Applying Symmetries to Solving a Nonlinear Acoustics Beam Model

Leila Hamedi Mobarra ⋆

Abstract

For an important nonlinear acoustic model, the (2+1)-dimensional Zabolotskaya –Khokhlov(ZK), a symmetry group and the optimal systems of the symmetry subalgebra have been introduced. Then related symmetry reductions and similarity solutions have been presented via two-stage using of the symmetry group method.

Keywords: Zabolotskaya–Khokhlov equation (ZK), symmetry of differential equations, reduction of differential equation, similarity solution.

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1. Introduction

1.1 The Zabolotskaya–Khokhlov Acoustics Model

In fluid mechanics, the NavierStokes equations are nonlinear partial differential equations which describe the motion of fluid substances, and they arise from applying Newton's second law to fluid motion, together with the assumption that the stress in the fluid is the sum of a diffusing viscous term and a pressure term, hence describing viscous flow. A solution of the NavierStokes equations is a description of the velocity of the fluid at a given point in space and time. The incompressible NavierStokes equation have the inconvenient feature that there is no explicit mechanism for advancing the pressure in time. The Zabolotskaya–Khokhlov equation

⋆Corresponding author (leila.hamedimobarra@fshiau.ac.ir)
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is a special type of the incompressible NavierStokes equation, and it is the mathematical model of the propagation phenomena in nonlinear acoustic beams. This equation has been found from focus of russian mathematicians Lyubov Sergeyevna Zabolotskaya and Rem Khokhlov, on developments in fluid mechanics and the corresponding models of nonlinear acoustic beams (ref to [4]). Indeed, the propagation of a bounded two-dimensional acoustic beam in nonlinear medias is described by the three dimensional Zabolotskaya–Khokhlov equation (ZK), with the following form:

$$\Delta(t, x, y, u) = u_{xt} - (u_x)^2 - uu_{xx} - u_{yy} = 0.$$

where $t, x > 0$ and $-\infty < y < +\infty$. In addition, the value of $u = u(t, x, y)$ is proportional to the deviation of the media density from the balanced density, while the dimensionless variables t, x, y are expressed via the temporal $\mathfrak{t}$ and spatial variables $\mathfrak{x}$, $\mathfrak{y}$, as follows (ref to [1, 3]):

$$t = \frac{\mathfrak{t} - \frac{\mathfrak{x}}{c_0}}{\sqrt{\gamma + 1}} \sqrt{\rho_0 c_0}, \quad x = \mu \mathfrak{x}, \quad y = \sqrt{\frac{2\mu}{c_0}} \mathfrak{y},$$

where c_0 is the sound velocity in the media,γ is the isentropic exponent, $\mathfrak{x}$ is the coordinate in the direction of beam propagation, μ is a small parameter, and ρ_0 is the balanced density.

1.2 Applying Symmetries to Solving Partial Differential Equations

The idea of studying the differential equations by applying the symmetries implied a new theory: the symmetry group theory, which is due to Sophus Lie[1].

Figure 1: Marius Sophus Lie (1842-1899), Elie Cartan (1869-1951), Emmy Noether (1882 -1935).

[1] Applying symmetries to solving differential equations was originally introduced by Sophus Lie, Emmy Noether, and Elie Cartan, in 19th century.

The symmetry group of a PDE is the largest Lie group of point transformations acting on the space of the independent and dependent variables of the PDE, with a principal property of conserving the set of solutions, namely, it leaves the equation invariant. A symmetry group of a PDE leads to an algorithm to determine the infinitesimal generators of the Lie group of point transformations admitted by the PDE. Let $\Delta = 0$ be a PDE of order k. Moreover, suppose that Δ has a Lie symmetry group G. A vector field X on G is known as an infinitesimal symmetry, if and only if it satisfys the condition of invariance of the equation under the infinitesimal prolonged infinitesimal generators, namely:

$$pr^{(k)}X(\Delta)\mid_{\Delta=0} = 0. \tag{1}$$

The relation (1) results the characteristic system of ODEs, that by integrating of this system, the corresponding ordinary invariants are followed.

