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Chebyshev Cardinal Wavelets for Nonlinear Volterra Integral Equations of the Second Kind

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Abstract

This study concentrated on the numerical solution of a nonlinear Volterra integral equation. The approach is accorded to a type of orthogonal wavelets named the Chebyshev cardinal wavelets. The undetermined solution is extended concerning the Chebyshev cardinal wavelets involving unknown coefficients. Hence, a system of nonlinear algebraic equations is drawn out by changing the introduced expansion to the predetermined problem, applying the generated operational matrix, and supposing the cardinality of the basis functions. Conclusively, the estimated solution is achieved by figuring out the mentioned system. Relatively, the convergence of the founded procedure process is reviewed in the Sobolev space. In addition, the results achieved from utilizing the method in some instances display the applicability and validity of the method.

Keywords: Volterra integral equation, Chebyshev wavelets, operational matrix, convergence, Sobolev space.

2020 Mathematics Subject Classification: 45D05, 45G10.

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1. Introduction

Integral equations of numerous kinds exist in different majors and subfields of science and engineering. Analytical and numerical study of integral equations (IEs)

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has been widely considered by researchers [3, 19, 38]. To achieve the solution of IEs, various basis functions are presented. The widely used methods are based on piecewise constant basis functions (PCBF) [34], Chebyshev polynomial [17] wavelet methods [9,49], radial basis functions (RBFs) [4,5], Toeplitz matrix method [2], the linear multistep method [28] and the triangular function method [31]. Also, with the achievement and success of creating generalized metric spaces and the results obtained from it, it can be mentioned that the metric fixed point theory has been of interest to researchers and has been applied to solve IEs [1,40]. Some of the mentioned methods are just useable to linear IEs during which others are applicable to specific cases of nonlinear IEs. The demand to progress an approach that is possible to be extended to a general kind of nonlinear IEs is considered to be vital in order to reach a single platform to be used for the numerical solution of such types of problem. In this study, we provide a new procedure based on the Chebyshev cardinal wavelets (ChC wavelets) that is planned for general types of nonlinear Volterra IEs. The introduction of the Haar function was the prelude to the beginning of the wavelet theory in 1910 [14]. The use of wavelets has achieved significance in the last three decades. They have vast applications in scientific computing, and also they have been widely applied in numerical estimation in the recent research. It is extensively related to the fact that wavelets cause a natural mechanism for decomposing the solution into a set of coefficients, that rely on scale and location. Based on this speciality, the proportion of wavelets for numerical estimation stays undisputed. Many researchers have utilized numerous methods in applying wavelets to numerical estimations such as the wavelet collocation method [44], the wavelet Galerkin method [27], and the wavelet meshless method [30]. The researchers have utilized wavelets for numerical integration [6, 43], and for numerical solutions of IEs [9, 49], integro-differential equations [41], ordinary differential equations [15], fractional differential equations [37, 39], partial differential equations [35] and fractional partial differential equations [26, 47]. These approchs include different kinds of wavelet. The instances involve Daubechies [16], Battle-Lemarie [50], B-spline [15], Chebyshev [8,11], Coifman [32], CAS [41], Legendre [45] and Haar wavelets [13]. Numerous kinds of wavelet have participated for numerical solution of various types of IE. These involve Haar [7, 9, 29], Legendre [49], trigonometric [18], CAS [48], Chebyshev [11], and Coifman [32] wavelets.

