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More on the Enumeration of Some Kind of Dominating Sets in Cactus Chains

Somayeh Jahari and Saeid Alikhani⋆

Abstract

A non-empty set $S \subseteq V$ is a dominating set, if every vertex not in S is adjacent to at least one vertex in S, and S is a total dominating set, if every vertex of V is adjacent to some vertices of S. We enumerate dominating sets, non-split dominating sets and total dominating sets in several classes of cactus chains.

Keywords: dominating sets, total dominating sets, generating function, cactus graphs, i-uniform.

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1. Introduction

Let $G = (V, E)$ be a simple graph. A non-empty set $S \subseteq V$ is a dominating set, if every vertex not in S is adjacent to at least one vertex in S, and S is a total dominating set, if every vertex of V is adjacent to some vertices of S. The domination number (total domination number) of the graph G, denoted by $\gamma(G)$ ($\gamma_t(G)$), is the minimum cardinality of all dominating sets (total dominating sets) of G. For a detailed treatment of domination theory, the reader is referred to [14]. Graph polynomials are the generating function for the number of subsets of vertices such S (or edges) such that S has a particular graph property. The

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concepts of enumeration of dominating sets in graphs were described about ten years ago, by introducing the domination polynomial of a graph. The domination polynomial $D(G, x)$ of G is defined as

$$D(G,x)=\sum_{i\geq 0} d(G,i)x^i,$$

where $d(G, i)$ is the number of dominating sets of G of cardinality i. This graph polynomial was introduced in the paper [3] that appeared in 2014 but numerous other papers on the polynomial appeared earlier.

Most of graph polynomials satisfies a linear recurrence relation, where graphs in the terms of recurrence are subgraphs which obtain from the graph using various vertex and edge elimination operations. For example almost all graph polynomials in the literature satisfy recurrence relations with respect to vertex and edge elimination operations, among them the matching polynomial, the independence polynomial, the chromatic polynomial and the vertex-cover polynomial, see e.g. [6]. Kotek et.al in [17] shown that the domination polynomial, $D(G, x)$ does not satisfy any linear recurrence relation which applies only the commonly used vertex operations of deletion, extraction, contraction and neighborhood-contraction. Nor does $D(G, x)$ satisfy any linear recurrence relation using only edge deletion, contraction and extraction. In [11] it is shown that computing the domination polynomial of a graph is NP-hard. So study of graphs whose domination polynomials satisfies simple recurrence relation has worth. Because of these reasons, in this paper we consider graphs with specific structures and study the number of dominating sets. The roots of graph polynomials reflect some important information about the structure of graphs. There are many papers on the location of the roots of graph polynomials such as chromatic polynomial, matching polynomial, independence polynomial, characteristic polynomial, domination polynomial and total domination polynomial. For example, in [1], there is a conjecture which states that every integer root of $D(G, x)$ is -2 or 0. Another natural question to ask is to what extent can a graph polynomial describe the underlying graph. Two graphs G and H are dominating equivalent or simply $\mathcal{D}$-equivalent (written $G \sim H$), if they have the same domination polynomial. As in [1], let $[G]$ denote the $\mathcal{D}$-equivalence class determined by G, that is $[G] = \{H|H \sim G\}$. A main problem arise: Can we determine the $\mathcal{D}$-equivalence class of a graph? To answer this main question, finding generating function and recurrence relation for the domination polynomial is necessary. For more information, refer to [1, 4, 5].

After counting dominating sets, the number of other kinds of dominating sets has studied [2, 7], especially the number of total dominating sets and independent dominating sets has studied well, see e.g. [9, 10]. The concept of nonsplit domination was introduced by Kulli and Janakiram [19]. In [19], authors obtained some bounds on the nonsplit domination number of a graph. A dominating set D of a graph G is called a nonsplit dominating set if the induced graph $\langle V \setminus D\rangle$ is connected. The nonsplit domination number $\gamma_{ns}(G)$ of the graph G is the minimum cardinality of a nonsplit domination set.

Since counting the number of dominating sets is # P-complete, even in restricted graph classes such as, e.g., split graphs and bipartite chordal graphs [18], so it is natural to consider the classes with specific constructions to obtain the number of their dominating sets. In this paper we consider graphs with simple connectivity patterns, for example cacti.

Cactus graphs were first known as Husimi trees; they appeared in the scientific literature some sixty years ago in papers by Husimi and Riddell concerned with cluster integrals in the theory of condensation in statistical mechanics [13, 16, 20]. In the meantime, they also found applications in chemistry [15, 22] and in the theory of electrical and communication networks [21], when it turned out that some computationally difficult problems can be solved on cacti in polynomial time. We refer the reader to papers [8] for some aspects of domination in cactus graphs and to [12] for some enumerative results on matchings and independent sets in chain cacti [7].

A cactus graph is a connected graph in which no edge lies in more than one cycle. Consequently, each block of a cactus graph is either an edge or a cycle. If all blocks of a cactus G are cycles of the same size m, the cactus is m-uniform.

The paper is ordered as follows: In the next section, the number of total dominating sets of triangular cactus and the number of total dominating sets of chain of squares were considered and in Section 3, the number of nonsplit dominating sets of chain of cacti graphs was investigated. In the last section, we study the number of dominating sets of chain of hexagonal cacti.

2. Counting Total Dominating Sets in Cactus Chains

In this section, the triangular cactus and chain of squares were considered and we investigate the number of total dominating sets in these graphs.

2.1. Triangular Chain T_n

Let us consider the way of labeling T_n in Figure 1, and symbolize the number of total dominating sets in T_n by t_n. Each total dominating set in T_n either does or does not contain vertex u_n. If the number of total dominating sets that contain u_n be represented by t'_n, and by t''_n the number of total dominating sets that do not contain u_n, we will have $t_n = t'_n + t''_n$.