Simpler forms of a differential equation are called its reductions. A reduction has lower order, or involves fewer independent variables, than the original equation. Indeed, the order of a partial differential equation, or the number of its independent variables can be reduced by one if it is invariant under a one-parameter symmetry group. The symmetry method shows how to construct solutions of partial differential equations (PDEs) by their symmetry groups. Similarity solutions of a PDE are the solutions which are invariant under the symmetries. These solutions are obtained by solving the reduction ODEs that are given from the PDE. In this work, the nonlinear Zabolotskaya–Khokhlov equation is reduced to PDEs with lesser number of independent variables, by the use of the symmetry group of the ZK. Then, the obtained reductions are reduced to ordinary differential equations reductions of same order, which their solutions are similarity solutions of the ZK equation.

2. Symmetry Reductions of the ZK Equation

Recall that the associated Lie algebra to a Lie symmetry group is named a symmetry algebra, and the generators of a symmetry algebra are the same related infinitesimal symmetries. We mention that the ZK is of order 2, the infinitesimal symmetries of the ZK equation are gotten(ref to [2]):

$$\begin{aligned} v_1 &= -\frac{1}{2}(y+t^2)\partial_x - t\partial_y + t\partial_u, \\ v_2 &= -t\partial_x - \partial_y + \partial_u, \\ v_3 &= \partial_x, \\ v_4 &= \partial_t. \end{aligned} \tag{2}$$

It is obvious that the vector fields (2) satisfy $pr^{(2)}v_i(\Delta)\mid_{\Delta=0} = 0$ $(i = 1, 2, 3, 4)$. On the other, the ZK equation involves three independent variables and one dependent variable, thus it can be accounted as the total space $E \simeq \mathbb{R}^{3+1}$ with coordinates t, x, y, u. The below theorem illustrates the behavior of the symmetry group on the space E.

Theorem 2.1. *The symmetry group of the ZK equation, contains the following transformations*

$$\begin{aligned}(t,x,y,u) \longmapsto &(t+c_1\varepsilon, x+(c_2+c_3t+c_4t^2+c_5y)\varepsilon+(c_6+c_7t)\varepsilon^2+c_8\varepsilon^3, y+\\ &(c_7+c_9t)\varepsilon+c_{10}\varepsilon^2+(c_{11}+c_{12}t)\varepsilon-c_7\varepsilon^2, u),\end{aligned}$$

where ε, c_i $(i=1,2,\dots,12)$ *are arbitrary numbers.*

Proof. The one- parameter Lie point symmetry transformations can be obtained as (ref to [2]):

$$\begin{aligned}\exp(\varepsilon v_1) &: (t,x,y,u) \longmapsto (t, x+(\varepsilon^2-\varepsilon)t-\varepsilon y, y-\varepsilon t, u+\varepsilon t),\\ \exp(\varepsilon v_2) &: (t,x,y,u) \longmapsto (t, x-\varepsilon t, y-\varepsilon, u+\varepsilon),\\ \exp(\varepsilon v_3) &: (t,x,y,u) \longmapsto (t, x+\varepsilon, y, u),\\ \exp(\varepsilon v_4) &: (t,x,y,u) \longmapsto (t+\varepsilon, x, y, u).\end{aligned}$$

Also, combining of the above transformations produces G 's elements, and this completes the proof. □

On the other, two s-dimensional symmetry subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ corresponded to H_1 and H_2 symmetry subgrous are conjugate, if for an element g of the symmetry group, the relation $H_2 = gH_1g^{-1}$ is existing. The equivalence categories of this relation are the optimal systems of symmetry subalgebras. Finding the optimal systems of the ZK's subalgebras gives:

