on the subintervals $[a, p]$ and $[p, b]$, such that

$$V_\alpha[a, b] = V_\alpha[a, p] + V_\alpha[p, b]. \quad (1)$$

Conversely, if α is of bounded variation on $[a, p]$ and $[p, b]$ for some $p \in [a, b]$, then α is of bounded variation on $[a, b]$, and (1) holds.

Theorem 3.2. Suppose that $\alpha : [a, b] \to R$ is not of bounded variation on $[a, b]$. Then there exists a point $p \in [a, b]$ such that α is not of bounded variation on every subinterval $[x, y] \subseteq [a, b]$ containing p as an interior point.

Proof. Let $c = \frac{a+b}{2}$. The intervals $[a, c]$ and $[c, b]$ then determines two subintervals Q_1, Q_2 whose union is $[a, b]$. Theorem 3.1 implies that, at least one of the sets Q_i, call it I_1 admits the following properties:

1. $I_1 \subset [a, b]$,
2. α is not of bounded variation on I_1,
3. if $x, y \in I_1$, then $|x - y| \leq \frac{b-a}{2}$.

Continuing the process, we obtain a sequence $\{I_n\}_{n\in N}$ of closed subintervals of $[a, b]$ with the following properties:

(a) $I_{n+1} \subset I_n$ for $n \in N$,

(b) α is not of bounded variation on I_n for $n \in N$,

(c) if $x, y \in I_n$, then $|x - y| \leq \frac{b-a}{2^n}$.

Therefore Cantor's intersection theorem implies that, there exists exactly one point p which lies in every I_n [14]. If $[x, y] \subseteq [a, b]$ is any subinterval and $x < p < y$, then there exists $r > 0$ such that $|z - p| < r$ implies that $z \in (x, y)$. If n is so large that $\frac{b-a}{2^n} < r$, then 3 implies that $I_n \subset [x, y]$. Now, (b) and Theorem 3.1 imply that α is not of bounded variation on $[x, y]$. □

Corollary 3.3. If $\alpha : [a, b] \to R$ is not of bounded variation on $[a, b]$, then there exists $p \in [a, b]$ such that α is not of bounded variation on every interval $[c, p]$ for all c, $(a \leq c < p)$, or, α is not of bounded variation on every interval $[p, d]$ for all d, $(p < d \leq b)$.

Proof. The proof of Theorem 3.2 implies that, there exists a point $p \in [a, b]$ and a sequence $\{I_n\}_{n \in N}$ of the closed subintervals of $[a, b]$ containing p, such that α is not of bounded variation on I_n for $n \in N$. If $p = a$ (res. $p = b$), then the structure of the proof of Theorem 3.2, shows that there exists a nonconstant sequence of the end points of the constructed intervals with terms bigger (res. smaller) than p where tends to p. Therefore, α is not of bounded variation on every interval $[p, d]$ for all d, $(p < d \leq b)$ (res. on every interval $[c, p]$ for all c, $(a \leq c < p)$). If $a < p < b$, then there exists a nonconstant subsequence of the end points of the intervals with terms bigger (and also smaller) than p where tends to p. Thus α is not of bounded variation on every interval $[c, p]$ for all c, $(a \leq c < p)$, (and also on every interval $[p, d]$ for all d, $(p < d \leq b)$). □

We now proceed to prove that if the integrator $\alpha : [a, b] \to R$ is a bounded function, and if every continuous function on $[a, b]$, is R-S integrable with respect to α on $[a, b]$, then α must be of bounded variation on $[a, b]$.

Theorem 3.4. Let the integrator $\alpha : [a, b] \to R$ be a bounded function. If α is not of bounded variation on $[a, b]$, then there exists a real continuous function, defined on $[a, b]$, which is not R-S integrable with respect to α on $[a, b]$.