Now we find recurrences for t'_n and t''_n.

It is clear that each total dominating set in T_n counted by t''_n can be extended to a total dominating set in T_{n+1} counted by t'_{n+1}. Moreover, a total dominating set in T_n counted by t'_n can be extended to a total dominating set in T_{n+1} counted by t'_{n+1} in only two ways. In addition, there are the number of sets that are dominating and not total dominating in T_n, but can be extended to a total

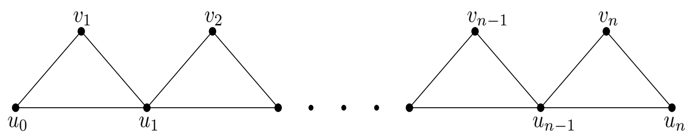

Figure 1: The chain triangular cactus.

dominating set in T_{n+1}. Clearly, such sets must include a single vertex u_n, and they are counted by t''_{n-1}, and can be extended to a total dominating set in T_{n+1} counted by t'_{n+1} in only two ways. Hence, we have $t'_{n+1} = t''_n + 2t'_n + 2t''_{n-1}$. Each total dominating set in T_n counted by t'_n can be extended to a total dominating set in T_{n+1} counted by t''_{n+1} in only two ways. Further, a total dominating set in T_{n-1} counted by t''_{n-1} can be extended to a total dominating set in T_{n+1} counted by t''_{n+1} in only one way, by including u_n and v_{n+1}. Hence,

$$t''_{n+1} = 2t'_n + t''_{n-1}.$$

The following system is obtained:

$$\begin{aligned} t'_{n+1} &= t''_n + 2t'_n + 2t''_{n-1}, \\ t''_{n+1} &= 2t'_n + t''_{n-1}. \end{aligned}$$

with the initial conditions $t'_1 = 3$ and $t''_1 = 1$.

Now we introduce two generating functions, $T'(x) = \sum_{n\geq 0} t'_{n+1}x^n$ and $T''(x) = \sum_{n\geq 0} t''_{n+1}x^n$. By multiplying both equations in the above system through by x^n and then summing over $n \geq 0$, the system can be translated into a linear system for two unknown generating functions. Considering $t'_0 = 1$, we have the following;

$$\begin{array}{rcl} (1-2x)T'(x) & - & (x+2x^2)T''(x) & = & 2 \\ (1-x^2)T''(x) & - & 2xT'(x) & = & 2. \end{array}$$

We obtain

$$T'(x) = \frac{2(1+x+x^2)}{1-2x-3x^2-2x^3}, \quad T''(x) = \frac{2}{1-2x-3x^2-2x^3}.$$

At last, by adding $T'(x)$ and $T''(x)$ and multiplying the sum by x we obtain the generating function for the sequence t_n. So we have the following theorem:

Theorem 2.1. *The generating function for the number of total dominating sets of T_n is yield by*

$$T(x) = \frac{1+2x-x^2}{1-2x-3x^2-2x^3}.$$

Because $T(x)$ is a rational function, it can be concluded that the numbers t_n satisfy a second order linear recurrence with constant coefficients. The initial conditions can be verified by direct computations. The following corollary gives the recurrence relation of t_n. The following corollary gives the recurrence relation of t_n.

Corollary 2.2. *For every $n \geq 4$, the number of total dominating sets in T_n, i.e., t_n is given by*

$$t_n = 2t_{n-1} + 3t_{n-2} + 2t_{n-3},$$

with the initial conditions $t_1 = 4, t_2 = 16$ and $t_3 = 46$.

2.2. Para-Chain Q_n

We consider a para-chain of length n, labeled as shown in Figure 2. We investigate the generating function for the number of total dominating sets of Q_n. Let state and prove the following theorem:

Theorem 2.3. *The generating function for the number of total dominating sets of Q_n is yield by*

$$Q(x) = \frac{1 + 3x + 6x^2}{1 - 3x - 18x^2 - 9x^3 + 9x^4}.$$

Proof. Consider the way of labeling Q_n in Figure 2 and symbolize the number of total dominating sets in Q_n by q_n. Each total dominating set in Q_n either does or does not contain vertex v_n. If the number of total dominating sets that contain v_n be represented by q'_n, and by q''_n the number of total dominating sets that do not contain v_n, we will have $q_n = q'_n + q''_n$.

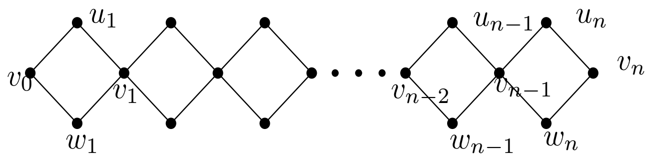

Figure 2: Labeled para-chain square cactus graphs.

Now we find recurrences for q'_n, q''_n. Each total dominating set in Q_n counted by q'_n and q''_n can be extended to a total dominating set in Q_{n+1} counted by q'_{n+1} in exactly three ways. In addition, there are the number of sets that are dominating and not total dominating in Q_n, but can be extended to a total dominating set in Q_{n+1}. Clearly, such sets must include a single vertex v_n, and they are counted by q''_{n-1} and q'_{n-1}, and can be extended to a total dominating set in Q_{n+1} counted by q'_{n+1} in only three ways. Further, there are the number of sets that are not total dominating in Q_n, but can be extended to a total dominating set in Q_{n+1}. Clearly, such sets do not dominate v_n, and they must include v_{n-1}, since the existence of this vertex is necessary to dominate u_n and w_n. Hence, they are counted by q'_{n-1}

