Mathematics Interdisciplinary Research **7** (2022) 139 – 153

Original Scientific Paper

Properties of Discrete Reversed Aging Intensity Function

Faranak Goodarzi ⋆

Abstract

In this paper, we discuss the properties of reversed aging intensity (RAI) function for discrete random variable and study its nature for some distributions. Further, using this function we characterize some discrete related distributions. The closure properties of the aging classes defined in terms of RAI function are also presented and study its closure properties under different reliability operations, viz., formation of k-out-of-n system. Moreover, we define an ordering, called reversed aging intensity ordering and study its relationship with some usual stochastic orderings. Also a numerical example is given to explain the theoretical results.

Keywords: characterization, k-out-of-n system, reversed aging intensity, stochastic orders, discrete reversed failure rate.

2020 Mathematics Subject Classification: 60E15, 60E05.

How to cite this article
F. Goodarzi, Properties of discrete reversed aging intensity function, *Math. Interdisc. Res.* **7** (2022) 139 – 153.

1. Introduction

An important phenomenon in reliability theory is aging which is an inherent property of a unit that may be a living organism or a system of components. By aging we generally mean positive aging. In the vast literature (see, among others, [4], [6], [9], [12] and [2]) failure rate, aging intensity, reversed failure rate and reversed aging intensity properties are analyzed for characterization of non-negative continuous random variables. In the reliability theory these continuous variables are

⋆Corresponding author (E-mail: f-goodarzi@kashanu.ac.ir)
Academic Editor: Ali Reza Ashrafi
Received 8 November 2021, Accepted 9 May 2022
DOI:10.22052/mir.2022.243268.1319

mainly used to describe the elements and systems life. However, quiet often we come across with situations where the product life can be described through non-negative integer valued random variable (see, [8] and [10]). Alzaatreh et al. [1] proposed a method for generating discrete distribution and Chakraborty in [3] also gave a comprehensive survey of the different methods of generating discrete probability distributions as analogues of continuous probability distributions along with their applications in construction of new discrete distributions. Dewan et al. [5] proposed a proportional hazards model for discrete data analogous to the version for continuous data and then studied its properties. They discussed, some ageing properties of the model and also introduced a new definition for reversed hazard rate. Now, motiviated by [7] and [11], we characterize a class of distriburions using two discrete reversed aging intensities.

Let T be a discrete random variable with support $\{1, 2, \ldots, b\}$, cumulative density function $F(k) = P(T \leq k)$ and probability mass function (pmf), $p(k) = P(T = k)$. Moreover, let us recall the definition of the mean inactivity time (MIT) function and the variance inactivity time (VIT) function of X, respectively as follows:

$$m(t) = E(k - X|X < k) = \frac{1}{\overline{F}(k-1)} \sum_{i=1}^{k} F(i-1),$$

and

$$\sigma^2(k) = (2k+1)m(k) - m^2(k) - \frac{2}{F(k)} \sum_{i=1}^{k} iF(i-1).$$

The discrete reversed failure rate and the discrete reversed failure rate average of T are defined as,

$$r^*(k) = \ln \frac{F(k)}{F(k-1)}, \tag{1}$$

$$H(k) = \frac{1}{b-k} \sum_{i=k+1}^{b} r^*(i).$$

By analogy with continuous distribution, the reversed aging intensity for the discrete random variable is defined as,

$$\overline{L}(k) = \frac{r^*(k)}{H(k)} = (b-k)\left[\frac{\ln F(k-1)}{\ln F(k)} - 1\right], \quad \text{for } k = 2, 3, \ldots, b. \tag{2}$$

By analogy with continuous distributions, we can define the following discrete alternative reversed aging intensity of T as

$$\overline{L}^*(k) = \frac{\ln \frac{\ln F(k)}{\ln F(k-1)}}{\ln \frac{b-k}{b-k+1}}. \tag{3}$$

We can simply show that

$$\overline{L}^*(k) = \frac{\ln \frac{b-k}{b-k-\overline{L}(k)}}{\ln \frac{b-k}{b-k+1}}, \quad \text{for } k = 2, 3, \ldots, b.$$

Although, characterization of distributions considered in the paper can be simultaneously performed through both discrete reversed aging intensity functions, in the same cases characterization through one of them seems to be easier than through the other one.

2. Characterization Through Discrete Reversed Aging Intensity and Discrete Reversed Alternative Aging Intensity

In reliability theory some functions characterize the associated distribution function. For example, Szymkowiak et al. [11] uniquely determined the discrete distribution function in terms of discrete failure rate. Discrete distribution can be characterized through its discrete reversed failure rate r^* determined by (1).

Lemma 2.1. *Given discrete reversed failure rate $r^*(k)$, for $k = 1, 2, \ldots, b$, the distribution function is defined as*

$$F(k) = \exp\left\{ - \sum_{i=k+1}^{b} r^*(i) \right\}. \tag{4}$$

Proof. Using (1), we have

$$r^*(t) = \ln F(t) - \ln F(t-1),$$

and thus

$$\sum_{i=k+1}^{b} r^*(i) = \sum_{i=k+1}^{b} \Big\{ \ln F(i) - \ln F(i-1) \Big\}$$
$$= \ln F(b) - \ln F(k) = -\ln F(k),$$

and consequently the Equation (4) is obtained. □

Moreover, let us present a theorem characterizing discrete random variable through discrete reversed aging intensity function.