Theorem 2.2. *The optimal systems of the ZK's symmetry subalgebras are according to* (*ref to* [2])*:*

$$\begin{aligned}\Theta_1 &= \{\langle v_3\rangle, \langle v_1+v_3\rangle, \langle v_2+v_3\rangle, \langle v_3+v_4\rangle, \langle v_1+v_2+v_3\rangle, \langle v_1+v_3+\frac{1}{2}v_4\rangle,\\ &\quad \langle v_2+v_3+v_4\rangle, \langle v_1+v_2+\frac{1}{4}v_3+\frac{1}{2}v_4\rangle\},\\ \Theta_2 &= \{\langle v_1+v_2+v_4, v_3\rangle\},\\ \Theta_3 &= \{v_1+v_4, v_2, v_3\rangle\}.\end{aligned}$$

Here, firstly the invariants associated with the members of the ZK's optimal systems are computed by integrating of the characteristic equations. Then with the use of invariants the reduction formulas are obtained, which with substitution of them the ZK's reductions are deduced.

The results for $\langle v_3\rangle = \langle \partial_x\rangle$ are gathered as Table 1.We explain a little about Table 1: By integrating of the characteristic equation $\frac{dt}{0} = \frac{dx}{1} = \frac{dy}{0} = \frac{du}{0}$, the corresponding ordinary invariants t, y, u are followed. This invariants results the reduction formulas $u(t,x,y) = s(w,z), t = w, y = z$, that substituting of them results Δ_1.

The results related to the rest members of the low dimensional optimal systems are listed in Tables 2-9 (ref to [2]):

Table 1: Reductions and similarity solutions based on $\langle v_3 \rangle$.

Stage 1: by $\langle v_3 \rangle$	Ordinary invariants of $\langle v_3 \rangle$: $t,\ y,\ u$ Reduction formulas: $u(t,x,y) = s(w,z), \quad t = w, \quad y = z$ Reduced equation: $\Delta_1 : s_{zz} = 0$ Similarity solution: $s = F_1(w)z + F_2(w)$, where F_1, F_2 are arbitrary C^∞ functions

Table 2: Reductions and similarity solutions based on $\langle v_3 + v_4 \rangle$.

Stage 1: by $\langle v_3 + v_4 \rangle$	Ordinary invariants of $\langle v_3 + v_4 \rangle$: $x - t, y, u$ Reduction formulas: $u(t,x,y) = s(w,z), \quad t = x - w, \quad y = z$ Reduced equation: $\Delta_4 : (s+1)s_{ww} + s_w^2 + s_{zz} = 0$ Symmetries of Δ_4 : $\beta_1 = w\partial_w + \frac{1}{2}z\partial_z + (1+s)\partial_s$, $\beta_2 = \partial_z,\ \beta_3 = \partial_w$
Stage 2-I: by $\langle \beta_1 - \beta_2 + \frac{1}{2}\beta_3 \rangle$	Ordinary invariants of $\langle \beta_1 - \beta_2 + \frac{1}{2}\beta_3 \rangle$: $\frac{z-2}{\sqrt{2w+1}},\ \frac{s+1}{2w+1}$ Reduction formulas: $s(w,z) = v(f)(1+2w) - 1$, $z = f\sqrt{1+2w} + 2$ Reduced equation: $\Delta_{4.1} : (f^2 v + 1)v_{ff} - 5fvv_f + f^2 v_f^2 + 4v^2 = 0$ Similarity solution: very huge and abnormal
Stage 2-II: by $\langle \beta_2 \rangle$	Ordinary invariants of $\langle \beta_2 \rangle$: w, s Reduction formulas: $s(w,z) = v(f),\ w = f$ Reduced equation: $\Delta_{4.2} : (v+1)v_{ff} + v_f^2 = 0$ Similarity solution: $v = -\sqrt{c_1 f + c_2} - 1$
Stage 2-III: by $\langle \beta_3 \rangle$	Ordinary invariants of $\langle \beta_3 \rangle$: z, s Reduction formulas: $s(w,z) = v(f),\ z = f$ Reduced equation: $\Delta_{4.3} : v_{ff} = 0$ Similarity solution: $v = c_1 f + c_2$

Table 3: Reductions and similarity solutions based on $\langle v_1 + v_2 + v_3 \rangle$.