and can be extended to a total dominating set in Q_{n+1} counted by q'_{n+1} in only three ways. We have the recurrence for q'_n,

$$q'_{n+1} = 3q'_n + 3q''_n + 3q''_{n-1} + 6q'_{n-1}.$$

Now we need a recurrence for q''_n. Each total dominating set in Q_n counted by q'_n can be extended to a total dominating set in Q_{n+1} counted by q''_{n+1} in exactly three ways. Each total dominating set in Q_n counted by q''_{n-1} can be extended to a total dominating set in Q_{n+1} counted by q''_{n+1} in exactly three ways, and the same is valid for the sets counted by q'_{n-1} which contain a single vertex v_n. Hence,

$$q''_{n+1} = 3q'_n + 3q''_{n-1} + 3q'_{n-1}.$$

Finally, we have obtained the system

$$\begin{aligned} q'_{n+1} &= 3q''_n + q''_{n+1} + 3q'_{n-1}, \\ q''_{n+1} &= 3q'_n + 3q''_{n-1} + 3q'_{n-1}. \end{aligned}$$

with the initial conditions $q'_1 = 6$ and $q''_1 = 3$.

Again, we introduce three generating functions, $Q'(x) = \sum_{n\geq 0} q'_{n+1}x^n$ and $Q''(x) = \sum_{n\geq 0} q''_{n+1}x^n$. By multiplying all equations in the above system through by x^n and then summing over $n \geq 0$, the system can be translated into a linear system for two unknown generating functions. Considering $q'_0 = 1$, we have the following;

$$\begin{aligned} (1-3x^2)Q'(x) &- (1+3x)Q''(x) = 3x, \\ (1-3x^2)Q''(x) &- (3x+3x^2)Q'(x) = 3+3x. \end{aligned}$$

We obtain

$$Q'(x) = \frac{3+15x+9x^2-9x^3}{1-3x-18x^2-9x^3+9x^4}, \quad Q''(x) = \frac{3+3x}{1-3x-18x^2-9x^3+9x^4}.$$

Finally, by adding $Q'(x)$ and $Q''(x)$ and multiplying the sum by x we obtain the generating function for the sequence q_n. □

The following corollary gives the recurrence relation of q_n.

Corollary 2.4. *For every $n \geq 5$, the number of total dominating sets in Q_n, i.e., q_n is given by*

$$q_n = 3q_{n-1} + 18q_{n-2} + 9q_{n-3} - 9q_{n-4},$$

with the initial conditions $q_1 = 9,\ q_2 = 45,\ q_3 = 288$ and $q_4 = 1755$.

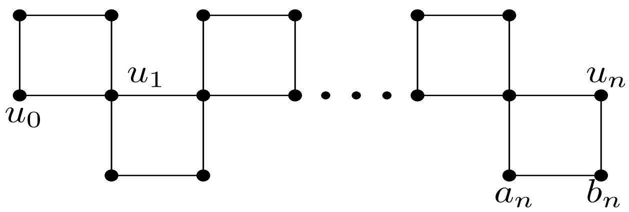

Figure 3: Labeled ortho-chain square S_n.

2.3. Ortho-Chain S_n

Consider the way of labeling S_n in Figure 3 and symbolize the number of total dominating sets in S_n by s_n. If the number of total dominating sets that contain u_n be represented by s'_n, and by s''_n the number of total dominating sets that do not contain s_n, we will have $s_n = s'_n + s''_n$.

Analogously, before we obtain the system of recurrences for s'_n, s''_n; have the system

$$s'_{n+1} = 4s'_n + 2s''_n,$$
$$s''_{n+1} = 2s'_n + s''_n.$$

with the initial conditions $s'_1 = 5$ and $s''_1 = 3$. Again, we introduce the corresponding generating functions, $S'(x) = \sum_{n\geq 0} s'_{n+1}x^n$, $S''(x) = \sum_{n\geq 0} s''_{n+1}x^n$ and obtain a linear system for them;

$$\begin{array}{rcrcl} S'(x) & - & 2S''(x) & = & 0 \\ (1-x)S''(x) & - & 2xS'(x) & = & 2. \end{array}$$

We obtain

$$S'(x) = \frac{4}{1-5x}, \quad S''(x) = \frac{2}{1-5x}.$$

Finally, by adding $S'(x)$ and $S''(x)$ and multiplying the sum by x we obtain the generating function for the sequence s_n and $S(x) = \sum_{n\geq 0} s_n x^n$. So we have the following result:

Theorem 2.5. *The generating function for the number of total dominating sets of S_n is yield by*

$$S(x) = \frac{1+x}{1-5x}.$$

The following corollary gives the recurrence relation of s_n.

Corollary 2.6. *For every $n \geq 2$, the number of total dominating sets in S_n, i.e., s_n is given by*

$$s_n = 5s_{n-1},$$

with the initial conditions $s_1 = 9$.

The sequence of the numbers $s_1, s_2, \ldots$ in the ortho-chain graphs, S_n, satisfies the conditions

$$s_{n+1} = 5s_n, \qquad (n \geq 1,\ s_1 = 9).$$

So for each natural number $n \geq 1$ have $s_n = 9 \cdot 5^{n-1}$.

3. Counting Nonsplit Dominating Sets in Cacti Chains

In this section, we investigate the number of nonsplit dominating sets in some classes of chain cacti graphs in the pervious part.

3.1. Triangular Chain T_n

Let us consider the way of labeling T_n in Figure 1, and we shall obtain a recurrence relation for the number of nonsplit dominating sets in T_n. For this purpose we symbolize the number of nonsplit dominating sets in T_n by t_n. Each nonsplit dominating set in T_n either does or does not contain vertex u_n. If the number of nonsplit dominating sets that contain u_n be represented by t'_n, and by t''_n the number of nonsplit dominating sets that do not contain u_n, we will have $t_n = t'_n + t''_n$.