In the present paper, while using the ChC wavelets, we keep this in mind that the ChC wavelets include numerous significant belongings, like orthogonality, cardinality and spectral accuracy. Due to the mentioned properties, it is considered as a candidate for a set of powerful basis functions in the approximation theory. In addition, both the wavelet features and the ChC polynomial are concurrently collected in the ChC wavelets. These wavelets have been used in [42] for solving nonlinear constrained optimal control problems. The key supremacy of the wavelets in comparison with other popular ones mentioned above is their cardinality. The cardinality feature rescues us from calculating integrals which often arise in obtaining the coefficients of the ChC wavelets expansion of a function. Actually, the aimed coefficients are acquired by calculating the values of the considered function at some grid points, which are also used in producing the wavelets [20, 21, 23, 24]. According to this feature, the nonlinear terms in the studied problems are calculated efficiently. This study deals with the numerical solution of nonlinear Volterra IE of second kind, which is as follows:

$$y(z) = g(z) + \int_0^z K(z,t)G(t,y(t))\mathrm{d}t, \qquad z \in [0,1]. \tag{1}$$

A general form of Equation (1) is considered as follows:

$$y(z) = g(z) + \int_0^z K(z,t,y(t))\mathrm{d}t, \qquad z \in [0,1], \tag{2}$$

where $g(z)$ is a known function defined on $[0,1]$ also $K(z,t,y(t))$ is a nonlinear function belonging to $[0,1] \times [0,1] \times \mathbb{R}$, and $y(z)$ is the unknown function to be computed.

The outline of the paper is as follows: In Section 2, properties of the ChC wavelets are described. Section 3 belongs to description of the proposed method. In Section 4, we explain the convergence analysis. Numerical examples of our study is presented in Section 5, and at the end, the conclusion is presented in Section 6.

2. Properties of the ChC Wavelets

The ChC wavelets are evaluated in summary, and focused properties are reported in this unit.

2.1 The ChC Wavelets

According to the process of building orthogonal polynomial wavelets that is stated in [22], we are able to explain the ChC wavelets over $[0,1]$ as follows:

$$\hat{\varphi}_{rs}(z) = \begin{cases} \sqrt{\frac{2S}{\pi}} 2^{\frac{k}{2}} C_s(2^{k+1}z - 2r + 1), & \frac{r-1}{2^k} \le z < \frac{r}{2^k}, \\ 0, & otherwise, \end{cases} \tag{3}$$

where $k \in \mathbb{R} \cup \{0\}$, z is an independent variable defined on $[0,1]$, $r = 1, 2, \ldots, 2^k$ and C_s is the ChC function of order s. Notice that the coefficient $\sqrt{\frac{2S}{\pi}}$ is applied for normalizing. The set $\{\hat{\varphi}_{rs}(z) | r = 1, 2, \ldots, 2^k, s = 1, 2, \ldots, S, S \in \mathbb{R}\}$ produces an orthonormal basis for $L^2_{w_r}[0,1]$, i.e.

$$\langle \hat{\varphi}_{rs}(z), \hat{\varphi}_{r's'}(z) \rangle_{w_r} = \int_0^1 \hat{\varphi}_{rs}(z)\hat{\varphi}_{r's'}(z) w_r(z)\mathrm{d}z = \begin{cases} 1, & (r,s) = (r',s'), \\ 0, & (r,s) \neq (r',s'), \end{cases}$$

where

$$w_r(z) = \begin{cases} \left(1-(2^{k+1}z-2r+1)^2\right)^{-\frac{1}{2}}, & \frac{r-1}{2^k} \le z < \frac{r}{2^k}, \\ 0, & otherwise. \end{cases} \tag{4}$$

By applying some simplifications, Equation (3) becomes as follows

$$\hat{\varphi}_{rs}(z) = \begin{cases} \sqrt{\frac{2S}{\pi}} 2^{\frac{k}{2}} \prod\limits_{\substack{l=1 \\ l\neq s}}^{S} \left(\frac{z-\gamma_{rl}}{\gamma_{rs}-\gamma_{rl}}\right), & \frac{r-1}{2^k} \le z < \frac{r}{2^k}, \\ 0, & otherwise, \end{cases} \tag{5}$$

in Equation (5) $\gamma_{rs} = \frac{1}{2^{k+1}}(\lambda_s + 2r - 1)$ for $r = 1, 2, \ldots, 2^k$ and $s = 1, 2, \ldots, S$, and the values $\lambda_s = -\cos\left(\frac{(2s-1)\pi}{2S}\right)$ are the roots of the Chebyshev polynomial [12] of order S defined on $[-1, 1]$ for $s = 1, 2, \ldots, S$.