Stage 1: by $\langle v_1 + v_2 + v_3 \rangle$	Ordinary invariants of $\langle v_1 + v_2 + v_3 \rangle$: $t, -x + \frac{1}{2}y(1+t) + \frac{1}{4}\frac{-6y+y^2}{1+t}, y+u$ Reduction formulas: $u(t,x,y) = s(w,z) - y,\ t = w,$ $x = -z + \frac{1}{2}y(1+w) + \frac{1}{4}\frac{-6y+y^2}{1+w}$ Reduced equation: $\Delta_5 : (2(w+1)^2 s - (w^2+2w-2)^2)s_{zz}$ $+2(w+1)s_{wz} + 2(w+1)s_z^2 + s_z = 0$ Symmetries of Δ_5 : $\gamma_1 = \left(z - \frac{1}{4}\left(\frac{1}{4}w^4 + w^3 - \frac{3}{2}w^2 - 5w\right) + 9\ln(w+1)\right)\partial_z + s\partial_s,$ $\gamma_2 = \partial_z.$
Stage 2-I:	Ordinary invariants of $\langle \gamma_1 \rangle$: $w, \frac{s}{-w^4-4w^3+6w^2+144\ln(w+1)+20w+16z}$ Reduction formulas: $s(w,z) = v(f)(16z - f^4 - 4f^3 + 6f^2 + 20f + 144\ln(f+1)),$ by $\langle \gamma_1 \rangle, w = f$ Reduced equation: $\Delta_{5.1} : 16(f+1)v_f + 256(f+1)v^2 + 8v = 0$ Similarity solution: $v = \frac{1}{32(f+1)+c_1\sqrt{f+1}}$
Stage 2-II: by $\langle \gamma_2 \rangle$	Ordinary invariants of $\langle \gamma_2 \rangle$: w, s Reduced equation: trivial

Table 4: Reductions and similarity solutions based on $\langle v_2 + v_3 + v_4 \rangle$.

Stage 1: by $\langle v_2 + v_3 + v_4 \rangle$	Ordinary invariants of $\langle v_2 + v_3 + v_4 \rangle$: $\frac{1}{2}t^2 - t + x, t + y, u - t$ Reduction formulas: $u(t,x,y) = s(w,z) + t,\ x = w + t - \frac{1}{2}t^2,$ $y = z - t$ Reduced equation: $\Delta_6 : s_{wz} - s_{zz} - (s+1)s_{ww} - s_w^2 = 0$ Symmetries of Δ_6 : $\lambda_1 = (z+4w)\partial_w + 2z\partial_z + (4s+3)\partial_s,$ $\lambda_2 = \partial_z,\ \lambda_3 = \partial_w$
Stage 2-I: by $\langle \lambda_1 \rangle$	Ordinary invariants of $\langle \lambda_1 \rangle$: $\frac{2w+z}{2z^2}, \frac{-4s-3}{4w+2z}$ Reduction formulas: $s(w,z) = -fz^2 v(f) - \frac{3}{4},\ w = fz^2 - \frac{1}{2}$ Reduced equation: $\Delta_{6.1} : (4f^3 - f^2 v)v_{ff} + (6f^2 - 4fv)v_f - f^2 v_f^2 - v^2 = 0$ Similarity solution: very huge and abnormal
Stage 2-II: by $\langle \lambda_2 \rangle$	Ordinary invariants of $\langle \lambda_2 \rangle$: w, s Reduction formulas: $s(w,z) = v(f),\ w = f$ Reduced equation: $\Delta_{6.2} : (v+1)v_{ff} + v_f^2 = 0$ Similarity solution: $v = -\sqrt{c_1 f + c_2} - 1$
Stage 2-III: $\langle \lambda_3 \rangle$.	Ordinary invariants of $\langle \lambda_3 \rangle$. : z, s Reduction formulas: $s(w,z) = v(f),\ w = z$ Reduced equation: $\Delta_{6.3} : v_{ff} = 0$ Similarity solution: $v = c_1 f + c_2$

Table 5: Reductions and similarity solutions based on $\langle v_1 + v_3 \rangle$.