Now we find recurrences for t'_n and t''_n.

It is clear that each nonsplit dominating set in T_n counted by t''_n can be extended to a nonsplit dominating set in T_{n+1} counted by t'_{n+1} in only two ways. Moreover, a nonsplit dominating set in T_n counted by t'_n can be extended to a nonsplit dominating set in T_{n+1} counted by t'_{n+1} in only one way, by including u_n and v_n. Also, there is one possibility of a nonsplit dominating set in T_n counted by t'_n can be extended to a nonsplit dominating set in T_{n+1} counted by t'_{n+1} by including u_n, and such set must include all vertices of T_n. Hence, we have

$$t'_{n+1} = 2t''_n + t'_n + 1.$$

Now we need a recurrence for t''_{n+1}. There are two possibility cases of a nonsplit dominating set in T_n counted by t'_n can be extended to a nonsplit dominating set in T_{n+1} counted by t''_{n+1}, and such sets must include all vertices of T_n. Further, a nonsplit dominating set in T_n counted by t''_n can be extended to a nonsplit dominating set in T_{n+1} counted by t''_{n+1} in only one way, by including v_n. Hence,

$$t''_{n+1} = 2 + t''_n.$$

The following system is obtained:

$$\begin{aligned} t'_{n+1} &= 2t''_n + t'_n + 1, \\ t''_{n+1} &= 2 + t''_n \end{aligned}$$

with the initial conditions $t'_1 = 4$ and $t''_1 = 3$.

Now we introduce two generating functions, $T'(x) = \sum_{n\geq 0} t'_{n+1}x^n$ and $T''(x) = \sum_{n\geq 0} t''_{n+1}x^n$. By multiplying both equations in the above system through by x^n and then summing over $n \geq 0$, the system can be translated into a linear system for two unknown generating functions. Considering $t'_0 = 1$, we have the following:

$$\begin{array}{rcll} (1-x)T'(x) & - & 2xT''(x) & = \frac{2-x}{1-x} \\ (1-x)T''(x) & & & = \frac{2}{1-x} \end{array}$$

We obtain

$$T'(x) = \frac{2+x+x^2}{1-3x+3x^2-x^3}, \quad T''(x) = \frac{2}{1-2x+x^2}.$$

At last, by adding $T'(x)$ and $T''(x)$ and multiplying the sum by x we obtain the generating function for the sequence t_n. So we have the following theorem:

Theorem 3.1. *The generating function for the number of nonsplit dominating sets of T_n is yield by*

$$T(x) = \frac{1+x+2x^2}{1-3x+3x^2-x^3}.$$

Because $T(x)$ is a rational function, it can be concluded that the numbers t_n satisfy a third order linear recurrence with constant coefficients. The initial conditions can be verified by direct computations. The following corollary gives the recurrence relation of t_n.

Corollary 3.2. *For every $n \geq 4$, the number of nonsplit dominating sets in T_n, i.e., t_n is given by*

$$t_n = 3t_{n-1} - 3t_{n-2} + t_{n-3},$$

with the initial conditions $t_1 = 7, t_2 = 16$ and $t_3 = 29$.

3.2. Para-Chain Q_n

Here, we investigate the generating function for the number of nonsplit dominating sets of Q_n. Consider the way of labeling Q_n in Figure 2 and symbolize the number of nonsplit dominating sets in Q_n by q_n. Each nonsplit dominating set in Q_n either does or does not contain vertex v_n. If the number of nonsplit dominating sets that contain v_n be represented by q'_n, and by q''_n the number of nonsplit dominating sets that do not contain v_n, we will have $q_n = q'_n + q''_n$.

Now we find recurrences for q'_n, q''_n. Each nonsplit dominating set in Q_n counted by q''_n can be extended to a nonsplit dominating set in Q_{n+1} counted by q'_{n+1} in exactly four ways. Further, a nonsplit dominating set, D, with cardinality $|V(Q_n)|$ in Q_n counted by q'_n can be extended to a nonsplit dominating set in Q_{n+1} counted by q'_{n+1} in only three ways; if this D has cardinality less than $|V(Q_n)|$ in Q_n

counted by q'_n can be extended to a nonsplit dominating set in Q_{n+1} counted by q'_{n+1} in only one way. These sets counted by q'_n. Therefore, the nonsplit dominating set $V(Q_{n+1})$ is double counted. In addition, there are the number of sets that are not dominating in Q_n, but can be extended to a nonsplit dominating set in Q_{n+1}. Clearly, such sets must include the vertex v_{n-1}, and do not dominate v_n. They can be extended to a nonsplit dominating set in Q_{n+1} counted by q'_{n+1} in only three ways. We have the recurrence for q'_n,

$$q'_{n+1} = q'_n + 4q''_n + 5.$$

Now we need a recurrence for q''_n. Each nonsplit dominating set in Q_n counted by q'_n with cardinality $|V(Q_n)|$ in Q_n can be extended to a nonsplit dominating set in Q_{n+1} counted by q''_{n+1} in only three ways. Hence,

$$q''_{n+1} = 3.$$

Finally, we have obtained the system

$$\begin{aligned} q'_{n+1} &= q'_n + 4q''_n + 5, \\ q''_{n+1} &= 3 \end{aligned}$$

with the initial conditions $q'_1 = 5$ and $q''_1 = 3$.