Theorem 2.2. *Discrete aging intensity $\overline{L}$ of discrete random variable T with support $\{1, 2, \ldots, b\}$ determines distribution function F through the following relationship:*

$$F(k) = \exp\left\{ \left[\prod_{i=2}^{k} \left(1 + \frac{\overline{L}(i)}{b-i} \right) \right]^{-1} \ln q \right\}, \quad \textit{for } k = 2, 3, \ldots, b,$$

where $q = F(1)$, $0 < q < 1$.

Proof. From Equation (2), we have

$$\overline{L}(i) = (b-i)\left[\frac{\ln F(i-1)}{\ln F(i)} - 1\right],$$

so, for $i = 1, 2, \ldots, b$,

$$\frac{\ln F(i-1)}{\ln F(i)} = 1 + \frac{\overline{L}(i)}{b-i},$$

and

$$\prod_{i=2}^{k} \frac{\ln F(i-1)}{\ln F(i)} = \prod_{i=2}^{k}\left(1 + \frac{\overline{L}(i)}{b-i}\right).$$

Therefore

$$\frac{\ln F(1)}{\ln F(k)} = \prod_{i=2}^{k}\left(1 + \frac{\overline{L}(i)}{b-i}\right),$$

and consequently

$$F(k) = \exp\left\{\left[\prod_{i=2}^{k}\left(1 + \frac{\overline{L}(i)}{b-i}\right)\right]^{-1} \ln F(1)\right\}.$$

Hence the result. □

Remark 1. Note that discrete reversed aging intensity $\overline{L}$ determines the family of distribution functions depending on parameter $q \in (0, 1)$.

Secondly, we state that discrete alternative reversed aging intensity $\overline{L}^*$, determined by formula (1) of discrete random variable T, characterizes its distribution function.

Theorem 2.3. *Discrete alternative reversed aging intensity* $\overline{L}^*$ *of discrete random variable* T *determines distribution function* F *through the following relationship:*

$$F(k) = \exp\left\{\left[\prod_{i=2}^{k}\left(\frac{b-i}{b-i+1}\right)^{\overline{L}^*(i)} \ln F(1)\right]\right\}.$$

Proof. From Equation (3), we have

$$\overline{L}^*(i) = \frac{\ln \frac{\ln F(i)}{\ln F(i-1)}}{\ln \frac{b-i}{b-i+1}},$$

so

$$\frac{\ln F(i)}{\ln F(i-1)} = \exp\left[\overline{L}^*(i) \ln \frac{b-i}{b-i+1}\right] = \left(\frac{b-i}{b-i+1}\right)^{\overline{L}^*(i)}, \quad \text{for } i = 2, 3, \ldots, b.$$

Therefore

$$\prod_{i=2}^{k} \frac{\ln F(i)}{\ln F(i-1)} = \prod_{i=2}^{k} \left(\frac{b-i}{b-i+1}\right)^{\overline{L}^*(i)},$$

and

$$\ln F(k) = \ln F(1) \prod_{i=2}^{k} \left(\frac{b-i}{b-i+1}\right)^{\overline{L}^*(i)},$$

consequently

$$F(k) = \exp\left\{\left[\prod_{i=2}^{k} \left(\frac{b-i}{b-i+1}\right)^{\overline{L}^*(i)} \ln F(1)\right]\right\}.$$

□

Corollary 2.4. *If for discrete random variable T, the discrete alternative reversed intensity $\overline{L}^*(k) = \alpha$, for $k = 2, 3, \ldots$ and $\alpha > 0$, then T follows*

$$F(k) = q^{\left(\frac{b-k}{b-1}\right)^{\alpha}}, \quad k = 1, 2, \ldots, b, \tag{5}$$

where $0 < q < 1$.

We can generalize the given distribution in (5) as:

$$F(k) = q^{\left(\frac{b-k}{b-1}\right)^{\alpha} c^{k-1}}, \quad k = 1, 2, \ldots, b,\ 0 < c \leq 1,\ \alpha > 0.$$

It's the discrete reversed aging intensity is given by

$$\overline{L}(k) = \frac{(b-k+1)^{\alpha} - c(b-k)^{\alpha}}{c(b-k)^{\alpha-1}},$$

and the discrete alternative reversed aging intensity is given by

$$\overline{L}^*(k) = \alpha + \frac{\ln c}{\ln \frac{b-k}{b-k+1}}, \quad \text{for } k = 1, 2, \ldots, b-1.$$

In the next theorem, we characterize a distribution in which the discrete reversed failure rate is proportional to 1.