To create a wavelet basis with interpolation property, we assume a modified form of Equation (5) as follows:

$$\varphi_{rs}(z) = \begin{cases} \prod\limits_{\substack{l=1 \\ l\neq s}}^{S} \left(\frac{z-\gamma_{rl}}{\gamma_{rs}-\gamma_{rl}}\right), & \frac{r-1}{2^k} \le z < \frac{r}{2^k}, \\ 0, & otherwise. \end{cases}$$

We remind that the set $\{\varphi_{rs}(z) | r = 1, 2, \ldots, 2^k, s = 1, 2, \ldots, S, S \in \mathbb{R}\}$ constructs an orthogonal basis due to the weight function $w_r(z)$ for $L^2_{w_r}[0, 1]$ and

$$\langle \varphi_{rs}(z), \varphi_{r's'}(z) \rangle_{w_r} = \int_0^1 \varphi_{rs}(z)\varphi_{r's'}(z) w_r(z) \mathrm{d}z = \begin{cases} \frac{\pi}{S2^{k+1}}, & (r, s) = (r', s'), \\ 0, & (r, s) \neq (r', s'). \end{cases}$$

2.2 Approximating a Function of a Variable

A function $y(z) \in L^2_{w_r}[0, 1]$ may be estimated by the ChC wavelets as follows:

$$y(z) \simeq \sum_{r=1}^{2^k} \sum_{s=1}^{S} c_{rs}\varphi_{rs}(z) = \mathbf{C}^T \mathbf{\Phi}(z), \tag{6}$$

where

$$\begin{aligned} \mathbf{C} &= [c_{11}, c_{12}, \ldots, c_{1S} | c_{21}, c_{22}, \ldots, c_{2S} | \ldots | c_{2^k 1}, c_{2^k 2}, \ldots, c_{2^k S}]^T, \\ \mathbf{\Phi}(z) &= [\varphi_{11}(z), \varphi_{12}(z), \ldots, \varphi_{1S}(z) | \varphi_{21}(z), \ldots, \varphi_{2S}(z) | \ldots | \varphi_{2^k 1}(z), \ldots, \varphi_{2^k S}(z)]^T \end{aligned} \tag{7}$$

and

$$c_{rs} = y(\gamma_{rs}), \qquad r = 1, 2, \ldots, 2^k, \quad s = 1, 2, \ldots, S.$$

As it is known, c_{rs} are the entries of vector C.

2.3 Approximating a Function of Two Variable

Suppose $y(z,t)$ is a function of two variables defined over the interval $z \in [0,1]$ and $t \in [0,1]$, then $y(z,t)$ can be extended as following,

$$y(z,t) \simeq \mathbf{\Phi}^T(z) Y \mathbf{\Phi}(t).$$

The above statement is clarified by the following explanation:

Remark 1. Equation (6) can be shown in a simpler form as follows

$$y(z) \simeq \sum_{p=1}^{\hat{s}} y_p \varphi_p(z) = \mathbf{Y}^T \mathbf{\Phi}(z),$$

where $\hat{s} = 2^k S$, $y_p = y_{rs}$ and $\varphi_p(z) = \varphi_{rs}$ for the index $p = (r-1)S + s$.

For example, we have shown the function $y(z) = \sin(z)$ and its approximation $y(z) \approx Y^T \Phi(z)$ for $S = 2$, $k = 2$ in Figure 1.