Again, we introduce three generating functions, $Q'(x) = \sum_{n\geq 0} q'_{n+1}x^n$ and $Q''(x) = \sum_{n\geq 0} q''_{n+1}x^n$. By multiplying all equations in the above system through by x^n and then summing over $n \geq 0$, the system can be translated into a linear system for two unknown generating functions. Considering $q'_0 = 1$, we have the following:

$$\begin{aligned} (1-x)Q'(x) &= \frac{18-x}{1-x}, \\ Q''(x) &= \frac{3}{1-x}. \end{aligned}$$

We obtain

$$Q'(x) = \frac{18-x}{1-2x+x^2}, \quad Q''(x) = \frac{3}{1-x}.$$

Finally, by adding $Q'(x)$ and $Q''(x)$ and multiplying the sum by x we obtain the generating function for the sequence q_n.

Theorem 3.3. *The generating function for the number of nonsplit dominating sets of Q_n is yield by*

$$Q(x) = \frac{1+19x-3x^2}{1-2x+x^2}.$$

The following corollary gives the recurrence relation of q_n.

Corollary 3.4. *For every $n \geq 3$, the number of nonsplit dominating sets in Q_n, i.e., q_n is given by*

$$q_n = 2q_{n-1} - q_{n-2},$$

with the initial conditions $q_1 = 9$ and $q_2 = 26$.

3.3. Ortho-Chain S_n

Consider the way of labeling S_n in Figure 3 and symbolize the number of nonsplit dominating sets in S_n by s_n. If the number of nonsplit dominating sets that contain u_n be represented by s'_n, and by s''_n the number of nonsplit dominating sets that do not contain s_n, we will have $s_n = s'_n + s''_n$. In addition, we denote by s'''_n the number of sets that are not dominating set in S_n, but can be extended to a nonsplit dominating set in S_{n+1}. Clearly, u_n is not dominated, then u_{n-1} and b_n are not contained in the nonsplit dominating sets. Such sets must include the vertex a_n to dominate b_n.

Analogously before, we obtain recurrences for s'_n, s''_n and s'''_n.

Each nonsplit dominating set in S_n counted by s''_n can be extended to a nonsplit dominating set in S_{n+1} counted by s'_{n+1} in exactly two ways. Further, a nonsplit dominating set, D, with cardinality $|V(S_n)|$ in S_n counted by s'_n can be extended to a nonsplit dominating set in S_{n+1} counted by s'_{n+1} in exactly four ways; if this D has cardinality less than $|V(S_n)|$ in S_n counted by s'_n can be extended to a nonsplit dominating set in S_{n+1} counted by s'_{n+1} in only one way. These sets counted by s'_n. Therefore, the nonsplit dominating set $V(S_{n+1})$ is double counted. Each set in S_n counted by s'''_n can be extended to a nonsplit dominating set in S_{n+1} counted by s'_{n+1} in only two ways.

Finally, we have the following recurrence for s'_n,

$$\begin{aligned} s'_{n+1} &= s'_n + 2s''_n + 4 - 1 + 2s'''_n \\ &= s'_n + 2s''_n + 3 + 2s'''_n. \end{aligned}$$

Now we need a recurrence for s''_n. Each nonsplit dominating set in S_n counted by s''_n can be extended to a nonsplit dominating set in S_{n+1} counted by s''_{n+1} in only two ways. Further, a nonsplit dominating set, D, with cardinality $|V(S_n)|$ in S_n can be extended to a nonsplit dominating set in S_{n+1} counted by s''_{n+1} in only two ways. Further, Each set in S_n counted by s'''_n can be extended to a nonsplit dominating set in S_{n+1} counted by s''_{n+1} in only one way by including a_{n+1} and b_{n+1}. Finally, we have the following recurrence for s'',

$$s''_{n+1} = 2s''_n + 2 + s'''_n.$$

Now we need a recurrence for s'''_n. Each nonsplit dominating set in S_n counted by s''_n can be extended to a nonsplit dominating set in S_{n+1} counted by s'''_{n+1} in only ony way by including a_{n+1}. Further, the same is valid for the sets counted by s'''_n. Hence

$$s'''_{n+1} = s''_n + s'''_n.$$

We have obtained the system of recurrences for s'_n, s''_n and s'''_n;

$$\begin{aligned} s'_{n+1} &= s'_n + 2s''_n + 3 + 2s'''_n, \\ s''_{n+1} &= 2s''_n + 2 + s'''_n, \\ s'''_{n+1} &= s''_n + s'''_n, \end{aligned}$$

with the initial conditions $s'_1 = 6$ and $s''_1 = 3$. Again, we introduce the corresponding generating functions, $S'(x) = \sum_{n\geq 0} s'_{n+1}x^n$, $S''(x) = \sum_{n\geq 0} s''_{n+1}x^n$ and $S'''(x) = \sum_{n\geq 0} s'''_{n+1}x^n$, we obtain a linear system for them;

$$\begin{array}{rcrcl} (1-x)S'(x) & - & 2S'''(x) & = & \frac{4-x}{1-x}, \\ (1-x)S''(x) & - & S'''(x) & = & \frac{2}{1-x}, \\ (1-x)S'''(x) & - & xS''(x) & = & 1. \end{array}$$

We obtain

$$S'''(x) = \frac{1+x^2}{1-4x+4x^2-x^3}, \quad S''(x) = \frac{3}{1-3x+x^2},$$

$$S'(x) = \frac{6-13x+9x^2-x^3}{1-5x+8x^2-5x^3+x^4}.$$

Finally, by adding $S'(x)$ and $S''(x)$ and multiplying the sum by x we obtain the generating function for the sequence s_n and $S(x) = \sum_{n\geq 0} s_n x^n$. So we have the following result:

Theorem 3.5. *The generating function for the number of nonsplit dominating sets of S_n is yield by*

$$S(x) = \frac{1+4x-11x^2+7x^3}{1-5x+8x^2-5x^3+x^4}.$$

The following corollary gives the recurrence relation of s_n.