Theorem 2.5. *Let T be a discrete random variable with support $\{1, 2, \ldots, b\}$, $E(\frac{1}{r^*(T)}) < \infty$ and $E(r^*(T)) < \infty$. Then*

$$E\left(\frac{1}{r^*(T)}\right) \geq \frac{1}{E(r^*(T))}, \tag{6}$$

The equality holds if and only if T follows the distribution

$$F(x) = q^{\frac{b-k}{b-1}}, \quad k = 1, \ldots, b. \tag{7}$$

Proof. The inequality (6) follows from the Cauchy-Schwarz inequality and equality holds if and only if there exists a constant $A > 0$ such that

$$\frac{p(k)}{r^*(k)} = Ar^*(k)p(k),$$

which is equivalent to the fact that $r^*(k) = c$, where c is constant. Now applying (4), we have $F(k) = \exp\{-(b-k)c\}$. On the other hand, taking into account $F(1) = q$, we have $c = \frac{-\ln q}{b-1}$ and thus $F(k) = \exp\{\frac{(b-k)\ln q}{b-1}\} = q^{\frac{b-k}{b-1}}$. □

It is to be noted that $\overline{L}(k) = 1$ for all $k = 1, 2, \ldots$ if and only if the reversed failure rate function $r^*(k)$ is constant. Thus $\overline{L}(k) = 1$ characterizes the distribution given in Equation (7). Further, $\overline{L}(k) < 1$ if $r^*(k)$ is decreasing in k (i.e., T is DRHR). On the basis of the monotonicity of the *RAI* function, we define the following nonparametric family of distribution.

Definition 2.6. A random variable T is said to be increasing in reversed aging intensity (IRAI) if the corresponding RAI function $\overline{L}(k)$ is increasing in $k = 1, 2, \ldots$. We call the random variable T as decreasing in reversed aging intensity (DRAI) if $\overline{L}(k)$ decreasing in $k = 1, 2, \ldots$.

Further, it is seen that the monotonic behavior of the reversed failure rate function is not, in general, transmitted to the monotonicity of the *RAI* function as is evident from the following two counterexamples.

Counterexample 2.7. *Let T be a discrete random variable with distribution function $F(k) = q^{(\frac{b-k}{b-1})^2}$, for $k = 1, 2, \ldots, b$, where $q = F(1)$. Then, $r^*(k) = -\frac{2(b-k)+1}{(b-1)^2}\ln q$ is decreasing in k. So, T is DRHR. Here $\overline{L}(k) = \frac{1}{b-k} + 2$, which increases in $k = 1, 2, \ldots, b$.*

Counterexample 2.8. *Let T be a discrete random variable with distribution function $F(k) = \frac{k}{b}$, for $k = 1, 2, \ldots, b$. Then, $r^*(k) = \ln(1 + \frac{1}{k-1})$ is decreasing in k. So, T is DRHR. Here $\overline{L}(k) = (b-k)\left[\frac{\ln\frac{k-1}{b}}{\ln\frac{k}{b}} - 1\right] = (b-k)\left[\frac{\ln\frac{k-1}{k}}{\ln\frac{k}{b}}\right]$, which decreases in $k = 1, 2, \ldots, b$.*

From the foregoing two counterexamples it is observed that an DRHR random variable can be *IRAI* or *DRAI*. For the random variable T with distribution function $F(x) = q^{\left(\frac{b-k}{b-1}\right)^3 0.5^{k-1}}$ for $k = 1, 2, \ldots, b$, $r^*(k) = \frac{0.5^{k-2}(0.5(b-k)^3-(b-k+1)^3)}{(b-1)^3}$ and thus T is DRHR. On the other hand, $\overline{L}(k) = \frac{(b-k+1)^3-0.5(b-k)^3}{0.5(b-k)^2}$ is nonmonotone in k. Thus, for a DRHR random variable, the RAI function could be nonmonotonic as well.

3. Analysis of Discrete Reversed Aging Intesity Function Through Failure Data

Let N units be put to test at $t = 0$. Further, let the number of units having survived at ordered times t_j be $N_s(t_j)$. Then a logical estimate for $\overline{L}^*(t)$, is

$$\hat{\overline{L}}^*(t) = \frac{\ln \frac{\ln\left(1-\frac{N_s(t_j)}{N}\right)}{\ln\left(1-\frac{N_s(t_j-1)}{N}\right)}}{\ln \frac{b-t_j}{b-t_j+1}}, \quad for \quad t_j \leq t < t_j + 1.$$

Example 3.1. Data are generated in R according to the probability distribution function given (5) for $\alpha = 2$ and $b = 10$, sample size $N = 1000$ and number of classes $k = 10$. Estimates of reversed failure rate $\hat{r}^*(t)$ and reversed aging intensity $\hat{\overline{L}}^*(t)$ are given in Table 1. The estimates of reversed failure rate and reversed aging intensity function for the data are plotted in Figures 2 and 3. The presented in Figure 3 function $\hat{\overline{L}}^*(t)$ can be considered to oscillate around the constant 2.

Table 1: Generated grouped data.