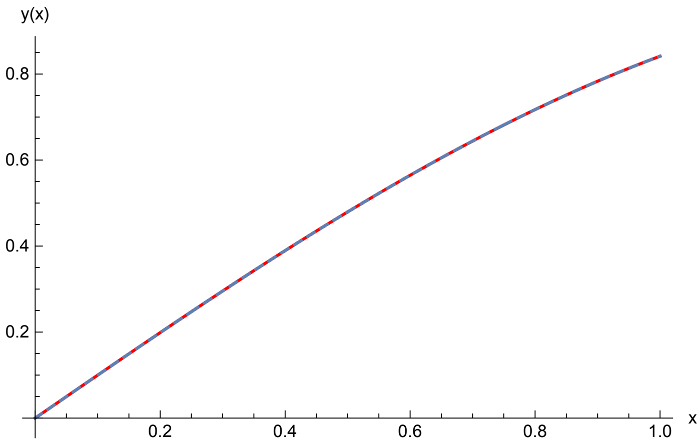

Figure 1: The function $\sin(z)$ and its approximatin with the ChC wavelets for $S = 2$, $k = 2$.

Remark 2. The ChC wavelets could be utilized to develop all kind of function $y(z,t) \in L^2_{w_{r,r'}}([0,1] \times [0,1])$

$$y(z,t) \simeq \sum_{p=1}^{\hat{s}} \sum_{q=1}^{\hat{s}} y(z_p, t_q) \varphi_p(z) \varphi_q(t) = \mathbf{\Phi}^T(z) Y \mathbf{\Phi}(t), \tag{8}$$

where $\hat{s} = 2^k S$, $Y = [y_{pq}]$ and its elements are determined as $y_{pq} = y(z_p, t_q)$.

For example, the function $y(z,t) = t - \cos(10zt)$ and its approximations with $S = 3$, $k = 1$ and $S = 3$, $k = 2$, are shown in Figures 2, 3 and 4, respectively.

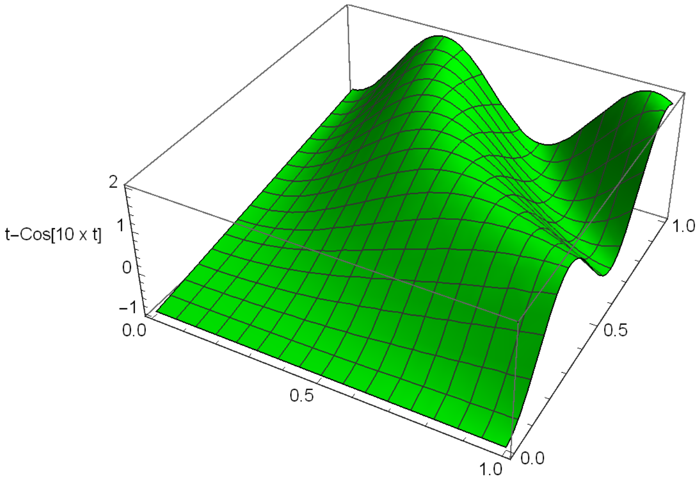

Figure 2: $y(z,t) = t - \cos(10zt)$.

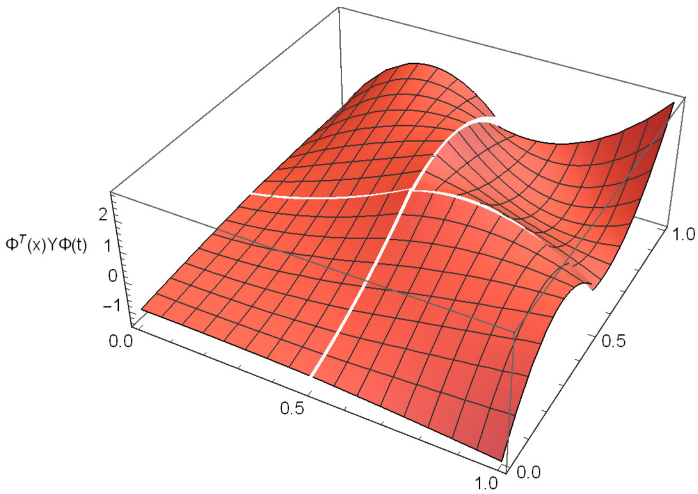

Figure 3: $y(z,t) \simeq \Phi^T(z) Y \Phi(t)$ with $S = 3$, $k = 1$.