Stage 1: by $\langle v_1 + v_3 \rangle$	Ordinary invariants of $\langle v_1 + v_3 \rangle$: $t, -x - \frac{y}{t} + \frac{1}{4}\frac{y^2}{t} + \frac{1}{2}ty, y + u$ Reduction formulas: $u(t,x,y) = s(w,z) - y$, $t = w,\ x = -z - \frac{y}{w} + \frac{1}{4}\frac{y^2}{w} + \frac{1}{2}wy$ Reduced equation: $\Delta_2 : (4w^2 s + (w^2-2)^2)s_{zz} + 4w^2 s_{wz} + 4w^2 s_z^2 + 2ws_z = 0$ Symmetries of Δ_2 : $\omega_1 = w\partial_w - \frac{1}{4}\frac{(w^4-4w^2+4)}{w}\partial_z + s\partial_s$, $\omega_2 = \partial_z$, $\omega_3 = w\partial_w + \frac{1}{6}\frac{(6zw+w^4+12)}{w}\partial_z$
Stage 2-I: by $\langle \omega_1 \rangle$	Ordinary invariants of $\langle \omega_1 \rangle$: $\frac{1}{12}(\frac{-w^4+12w^2+12zw+12}{w}),\ sw$ Reduction formulas: $s(w,z) = \frac{v}{w}, z = f + \frac{1}{12}w^3 - \ln(w) - \frac{1}{w} - w$ Reduced equation: $\Delta_{2.1} : 2v_{ff}v - v_f + 2v_f^2 = 0$ Similarity solution: $v = c_1\Big(LambertW\left(\frac{1}{c_1}e^{\frac{1}{2c_1}(f+c_2)}\right) + 1\Big)$
Stage 2-II: by $\langle \omega_1 + \omega_2 \rangle$	Ordinary invariants of $\langle \omega_1 + \omega_2 \rangle$: $\frac{1}{12}(\frac{-w^4+12w\ln w+12w^2+12zw+12}{w}),\ sw$ Reduction formulas: $s(w,z) = \frac{v}{w}, z = f + \frac{1}{12}w^3 - \ln(w) - \frac{1}{w} - w$ Reduced equation: $\Delta_{2.2} : 2(v+1)v_{ff} - v_f + 2v_f^2 = 0$ Similarity solution: $v = c_1\left(LambertW\left(\frac{1}{c_1}e^{\frac{1}{c_1}(\frac{1}{2}(f+c_2)+1)-1}\right)\right) + c_1 - 1$
Stage 2-III: by $\langle \omega_1 + \omega_3 \rangle$	Ordinary invariants of $\langle \omega_1 + \omega_3 \rangle$: $w,\ \frac{s}{w^4-12w^2-12zw-12}$ Reduction formulas: $s(w,z) = v(f)(f^4 - 12f^2 - 12zf - 12),\ w = f$ Reduced equation: $\Delta_{2.3} : 12f^3 v_f - 144f^4v^2 + 19f^2 v = 0$ Similarity solution: $v = \frac{1}{-24f^2 + c_1 f^{3/2}}$

Table 6: Reductions and similarity solutions based on $\big\langle v_1 + v_2 + v_4, v_3 \big\rangle$.

Stage 1: by $\big\langle v_1 + v_2 + v_4, v_3 \big\rangle$	Ordinary invariants of $\langle v_1 + v_2 + v_4, v_3 \rangle$: $\frac{1}{2}t^2 + t + y, -\frac{1}{2}t^2 - t + u$ Reduction formulas: $u(t,x,y) = s(w) + \left(\frac{1}{2}t^2 + t\right)$, $y = w - \frac{1}{2}t^2 - t$ Reduced equation: $\Delta_9 : s_{ww} = 0$ Similarity solution: $v = c_1 w + c_2$

Table 7: Reductions and similarity solutions based on $\langle v_2 + v_3 \rangle$.