Calss	$t \in [t_j, t_j+1)$	$N_s(t_j)$	$1 - \frac{N_s(t_j)}{N}$	$\hat{r}^*(t)$	$\hat{\overline{L}}^*(t)$
1	0-1	1000	1	–	–
2	1-2	502	0.498	–	–
3	2-3	407	0.593	0.1748	2.447
4	3-4	341	0.659	0.1053	1.691
5	4-5	269	0.731	0.1035	1.855
6	5-6	195	0.805	0.09622	2.016
7	6-7	133	0.867	0.07418	1.877
8	7-8	89	0.911	0.04974	1.48
9	8-9	33	0.967	0.05921	2.521
10	9-10	13	0.987	0.02078	1.359

4. Clousure Properties of IRAI and DRAI Classes

In this section, we study whether IRAI and DRAI classes are closed under different reliability operations, viz., geometric mean and harmonic mean of distributions and formation of k-out-of-n system.

The following counterexamples show that IRAI class is not closed under the aforesaid operations.

Counterexample 4.1. *Let T_1 and T_2 be two random variables having respective distribution functions $F_1(k) = 0.5^{\left(\frac{b-k}{b-1}\right)^2}$ and $F_2(k) = 0.5^{\left(\frac{b-k}{b-1}\right)^3}$ for $k = 1, 2, \ldots, b$.*

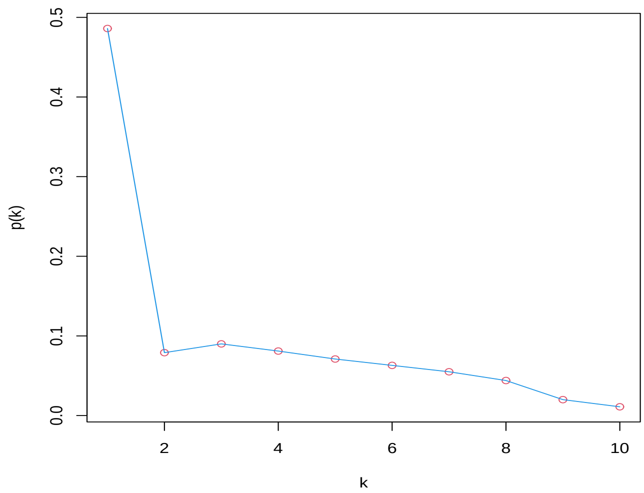

Figure 1: Empirical mass function.

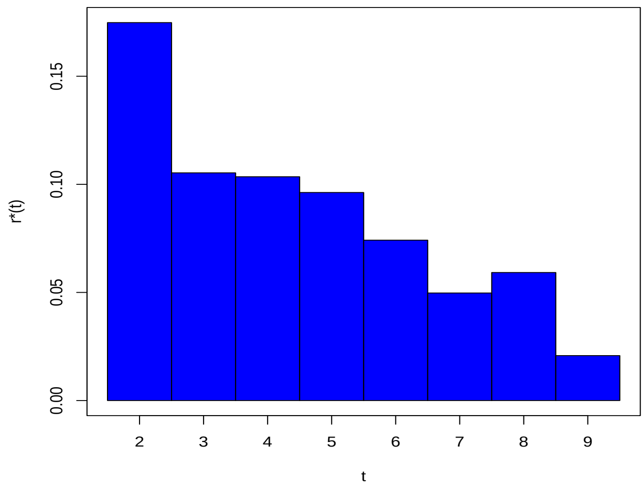

Figure 2: Discrete reversed failure rate estimator $\hat{r^*}(t)$.

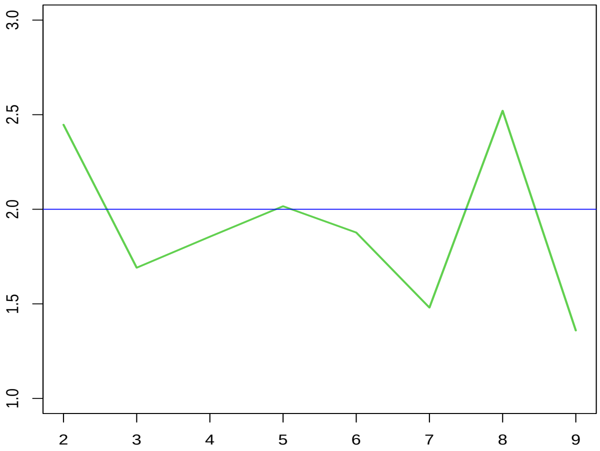

Figure 3: Discrete reversed aging intensity estmator $\hat{\overline{L}}^*(t)$.

Then the reversed hazard function of the random variables T_1 and T_2, are decreasing. Now, if T be a random variable having distribution function $F(k) = (F_1(k)F_2(k))^{\frac{1}{2}}$, then, for $k = 1, 2, \ldots, b$,

$$\overline{L}_T(k) = (b-k)\left(\frac{\left(\frac{b-k+1}{b-1}\right)^2 + \left(\frac{b-k+1}{b-1}\right)^3}{\left(\frac{b-k}{b-1}\right)^2 + \left(\frac{b-k}{b-1}\right)^3} - 1\right).$$

Although both T_1 and T_2 are IRAI, the RAI function of T is nonmonotone. Thus, IRAI is not closed with respect to geometric mean of distributions.