Corollary 3.6. *For every $n \geq 5$, the number of nonsplit dominating sets in S_n, i.e., s_n is given by*

$$s_n = 5s_{n-1} - 8s_{n-2} + 5s_{n-3} - s_{n-4},$$

with the initial conditions $s_1 = 9$, $s_2 = 26$, $s_3 = 70$ and $s_4 = 186$.

4. Counting Dominating Sets in the Chain Hexagonal Cacti

Counting the dominating sets of some cactus chains has studied in [7], but there is no result for the number of dominating sets in the chain hexagonal cacti. In this section we investigate the number of dominating sets in three classes of chain hexagonal cacti. The ortho-chain of length n is denoted by O_n, and the meta-chain and the para-chain of length n are denoted by M_n and L_n, respectively.

Approach for enumeration of dominating sets of these three families are similar but we think should prove and state all details.

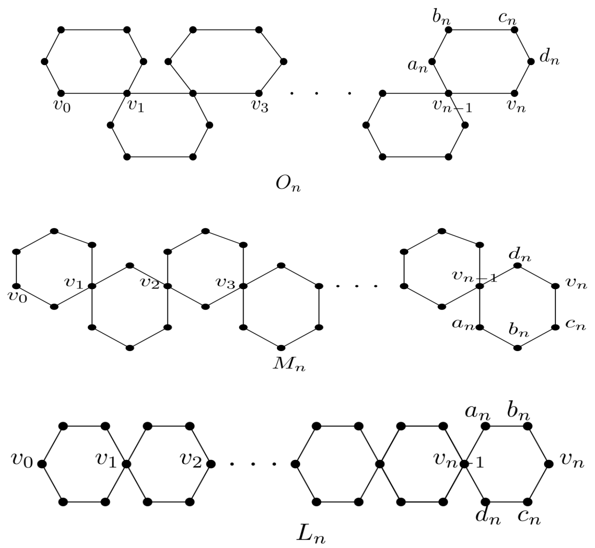

Figure 4: An ortho-, meta-, and para-chain hexagonal cacti of length n.

4.1. Ortho-Chain

We consider an ortho-chain hexagonal cacti of length n, labeled as shown in Figure 4. The number of dominating sets in O_n denoted by o_n, and the number of dominating set in O_n containing and not containing vertex v_n are denoted by o'_n and o''_n. Again, $o_n = o'_n + o''_n$. In addition, we denote by o'''_n the number of sets that are not dominating set in O_n, but can be extended to a dominating set in O_{n+1}. Clearly, such sets do not dominate v_n, but dominate other vertices of the last hexagon.

Now we find three recurrences for o'_n, o''_n and o'''_n.

Each dominating set in O_n counted by o'_n can be extended to a dominating set in O_{n+1} counted by o'_{n+1} in exactly thirteen ways. These are the dominating set $D \subset V(O_n)$ containing v_n together with the following cases;

- c_{n+1}, v_{n+1},
- b_{n+1}, v_{n+1},
- v_{n+1} together with at least two vertices of $\{a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}\}$.

Further, a subset of vertices in O_n counted by o''_n and o'''_n can be extended to a dominating set in O_{n+1} counted by o'_{n+1} in exactly eleven ways. These are the following cases;

- Choose b_{n+1}, v_{n+1},

- Choose v_{n+1} together with at least two vertices of $\{a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}\}$ except c_{n+1}, d_{n+1}.

By adding all contributions we obtain the recurrence for o'_n ,

$$o'_{n+1} = 13o'_n + 11o''_n + 11o'''_n.$$

Now we need a recurrence for o''_n. Each dominating set in O_n counted by o'_n can be extended to a dominating set in O_{n+1} counted by o''_{n+1} in exactly eleven ways. These are the dominating set $D \subset V(O_n)$ containing v_n together with the following

- c_{n+1},
- Choose at least two vertices of $\{a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}\}$ except a_{n+1}, b_{n+1}.

Further, a dominating set in O_n counted by o''_n can be extended to a dominating set in O_{n+1} counted by o''_{n+1} in exactly six ways. These are the dominating set $D \subset V(O_n)$ containing the following;

- a_{n+1}, d_{n+1},
- b_{n+1}, d_{n+1},
- Choose at least three vertices of $\{a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}\}$ except $a_{n+1}, b_{n+1}, c_{n+1}$.

Further, a subset of vertices in O_n counted by o'''_n can be extended to a dominating set in O_{n+1} counted by o''_{n+1} in exactly four ways. These are the following cases;

- a_{n+1}, d_{n+1},
- Choose at least three vertices of $\{a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}\}$ except $a_{n+1}, b_{n+1}, c_{n+1}$ and $b_{n+1}, c_{n+1}, d_{n+1}$.

By adding all contributions we obtain the recurrence for o''_n ,

$$o''_{n+1} = 11o'_n + 6o''_n + 4o'''_n.$$

Finally, a dominating set in O_n counted by o''_n can be extended to a set in O_{n+1} counted by o'''_{n+1} in exactly three ways. These are the following cases;

- Choose a_{n+1}, c_{n+1},
- Choose b_{n+1}, c_{n+1},
- Choose $a_{n+1}, b_{n+1}, c_{n+1}$.

Further, the same except b_{n+1}, c_{n+1} is valid for the sets counted by o'''_n. Hence

$$o'''_{n+1} = 3o''_n + 2o'''_n.$$

We have obtained the system

$$\begin{aligned} o'_{n+1} &= 13o'_n + 11o''_n + 11o'''_n, \\ o''_{n+1} &= 11o'_n + 6o''_n + 4o'''_n, \\ o'''_{n+1} &= 3o''_n + 2o'''_n, \end{aligned}$$

with the initial conditions $o'_1 = 24$ and $o''_1 = 15$.