Proof. Because of boundedness of α, there exists $M > 1$ such that $|\alpha(x)| < M$ for all $x \in [a, b]$. By corollary 3.3, there exists $p \in [a, b]$ such that α is not of bounded variation on $[c, p]$ for all c $(a \leq c < p)$, or, on $[p, d]$ for $(p < d \leq b)$. Without loss of generality we assume the first case, so there exists a partition $\{x_0, x_1, \ldots, x_{l_1}, p\} \in \mathbb{P}[a, p]$ such that $a = x_0 < x_1 < x_2 < \cdots < x_{l_1} < p$ and $\Sigma_{i=1}^{i=l_1} |\alpha(x_i) - \alpha(x_{i-1})| + |\alpha(p) - \alpha(x_{l_1})| > 3M$. Hence there exists an ordered set $x_0 < x_1 < x_2 < \cdots < x_{l_1} < p$ of the points in $[a, p]$ such that $\Sigma_{i=1}^{i=l_1} |\alpha(x_i) - \alpha(x_{i-1})| > 3M - |\alpha(p) - \alpha(x_{l_1})| > M$. Since α is not of bounded variation on $[x_{l_1}, p]$, repetition the process for $[x_{l_1}, p]$ shows that there exists an ordered set $x_{l_1} < x_{(l_1+1)} < \cdots < x_{l_2} < p$ of the points in $[x_{l_1}, p]$ such that $\Sigma_{i=l_1+1}^{i=l_2} |\alpha(x_i) - \alpha(x_{i-1})| > M$. Let $k \in N (k \geq 2)$ and the process is repeated for $[x_{l_k}, p]$. Thus there exists an ordered set $x_{l_k} < x_{(l_k+1)} < \cdots < x_{l_{k+1}} < p$ of the points in $[x_{l_k}, p]$ such that $\Sigma_{i=l_k+1}^{i=l_{k+1}} |\alpha(x_i) - \alpha(x_{i-1})| > M$. Since α is not of bounded variation on $[x_{l_{k+1}}, p]$, repetition the process for $[x_{l_{k+1}}, p]$ shows that there exists an ordered set $x_{l_{k+1}} < x_{(l_{k+1}+1)} < \cdots < x_{l_{k+2}} < p$ of the points in $[x_{l_{k+1}}, p]$ such that $\Sigma_{i=l_{k+1}+1}^{i=l_{k+2}} |\alpha(x_i) - \alpha(x_{i-1})| > M$. Continuing the process, produces an increasing sequence

$$a = x_0, x_1, \ldots, x_{l_1}, x_{(l_1+1)}, \ldots, x_{l_2}, x_{(l_2+1)}, \ldots, x_{l_3}, \ldots, x_{(l_k+1)}, \ldots, x_{l_{k+1}}, \ldots$$
$$, x_{(l_{k+u-1}+1)}, \ldots, x_{l_{k+u}}, \ldots$$

of the points in $[a, b]$. Without loss of generality, it can be assumed that the sequence $x_{l_1}, x_{l_2}, \ldots, x_{l_k}, \ldots$ tends to p. Let $r > 0$ be arbitrary and define the functions $\theta : R \to R$, and $f_r : [a, b] \to R$ such that $x\theta(x) = |x|$, $\theta(0) = 0$ and

$$f_r(x) = \begin{cases} 0, & if \quad x \in [a, x_{l_1}], \\ \frac{r}{q+1}\theta(\alpha(x_i) - \alpha(x_{i-1})), & if \quad x = x_i, l_q + 1 \le i \le l_{q+1}, q = 1, 2, 3, \ldots \\ 0, & if \quad x \in [p, b]. \end{cases}$$

Obviously, one can extend f_r to be continuous on $[a, b]$. It suffices to show that, this extension is not R-S integrable with respect to α on $[a, b]$. For this, let

$$P_0 = \{a = y_0, y_1, \ldots, y_{s-1}, y_s, y_{s+1}, \ldots, y_m = b\}$$

be a partition of $[a, b]$ with $y_s = p$, and let $k \in N, k > 1$ be such that $y_{s-1} < x_{l_k}$. For $u \in N$ the partition

$$P_{(k,u)} = \{a = y_0, y_1, \ldots, y_{s-1}, x_{l_k}, x_{(l_k+1)}, \ldots, x_{l_{k+1}}, \ldots$$
$$, x_{l_{k+u-1}}, x_{(l_{k+u-1}+1)}, \ldots, x_{l_{k+u}}, p = y_s, y_{s+1}, \ldots, y_m = b\}$$