Counterexample 4.2. *Let T_1 and T_2 be two random variables having the distribution functions given in previous counterexample. Let T have distribution function $F(k) = \left(0.5\left(\frac{1}{0.5^{(\frac{b-k}{b-1})^2}} + \frac{1}{0.5^{(\frac{b-k}{b-1})^3}}\right)\right)^{-1}$ that is harmonic mean of distributions T_1 and T_2. We should note that, although T_1 and T_2 are IRAI, however the reversed aging intensity function of T is nonmonotone. Thus, IRAI is not closed with respect to harmonic mean of distributions.*

Next, we cite a counterexample to show that IRAI class is not closed under the formation of a k-out-of-n.

Counterexample 4.3. *Let T be a random variable having distribution function $F(k) = 0.3^{(\frac{b-k}{b-1})^{0.25}}$ for $k = 1, 2, \ldots, b$ and $F_{T_{3,4}}(k)$ be the distribution function of* *the* $3nd$ *order statistic* $T_{3,4}$ *in a sample of size* 4 *from this distribution. Then, the corresponding RAI function is*

$$\overline{L}_{T_{3,4}}(k) = (b-k)\left(\frac{\ln\left(4\left(0.3^{\left(\frac{b-k+1}{b-1}\right)^{0.25}}\right)^3 - 3\left(0.3^{\left(\frac{b-k+1}{b-1}\right)^{0.25}}\right)^4\right)}{\ln\left(4\left(0.3^{\left(\frac{b-k}{b-1}\right)^{0.25}}\right)^3 - 3\left(0.3^{\left(\frac{b-k}{b-1}\right)^{0.25}}\right)^4\right)} - 1\right).$$

For example, we can show that, for $b = 15$, $\overline{L}_{T_{3,4}}(k)$ *is nonmonotone.*

From the counterexample given below, we see that DRAI class is not closed under the formation of a k-out-of-n system.

Counterexample 4.4. *Let* T *be a random variable having distribution function* $F(k) = \dfrac{k^2}{b^2}$ *for* $k = 1, 2, \ldots, b$ *and* $F_{T_{3,4}}(k)$ *be the distribution function of the* $3nd$ *order statistic* $T_{3,4}$ *in a sample of size* 4 *from this distribution. Then, the corresponding RAI function is*

$$\overline{L}_{T_{3,4}}(k) = (b-k)\left(\frac{\ln\left(\frac{4(k-1)^6}{b^6} - \frac{3(k-1)^8}{b^8}\right)}{\ln\left(\frac{4k^6}{b^6} - \frac{3k^8}{b^8}\right)} - 1\right).$$

For example, we can show that, for $b = 10$, $\overline{L}_{T_{3,4}}(k)$ *is nonmonotone.*

Proposition 4.5. *Let* T_1 *and* T_2 *be lifetimes of components in a two-component parallel system and* T_1 *and* T_2 *are independent and have the distribution functions* $F_1(k)$ *and* $F_2(k)$ *respectively. If we show reversed aging intensities of components respectively by* $\overline{L}_1$ *and* $\overline{L}_2$, *then the reversed aging intensity function of system is given by*

$$\overline{L}_{T_{2:2}}(k) = \frac{\frac{\overline{L_1}(k)}{\ln F_{T_1}(k)} + \frac{\overline{L_2}(k)}{\ln F_{T_2}(k)}}{\frac{1}{\ln F_{T_1}(k)} + \frac{1}{\ln F_{T_2}(k)}}.$$

Proof. We know that $T_{2:2} = \max(T_1, T_2)$ is the lifetime of a parallel system. Hence $F_{T_{2:2}}(k) = F_{T_1}(k)F_{T_2}(k)$, thus the reversed aging intensity function of the system would have

$$\begin{aligned}\overline{L}_{T_{2:2}}(k) &= (b-k)\left\{\frac{\ln F_{T_1}(k-1)F_{T_2}(k-1)}{\ln F_{T_1}(k)F_{T_2}(k)} - 1\right\}\\ &= (b-k)\left\{\frac{\ln F_{T_1}(k-1) + \ln F_{T_2}(k-1) - \ln F_{T_1}(k) - \ln F_{T_2}(k)}{\ln F_{T_1}(k) + \ln F_{T_2}(k)}\right\}\\ &= (b-k)\left\{\frac{\ln \frac{F_{T_1}(k-1)}{F_{T_1}(k)} + \ln \frac{F_{T_2}(k-1)}{F_{T_2}(k)}}{\ln F_{T_1}(k) + \ln F_{T_2}(k)}\right\}\end{aligned}$$

$$= (b-k)\left\{\frac{-r_1^*(k)-r_2^*(k)}{\ln F_{T_1}(k)+\ln F_{T_2}(k)}\right\}$$