Stage 1: by $\langle v_2 + v_3 \rangle$	Ordinary invariants of $\langle v_2 + v_3 \rangle$: $t, y - \frac{x}{t-1}, u + \frac{x}{t-1}$ Reduction formulas: $u(t,x,y) = s(w,z) - \frac{x}{w-1}, \quad t = w,$ $y = z + \frac{x}{w-1}$ Reduced equation: $\Delta_3 : (w-1)s_{wz} + (s + (w-1)^2)s_{zz} + s_z^2 + s_z = 0$ Symmetries of Δ_3: $\alpha_1 = \left(\frac{1}{2}w - \frac{1}{2}\right)\partial_w + z\partial_z + s\partial_s,$ $\alpha_2 = \left(\frac{1}{2}w - \frac{1}{2}\right)\partial_w + \left(\frac{1}{2}w^2 - w + 1\right)\partial_z,\ \alpha_3 = \partial_z$
Stage 2-I: by $\langle \alpha_1 - \alpha_2 + \frac{1}{2}\alpha_3 \rangle$	Ordinary invariants of $\langle \alpha_1 - \alpha_2 + \frac{1}{2}\alpha_3 \rangle$: $w, \frac{s}{w^2-2w-2z+1}$ Reduction formulas: $s(w,z) = v(f)(f^2 - 2f + 1 - 2z),\ w = f$ Reduced equation: $\Delta_{3.1} : (1-f)v_f + 2v^2 - v = 0$ Similarity solution: $v = \frac{1}{c_1(f-1)+2}$
Stage 2-II: by $\langle \alpha_2 \rangle$	Ordinary invariants of $\langle \alpha_2 \rangle$: $-\frac{1}{2}w^2 - \ln(w-1) + w - z, s$ Reduction formulas: $s(w.z) = v(f),\ z = \left(-f + \frac{1}{2}(w-1)^2 + \ln(w-1)\right)$ Reduced equation: $\Delta_{3.2} : (v-1)v_{ff} + v_f^2 - v_f = 0$ Similarity solution: $v = c_1\left(LambertW\left(\frac{e^{\frac{1}{c_1}-1}}{c_1 e^{-\frac{1}{c_1}(f+c_2)}}\right) + 1\right) + 1$
Stage 2-III: by $\langle \alpha_3 \rangle$	Ordinary invariants of $\langle \alpha_3 \rangle$: w, s Reduction formulas: $s(w,z) = v(f),\ w = f$ Reduced equation: trivial

Table 8: Reductions and similarity solutions based on $\langle v_1 + v_3 + \frac{1}{2}v_4 \rangle$.

Stage 1: by $\langle v_1 + v_3 + \frac{1}{2}v_4 \rangle$	Ordinary invariants of $\langle v_1 + v_3 + \frac{1}{2}v_4 \rangle$: $y + t^2,$ $-2t + t(y + t^2) + x, -t^2 + u$ Reduction formulas: $u(t,x,y) = s(w,z) + t^2,$ $x = z + 3t - tw,\ y = w - t^2$ Reduced equation: $\Delta_7 : (s - w + 3)s_{zz} + s_{ww} + s_z^2 = 0$ Symmetries of Δ_7 : $\kappa_1 = \partial_w + \partial_s,$ $\kappa_2 = \left(\frac{1}{2}w - 3\right)\partial_w + z\partial_z + \left(s - \frac{1}{2}w\right)\partial_s,$ $\kappa_3 = \partial_z$
Stage 2-I: by $\langle \kappa_1 \rangle$	Ordinary invariants of $\langle \kappa_1 \rangle$: $z, s - w$ Reduction formulas: $s(w,z) = v(f) + w,\ z = f$ Reduced equation: $\Delta_{7.1} : (v+3)v_{ff} + v_f^2 = 0$ Similarity solution: $v = -\sqrt{c_1 f + c_2} - 3$
Stage 2-II: by $\langle \kappa_2 \rangle$	Ordinary invariants of $\langle \kappa_2 \rangle$: $\frac{z}{(w-6)^2}, \frac{s-w+3}{w^2-12w+36}$ Reduction formulas: $s(w,z) = v(f)(w^2 - 12w + 36) + w - 3,$ $z = f(w-6)^2$ Reduced equation: $\Delta_{7.2} : 4\left(f^2 + \frac{1}{4}v\right)v_{ff} + v_f^2 - 2fv_f + 2v = 0$ Similarity solution: very huge and abnormal
Stage 2-III: by $\langle \kappa_3 \rangle$	Ordinary invariants of $\langle \kappa_3 \rangle$: w, s Reduction formulas: $w = f, s(w,z) = v(f)$ Reduced equation: $\Delta_{7.3} : v_{ff} = 0$ Similarity solution: $v = c_1 f + c_2$