As before, we introduce the corresponding generating functions, $O'(x) = \sum_{n\geq 0} o'_{n+1}x^n$, $O''(x) = \sum_{n\geq 0} o''_{n+1}x^n$ and $O'''(x) = \sum_{n\geq 0} o'''_{n+1}x^n$ and obtain a linear system for them;

$$\begin{array}{lllllll} (1-13x)O'(x) & - & 11xO''(x) & - & 11xO'''(x) & = & 24 \\ (1-6x)O''(x) & - & 11xO'(x) & - & 4xO'''(x) & = & 15 \\ (1-2x)O'''(x) & - & 3xO''(x) & & & = & 2. \end{array}$$

We obtain

$$O'(x) = \frac{24 - 5x + 121x^2}{1 - 21x - 17x^2 - 121x^3}, \quad O''(x) = \frac{15 + 47x}{1 - 21x - 17x^2 - 121x^3},$$

$$O'''(x) = \frac{2 + 7x + 121x^2}{1 - 21x - 17x^2 - 121x^3}.$$

Finally, by adding $O'(x)$ and $O''(x)$ and multiplying the sum by x we obtain the generating function for the sequence o_n and $O(x) = \sum_{n\geq 0} o_n x^n$.

Theorem 4.1. *The generating function for the number of dominating sets of* O_n *is given by*

$$O(x) = \frac{1 + 18x + 25x^2}{1 - 21x - 17x^2 - 121x^3}.$$

The following corollary gives the recurrence relation of o_n.

Corollary 4.2. *For every* $n \geq 3$, *the number of dominating sets in the chain hexagonal cacti* O_n, *i.e.,* o_n *is given by*

$$o_n = 21o_{n-1} + 17o_{n-2} + 121o_{n-3},$$

with the initial conditions $o_0 = 1,\ o_1 = 39$ *and* $o_2 = 861.$

4.2. Meta-Chain

Now, we consider a meta-chain hexagonal cacti of length n, labeled as shown in Figure 4. The number of dominating sets in M_n denoted by m_n, and the number of dominating set in M_n containing and not containing vertex v_n are denoted by m'_n and m''_n. Finally, we denote by m'''_n the number of sets that are not dominating set in M_n, but can be extended to a dominating set in M_{n+1}. Clearly, such sets do not dominate v_n, and they must include v_{n-1}, since this vertex is necessary to dominate d_n. Hence, they are counted by m'_{n-1} in two ways, by including b_n, or a_n, b_n, and we have $m'''_n = 2m'_{n-1}$.

Now we find two recurrences for m'_n and m''_n.

Each dominating set in M_n counted by m'_n can be extended to a dominating set in M_{n+1} counted by m'_{n+1} in exactly fourteen ways. These are the dominating set $D \subset V(M_n)$ containing v_n together with the following cases;

- a_{n+1}, v_{n+1},
- b_{n+1}, v_{n+1},
- c_{n+1}, v_{n+1},
- v_{n+1} together with at least two vertices of $\{a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}\}$.

Further, a dominating set in M_n counted by m''_n can be extended to a dominating set in M_{n+1} counted by m'_{n+1} in exactly twelve ways. These are the following cases;

- a_{n+1}, v_{n+1},
- b_{n+1}, v_{n+1},
- v_{n+1} together with at least two vertices of $\{a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}\}$ except c_{n+1}, d_{n+1}.

The same is valid for m''' except b_{n+1}, v_{n+1} and b_{n+1}, c_{n+1}. By adding all contributions we obtain the recurrence for m'_n ,

$$m'_{n+1} = 14m'_n + 12m''_n + 10m'''_n.$$

Now we need a recurrence for m''_n. Each dominating set in M_n counted by m'_n can be extended to a dominating set in M_{n+1} counted by m''_{n+1} in exactly ten ways. These are the dominating set $D \subset V(M_n)$ containing v_n together with the following

- c_{n+1},
- Choose at least two vertices of $\{a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}\}$ except a_{n+1}, b_{n+1} and a_{n+1}, d_{n+1}.

Further, a dominating set in M_n counted by m''_n can be extended to a dominating set in M_{n+1} counted by m''_{n+1} in exactly five ways. These are the following;

- b_{n+1}, d_{n+1},
- Choose at least three vertices of $\{a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}\}$ except $a_{n+1}, b_{n+1}, c_{n+1}$.

The same is valid for m'''_n.

By adding all contributions we obtain the recurrence for m''_n,

$$m''_{n+1} = 10m'_n + 5m''_n + 5m'''_n.$$

Analogously, before we obtain the system of recurrences for m'_n, m''_n and m'''_n; have the system

$$\begin{aligned} m'_{n+1} &= 14m'_n + 12m''_n + 10m'''_n, \\ m''_{n+1} &= 10m'_n + 5m''_n + 5m'''_n, \\ m'''_{n+1} &= 2m'_n, \end{aligned}$$

with the initial conditions $m'_1 = 24$ and $m''_1 = 15$.

Again, we introduce the corresponding generating functions, $M'(x) = \sum_{n\geq 0} m'_{n+1}x^n$, $M''(x) = \sum_{n\geq 0} m''_{n+1}x^n$ and $M'''(x) = \sum_{n\geq 0} m'''_{n+1}x^n$ and obtain a linear system for them;

$$\begin{array}{lllllll} (1-14x)M'(x) & - & 12xM''(x) & - & 10xM'''(x) & = & 24 \\ (1-5x)M''(x) & - & 10xM'(x) & - & 5xM'''(x) & = & 15 \\ M'''(x) & - & 2xM'(x) & & & = & 2. \end{array}$$

We obtain

$$M'(x) = \frac{4(6+20x+5x^2)}{1-19x-70x^2-20x^3}, \quad M''(x) = \frac{5(8x+3)}{1-19x-70x^2-20x^3},$$

$$M'''(x) = \frac{2(1+5x+10x^2)}{1-19x-70x^2-20x^3}.$$

Finally, by adding $M'(x)$ and $M''(x)$ and multiplying the sum by x we obtain the generating function for the sequence m_n and $M(x) = \sum_{n\geq 0} m_n x^n$.