$$= (b-k)\left\{\frac{\frac{-r_1^*(k)}{\ln F_{T_1}(k)\ln F_{T_2}(k)}+\frac{-r_2^*(k)}{\ln F_{T_1}(k)\ln F_{T_2}(k)}}{\frac{\ln F_{T_1}(k)+\ln F_{T_2}(k)}{\ln F_{T_1}(k)\ln F_{T_2}(k)}}\right\}$$

$$= \frac{\frac{\overline{L_1}(k)}{\ln F_{T_2}(k)}+\frac{\overline{L_2}(k)}{\ln F_{T_1}(k)}}{\frac{1}{\ln F_{T_1}(k)}+\frac{1}{\ln F_{T_2}(k)}}.$$

□

Corollary 4.6. *If T_1 and T_2 are identically independent distributed, then $\overline{L}_{T_{2:2}}(k) = \overline{L}_1(k)$.*

5. Some Properties of Discrete Reversed Aging Intensity Order

Analogous to continuous case, reversed aging intensity order for discrete distributions can be determined.

5.1 Discrete Reversed Aging Intensity Order

For random variables X, Y with support $\{1, 2, \ldots, b\}$ we say that $X \leq_{DRAI} Y$, if for all $k = 1, 2, \ldots, b$, $\overline{L}_X^*(k) \leq \overline{L}_Y^*(k)$. The choice of $\overline{L}$ or $\overline{L}^*$ used to determine reversed aging intensity order in the particular class of distributions depends on the specific forms of these functions.

It means that if one random variable has larger discrete reversed aging intensity function (or discrete alternative reversed aging intensity function), then it is greater (better) in the discrete aging intensity DRAI order-it has the weaker tendency of aging.

Theorem 5.1. *If T_1 have the distribution $F_1(k) = q_1^{\left(\frac{b-k}{b-1}\right)^{\alpha_1} c_1^{k-1}}$, for $k = 1, 2, \ldots, b$, and T_2 have the distribution $F_2(k) = q_2^{\left(\frac{b-k}{b-1}\right)^{\alpha_2} c_2^{k-1}}$, for $k = 1, 2, \ldots, b$, with $\alpha_1 \leq \alpha_2$ and $c_1 \geq c_2$, then $T_1 \leq_{DRAI} T_2$.*

Proof. We must shown that $\overline{L}_{T_1}(k) \leq \overline{L}_{T_2}(k)$. By using Equation (3), since $\alpha_1 \leq \alpha_2$ and $\ln c_1 \leq \ln c_2$ and also $\ln \frac{b-k}{b-k+1} < 0$, the result is obtained. □

The following counterexample shows that the condition $\alpha_1 \leq \alpha_2$ in Theorem 5.1 cannot be relaxed.

Counterexample 5.2. *Let T_1 and T_2 have distributions F_1 and F_2 given in the Theorem* 5.1. *If $\alpha_1 = 4 > \alpha_2 = 3$ and $c_1 = 0.4 > c_2 = 0.3$ then for $b = 10$, $\overline{L}_{T_1}^*(2) = 11.78 < \overline{L}_{T_2}^*(2) = 13.22$ but $\overline{L}_{T_1}^*(9) = 5.322 > \overline{L}_{T_2}^*(9) = 4.737$. Hence $T_1 \nleq_{DRAI} T_2$.*

The next counterexample shows that the condition $c_1 > c_2$ in Theorem 5.1 cannot be removed.

Counterexample 5.3. *Let T_1 and T_2 have distributions F_1 and F_2 given in the previous Theorem. If $\alpha_1 = 3 < \alpha_2 = 4$ and $c_1 = 0.3 < c_2 = 0.4$ then for $b = 10$, $\overline{L}^*_{T_1}(2) = 13.222 > \overline{L}^*_{T_2}(2) = 11.779$ but $\overline{L}^*_{T_1}(9) = 4.737 < \overline{L}^*_{T_2}(9) = 5.322$. Hence $T_1 \nleq_{DRAI} T_2$.*

6. Relationship Between DRAI Order and other Stochastic Orders

To study the relationship between DRAI order and other existing stochastic orders, let us first recall that for discrete variables T_1 and T_2 with support $\{1, 2, \ldots, b\}$, T_1 is smaller than T_2 in

(i) stochastic order (denoted by $T_1 \leq_{st} T_2$) if $\overline{F}_{T_1}(k) \leq \overline{F}_{T_2}(k)$, for all k;

(ii) likelihood ratio order (denoted by $T_1 \leq_{lr} T_2$) if $\frac{p_{T_1}(k)}{p_{T_2}(k)}$ is nondecreasing for all k;

(iii) reversed failure rate order (denoted by $T_1 \leq_{rfr} T_2$) if $r^*_{T_1}(k) \leq r^*_{T_2}(k)$, for all k;

(iv) discrete increasing concave order (denoted by $T_1 \leq_{dicv} T_2$) if $\sum_{i=1}^{k} F_{T_1}(i) \geq \sum_{i=1}^{k} F_{T_2}(i)$, for all k;

(v) mean inactivity time order (denoted by $T_1 \leq_{MIT} T_2$) if and only if $\frac{\sum_{i=1}^{k} F_{T_1}(i)}{\sum_{i=1}^{k} F_{T_2}(i)}$ is decreasing in k;

(vi) variance inactivity time order (denoted by $T_1 \leq_{VIT} T_2$) if and only if $\frac{\sum_{i=1}^{k} iF_{T_1}(i)}{\sum_{i=1}^{k} F_{T_1}(i)} \leq \frac{\sum_{i=1}^{k} iF_{T_2}(i)}{\sum_{i=1}^{k} F_{T_2}(i)}$, for all k.