Table 9: Reductions and similarity solutions based on $\left\langle v_1 + v_2 + \frac{1}{4}v_3 + \frac{1}{2}v_4 \right\rangle$.

Stage 1: by $\left\langle v_1 + v_2 + \frac{1}{4}v_3 + \frac{1}{2}v_4 \right\rangle$	Ordinary invariants of $\left\langle v_1 + v_2 + \frac{1}{4}v_3 + \frac{1}{2}v_4 \right\rangle$: $t^2 + 2t + y$, $-\frac{1}{2}t + t(t^2 + 2t + y) + x$, $-t^2 - 2t + u$ Reduction formulas: $u(t, x, y) = s(w, z) + t^2 + 2t$, $x = z + \frac{1}{2}t - tw$, $y = w - t^2 - 2t$ Reduced equation: $\Delta_8 : \frac{1}{2}(2s - 2w + 1)s_{zz} - 2s_{wz} + s_{ww} + s_z^2 = 0$ Symmetries of Δ_8 : $\zeta_1 = \partial_w + \partial_s$, $\zeta_2 = \frac{1}{2}(w + 1)\partial_w + \left(z + \frac{1}{2}w\right)\partial_z + \left(s - \frac{1}{2}w\right)\partial_s$, $\zeta_3 = \partial_z$
Stage 2-I: by $\langle \zeta_1 \rangle$	Ordinary invariants of $\langle \zeta_1 \rangle$: $z, s - w$ Reduction formulas: $s(w, z) = v(f) + w,\ z = f$ Reduced equation: $\Delta_{8.1} : \frac{1}{2}(2v + 1)v_{ff} + v_f^2 = 0$ Similarity solution: $v = -\frac{1}{2}\sqrt{c_1 f + c_2} - \frac{1}{2}$
Stage 2-II: by $\langle \zeta_2 \rangle$	Ordinary invariants of $\langle \zeta_2 \rangle$: $\frac{1+2(w+z)}{2(w+1)^2}, \frac{2s-2w-1}{2(w+1)^2}$ Reduction formulas: $s(w, z) = v(f)(w^2 + 2w + 1) + w + \frac{1}{2}$, $z = f(w^2 + 2w + 1) - w - \frac{1}{2}$ Reduced equation: $\Delta_{8.2} : \left(v + 4f^2\right)v_{ff} - 2fv_f + v_f^2 + 2v = 0$ Similarity solution: very huge and abnormal
Stage 2-III: by $\langle \zeta_3 \rangle$	Ordinary invariants of $\langle \zeta_3 \rangle$: w, s Reduction formulas: $s(w, z) = v(f),\ w = f$ Reduced equation: $\Delta_{8.3} : v_{ff} = 0$ Similarity solution: $v = c_1 f + c_2$

3. Conclusion

This research devotes to introducing a well-known nonlinear acoustics model: 3D-ZK equation, and illustrating its similarity reductions and solutions. The solutions may be used to clarifying the propagation of a bounded two-dimensional acoustic beam in nonlinear medias.

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Leila Hamedi Mobarra
Department of Mathematics,
Fouman and Shaft Branch,
Islamic Azad University,
Fouman, I. R. Iran
e-mail: leila.hamedimobarra@fshiau.ac.ir