is a refinement of P_0. Now for $t_\nu \in [y_{\nu-1}, y_\nu] (1 \le \nu \le s-1, s+2 \le \nu \le m), t_{l_k} \in [y_{s-1}, x_{l_k}], t_{s+1} \in [p, y_{s+1}]$ and $t_{l_{k+u}} \in [x_{l_{k+u}}, p]$, we have

$$\begin{aligned} S(P_{(k,u)}, f_r, \alpha) &= \sum_{\nu=1}^{\nu=s-1} f_r(t_\nu)(\alpha(y_\nu) - \alpha(y_{\nu-1})) + f_r(t_{l_k})(\alpha(x_{l_k}) - \alpha(y_{s-1})) \\ &+ \sum_{\mu=l_k+1}^{\mu=l_{k+u}} f_r(x_\mu)(\alpha(x_\mu) - \alpha(x_{\mu-1})) + f_r(t_{l_{k+u}})(\alpha(p) - \alpha(x_{l_{k+u}})) \\ &+ f_r(t_{s+1})(\alpha(y_{s+1}) - \alpha(p)) + \sum_{\nu=s+2}^{\nu=m} f_r(t_\nu)(\alpha(y_\nu) - \alpha(y_{\nu-1})) \\ &= \sum_{\nu=1}^{\nu=s-1} f_r(t_\nu)(\alpha(y_\nu) - \alpha(y_{\nu-1})) + f_r(t_{l_k})(\alpha(x_{l_k}) - \alpha(y_{s-1})) \\ &+ \sum_{q=k}^{q=k+u-1} \sum_{\mu=l_q+1}^{\mu=l_{q+1}} \frac{r}{q+1}\theta(\alpha(x_\mu) - \alpha(x_{\mu-1}))(\alpha(x_\mu) - \alpha(x_{\mu-1})) \\ &+ f_r(t_{l_{k+u}})(\alpha(p) - \alpha(x_{l_{k+u}})) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\nu=1}^{\nu=s-1} f_r(t_\nu)(\alpha(y_\nu) - \alpha(y_{\nu-1})) + f_r(t_{l_k})(\alpha(x_{l_k}) - \alpha(y_{s-1})) \\
&+ r \sum_{q=k}^{q=k+u-1} \sum_{\mu=l_q+1}^{\mu=l_{q+1}} \frac{1}{q+1} |\alpha(x_\mu) - \alpha(x_{\mu-1})| \\
&+ f_r(t_{k+u})(\alpha(p) - \alpha(x_{l_{k+u}})) \\
&> \sum_{\nu=1}^{\nu=s-1} f_r(t_\nu)(\alpha(y_\nu) - \alpha(y_{\nu-1})) + f_r(t_{l_k})(\alpha(x_{l_k}) - \alpha(y_{s-1})) \\
&+ rM \sum_{q=k}^{q=k+u-1} \frac{1}{q+1} + f_r(t_{l_{k+u}})(\alpha(p) - \alpha(x_{l_{k+u}})).
\end{aligned}
$$

Since α and f_r are bounded functions and $lim_{u\to+\infty} \sum_{q=k}^{q=k+u-1} \frac{1}{q+1} = +\infty$, so, the set $\{S(P, f_r, \alpha)|P_0 \subseteq P\}$ is not bounded, and f_r is not R-S integrable with respect to α on $[a, b]$. □

Corollary 3.5. If a bounded function $\alpha : [a, b] \to R$ is not of bounded variation on $[a, b]$, then there exist infinitely many real continuous functions on $[a, b]$, that are not R-S integrable with respect to α on $[a, b]$.

A piecewise linear function defined on an interval, is a function composed of some number of linear segments defined over a number of subintervals, not essentially of equal lengths. Since f_r, in the preceeding proof, can be extended to a continuous piecewise linear function on $[a, b]$, the following Corollary is hold.

Corollary 3.6. If all real continuous piecewise linear functions are R-S integrable with respect to a bounded integrator $\alpha : [a, b] \to R$ on the interval $[a, b]$, then α must be of bounded variation on $[a, b]$.

Theorem 3.7. If a bounded funtion $f : [a, b] \to R$ is R-S integrable with respect to every continuous piecewise linear function $\alpha : [a, b] \to R$ on $[a, b]$, then f must be of bounded variation on $[a, b]$.

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