Theorem 6.1. *For two random variables T_1 and T_2, the following conditions are equivalent.*

(i) $T_1 \geq_{DRAI} T_2$.

(ii) $\frac{\sum_{i=k+1}^{b} r^*_{T_1}(i)}{\sum_{i=k+1}^{b} r^*_{T_2}(i)}$ *is decreasing in* $k = 1, 2, \ldots, b$.

(iii) $\frac{\ln F_{T_1}(k)}{\ln F_{T_2}(k)}$ *is decreasing in* $k = 1, 2, \ldots, b$.

Proof. We first show that (i) and (iii) are equivalent.

$$\begin{aligned}T_1 \geq_{DRAI} T_2 &\Longleftrightarrow \frac{\ln \frac{\ln F_{T_1}(k)}{\ln F_{T_1}(k-1)}}{\ln \frac{b-k}{b-k+1}} \geq \frac{\ln \frac{\ln F_{T_2}(k)}{\ln F_{T_2}(k-1)}}{\ln \frac{b-k}{b-k+1}} \\ &\Longleftrightarrow \ln \frac{\ln F_{T_1}(k)}{\ln F_{T_1}(k-1)} \leq \ln \frac{\ln F_{T_2}(k)}{\ln F_{T_2}(k-1)} \\ &\Longleftrightarrow \frac{\ln F_{T_1}(k)}{\ln F_{T_1}(k-1)} \leq \frac{\ln F_{T_2}(k)}{\ln F_{T_2}(k-1)} \\ &\Longleftrightarrow \frac{\ln F_{T_1}(k)}{\ln F_{T_2}(k)} \leq \frac{\ln F_{T_1}(k-1)}{\ln F_{T_2}(k-1)}.\end{aligned}$$

The equivalence of (ii) and (iii) results from Equation (4). □

The reflexive, commutative, and antisymmetric properties of the RAI order are given below. The proof is omitted.

Proposition 6.2. (i) $T_1 \leq_{DRAI} T_1$.

(ii) *If* $T_1 \leq_{DRAI} T_2$ *and* $T_2 \leq_{DRAI} T_3$*, then* $T_1 \leq_{DRAI} T_3$.

(iii) *If* $T_1 \leq_{DRAI} T_2$ *and* $T_2 \leq_{DRAI} T_1$*, then* T_1 *and* T_2 *have proportional reversed failure rates.*

Now, we want to express relationship between RAI order and some stochastic orders.

The following counterexample shows that DRAI order does not imply increasing concave order.

Counterexample 6.3. *Let* T_1 *and* T_2 *have distributions* $F_1(k) = 0.5^{(\frac{b-k}{b-1})^2}$ *and* $F_2(k) = 0.4^{(\frac{b-k}{b-1})^3}$ *for* $k = 1, 2, \dots, b$*, respectively. Then by using Theorem* 6.1, $T_1 \leq_{DRAI} T_2$. *Now, for example if we assume* $b = 8$*, then* $\sum_{i=1}^{2} 0.5^{(\frac{b-i}{b-1})^2} = 1.101 > \sum_{i=1}^{2} 0.4^{(\frac{b-i}{b-1})^3} = 0.962$ *and* $\sum_{i=1}^{7} 0.5^{(\frac{b-i}{b-1})^2} = 5.412 < \sum_{i=1}^{7} 0.4^{(\frac{b-i}{b-1})^3} = 5.427$. *Hence* $T_1 \nleq_{dicv} T_2$*. This shows that* $T_1 \leq_{DRAI} T_2 \nRightarrow T_1 \leq_{dicv} T_2$.

Since DRAI ordering does not imply increasing concave ordering, it is obvious that DRAI ordering implies none of likelihood ratio ordering , reversed failure rate ordering, and stochastic ordering.

Counterexample 6.4. *Let* T_1 *and* T_2 *have distribution functions* $F_1(k) = 0.5^{(\frac{b-k}{b-1})^3 0.3^{k-1}}$ *for* $k = 1, 2, \dots, b$ *and* $F_2(k) = 0.5^{(\frac{b-k}{b-1})^4 0.4^{k-1}}$*, for* $k = 1, 2, \dots, b$ *respectively. By using counterexample* 5.3*, we know that* $T_1 \nleq_{DRAI} T_2$*. It is easy to verify that for* $b = 10$,

$$\frac{F_2(k)}{F_1(k)} = 0.5^{(\frac{10-k}{9})^3 \left\{\frac{10-k}{9} 0.4^{k-1} - 0.3^{k-1}\right\}},$$

increases with $k = 2, \dots, 10$*. This means* $T_1 \leq_{rfr} T_2$*, but* $T_1 \nleq_{DRHR} T_2$.