Theorem 4.3. *The generating function for the number of dominating sets of M_n is given by*

$$M(x) = \frac{1+20x+50x^2}{1-19x-70x^2-20x^3}.$$

The following corollary gives the recurrence relation of m_n.

Corollary 4.4. *For every $n \geq 3$, the number of dominating sets in the chain hexagonal cacti M_n, i.e., m_n is given by*

$$m_n = 19m_{n-1} + 70m_{n-2} + 20m_{n-3},$$

with the initial conditions $m_0 = 1,\ m_1 = 39$ and $m_2 = 861$.

4.1 Para-Chain

Now, we consider a para-chain hexagonal cacti of length n, labeled as shown in Figure 4. The number of dominating sets in L_n denoted by l_n, and the number of dominating set in L_n containing and not containing vertex v_n are denoted by l'_n and l''_n. Finally, we denote by l'''_n the number of sets that are not dominating set in L_n, but can be extended to a dominating set in L_{n+1}. Clearly, such sets do not dominate v_n, and they must include a_n and d_n, since these vertices are necessary to dominate b_n and c_n. Hence, they are counted by l'_{n-1}, l''_{n-1} and l'''_{n-1}, and we have $l'''_n = l'_{n-1} + l''_{n-1} + l'''_{n-1}$.

Now we find three recurrences for l'_n and l''_n.

Each dominating set in L_n counted by l'_n can be extended to a dominating set in L_{n+1} counted by l'_{n+1} in exactly twelve ways. These are the dominating set $D \subset V(L_n)$ containing v_n together with the following cases;

- v_{n+1},
- v_{n+1} together with at least two vertices of $\{a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}\}$.

Further, a dominating set in L_n counted by l''_n can be extended to a dominating set in L_{n+1} counted by l'_{n+1} in exactly nine ways. These are the following cases;

- v_{n+1} together with at least two vertices of $\{a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}\}$ except a_{n+1}, b_{n+1} and c_{n+1}, d_{n+1}.

The same is valid for l''' except b_{n+1}, c_{n+1}. By adding all contributions we obtain the recurrence for l'_n ,

$$l'_{n+1} = 12l'_n + 9l''_n + 8l'''_n.$$

Now we need a recurrence for l''_n. Each dominating set in L_n counted by l'_n can be extended to a dominating set in L_{n+1} counted by l''_{n+1} in exactly eight ways. These are the dominating set $D \subset V(L_n)$ containing v_n together with the following

- Choose at least two vertices of $\{a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}\}$ except $\{a_{n+1}, b_{n+1}\}$, $\{c_{n+1}, d_{n+1}\}$ and $\{a_{n+1}, d_{n+1}\}$.

The same is valid for l''_n. Also the same is valid for l'''_n except b_{n+1}, c_{n+1}.

By adding all contributions we obtain the recurrence for l''_n ,

$$l''_{n+1} = 8l'_n + 8l''_n + 7l'''_n.$$

Analogously, before we obtain the system of recurrences for l'_n, l''_n and l'''_n; have the system

$$\begin{aligned} l'_{n+1} &= 12l'_n + 9l''_n + 8l'''_n, \\ l''_{n+1} &= 8l'_n + 8l''_n + 7l'''_n, \\ l'''_{n+1} &= l'_n + l''_n + l'''_n, \end{aligned}$$

with the initial conditions $l'_1 = 24,\ l''_1 = 15$.

Again, we introduce the corresponding generating functions, $L'(x) = \sum_{n\geq 0} l'_{n+1}x^n$, $L''(x) = \sum_{n\geq 0} l''_{n+1}x^n$ and $L'''(x) = \sum_{n\geq 0} l'''_{n+1}x^n$ and obtain a linear system for them;

$$\begin{array}{lclclcl} (1-12x)L'(x) & - & 9xL''(x) & - & 8xL''' & = & 20 \\ (1-8x)L''(x) & - & 8xL'(x) & - & 7xL''' & = & 15 \\ (1-x)L''' & - & xL''(x) & - & xL'(x) & = & 2. \end{array}$$

We obtain

$$L'(x) = \frac{20 - 29x + 3x^2}{1 - 21x + 29x^2 - 3x^3}, \quad L''(x) = \frac{15 - 21x}{1 - 21x + 29x^2 - 3x^3},$$

$$L'''(x) = \frac{2 - 5x + 3x^2}{1 - 21x + 29x^2 - 3x^3}.$$

Finally, by adding $L'(x)$ and $L''(x)$ and multiplying the sum by x we obtain the generating function for the sequence l_n and $L(x) = \sum_{n\geq 0} l_n x^n$.

Theorem 4.5. *The generating function for the number of dominating sets of L_n is given by*

$$L(x) = \frac{1 + 14x - 21x^2}{1 - 21x + 29x^2 - 3x^3}.$$

The following corollary gives the recurrence relation of l_n.

Corollary 4.6. *For every $n \geq 4$, the number of dominating sets in the chain hexagonal cacti L_n, i.e., l_n is given by*

$$l_n = 21l_{n-1} - 29l_{n-2} + 3l_{n-3},$$

with the initial conditions $l_1 = 39,\ l_2 = 861$ and $l_3 = 18997$.

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