The following counterexample shows that DRAI orderig does not imply variance inactivity time ordering.

Counterexample 6.5. *If* T_1 *have the distribution* $F_1(k) = (0.4)^{\left(\frac{10-k}{9}\right)^3 (0.5)^{k-1}}$, *for* $k = 1, 2, \ldots, 10$, *and* T_2 *have the distribution* $F_2(k) = (0.3)^{\left(\frac{10-k}{9}\right)^4 (0.2)^{k-1}}$, *for* $k = 1, 2, \ldots, 10$*, then* $T_1 \leq_{DRAI} T_2$*, however since,*

$$\frac{\sum_{i=1}^{k} i(0.4)^{\left(\frac{10-i}{9}\right)^3 (0.5)^{i-1}}}{\sum_{i=1}^{k} (0.4)^{\left(\frac{10-i}{9}\right)^3 (0.5)^{i-1}}} \nleq \frac{\sum_{i=1}^{k} i(0.3)^{\left(\frac{10-i}{9}\right)^4 (0.2)^{i-1}}}{\sum_{i=1}^{k} (0.3)^{\left(\frac{10-i}{9}\right)^4 (0.2)^{i-1}}},$$

therefore $T_1 \nleq_{VIT} T_2$*. For example, for* $k = 3$ *the left ratio is equal to* 2.246 *and the right ratio is equal to* 2.318*, whereas for* $k = 8$ *the left ratio is* 4.922 *and the right ratio is* 4.896*.*

We can also easily show that, $T_1 \nleq_{MIT} T_2$*.*

7. Conclusion

In this paper, discrete reversed aging intensity and discrete alternative reversed aging intensity functions were introduced. Further, using these functions we characterized the distribution functions F. For example in situation that $\overline{L}^*(k)$ is constant, we characterized the distribution function uniquely. Hence for analysis of real data, if estimation of the discrete alternative reversed failure rate oscillate around the constant, then we conclude that the data follows of the distribution (5). Also we introdued the discrete reversed aging intesity order and studied its properties and the relationships it has with some stochastic orders. Furthermore we studied whether class IRAI or DRAI clossed under different reliability operations.

Conflicts of Interest. The author declares that there are no conflicts of interest regarding the publication of this article.

References

[1] A. Alzaatreh, C. Lee and F. Famoye, On the discrete analogues of continuous distributions, *Stat. Methodol.* **9** (6) (2012) 589 – 603.

[2] F. Buono, M. Longobardi and M. Szymkowiak, On generalized reversed aging intensity functions, *Ricerche Mat.* **71** (2021) 85 – 108.

[3] S. Chakraborty, Generating discrete analogues of continuous probability distributions-A survey of methods and constructions, *J. Stat. Distrib. Appl.* **2** (1) (2015) 1 – 30.

[4] N. K. Chandra and D. Roy, Some results on reversed hazard rate, *Prob. Eng. Inf. Sci.* **15** (2001) 95 – 102.

[5] I. Dewan and K. K. Sudheesh, On proportional (reversed) hazard model for discrete data, Technical report. Indian Statistical Institute, New Delhi, India, (2009) 1 − 30.

[6] R. Jiang, P. Ji and X. Xiao, Aging property of univariate failure rate models, *Reliab. Eng. Syst. saf.* **79** (2003) 113 − 116.

[7] A. K. Nanda, S. Bhattacharjee and S. S. Alam, Properties of aging intesity funciton, *Stat. Prob. Lett.* **77** (4) (2007) 365 − 373.

[8] M. S. Noughabi, A. H. Rezaei Roknabadi and G. R. Mohtashami Borzadaran, Some discrete lifetime distributions with bathtub-shaped hazard rate functions, *Qual. Eng.* **25** (3) (2013) 225 − 236.

[9] M. Rezaei and V. A. Khalef, On the reversed average intensity order, *JSRI* **11** (1) (2014) 25 − 39.

[10] A. H. Rezaei Roknabadi, G. M. Borzadaran and M. Khorashadizadeh, Some aspects of discrete hazard rate function in telescopic families, *Econ. Qual. Control.* **24** (1) (2009) 3542.

[11] M. Szymkowiak and M. Iwińska, Characterizations of discrete weibull related distributions, *Stat. Prob. Lett.* **111** (C) (2016) 41 − 48.

[12] M. Szymkowiak, Characterizations of distributions through aging intensity, *IEEE Trans. Reliab.* **67** (2) (2018) 446 − 458.

Faranak Goodarzi
Department of Statistics,
University of Kashan,
Kashan, I. R. Iran
e-mail: f-goodarzi@kashanu.ac.ir