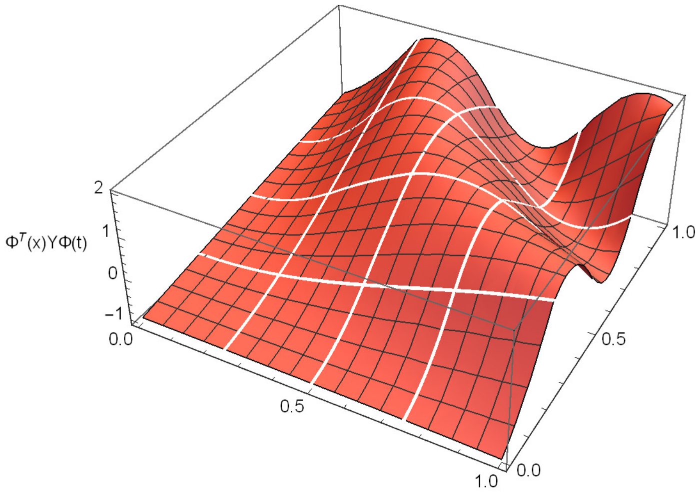

Figure 4: $y(z,t) \simeq \Phi^T(z)Y\Phi(t)$ with $S=3$, $k=2$.

2.4 ChC Wavelets Operational Matrix

The operational matrix of integration of the ChC wavelets has been derived in [22]. The integration of the vector $\mathbf{\Phi}(t)$ defined in Equation (7) can be obtained as

$$\int_0^z \mathbf{\Phi}(\tau)\mathrm{d}\tau \simeq \mathbf{P}\mathbf{\Phi}(z),$$

where $\mathbf{P}$ is an $\hat{s}\times\hat{s}$ operational matrix for integration and is given by

$$\mathbf{P} = \begin{pmatrix} W & V & V & V & \dots & V \\ 0 & W & V & V & \dots & V \\ 0 & 0 & \ddots & \ddots & \ddots & V \\ \vdots & \vdots & \ddots & W & V & V \\ 0 & 0 & \dots & 0 & W & V \\ 0 & 0 & 0 & \dots & 0 & W \end{pmatrix}_{\hat{s}\times\hat{s}}. \tag{9}$$

In Equation (9), $W=[w_{ij}]$ and $V=[v_{ij}]$ are $S\times S$ matrices, and their components are obtained via using the relations below:

$$w_{ij} = \frac{1}{2^{k+1}}\int_{-1}^{\lambda_j}\Big(\prod_{\substack{l=1\\ l\neq i}}^{S}\big(\frac{\tau-\lambda_l}{\lambda_i-\lambda_l}\big)\Big)\mathrm{d}\tau, \qquad v_{ij} = \frac{1}{2^{k+1}}\int_{-1}^{1}\Big(\prod_{\substack{l=1\\ l\neq i}}^{S}\big(\frac{\tau-\lambda_l}{\lambda_i-\lambda_l}\big)\Big)\mathrm{d}\tau.$$

As an explanatory instance for $k=1$, $S=2$ and $k=1$, $S=3$ we have

$$\mathbf{P}=\begin{pmatrix} \frac{1}{8}-\frac{1}{16\sqrt{2}} & \frac{1}{8}+\frac{3}{16\sqrt{2}} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8}-\frac{3}{16\sqrt{2}} & \frac{1}{8}+\frac{1}{16\sqrt{2}} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{8}-\frac{1}{16\sqrt{2}} & \frac{1}{8}+\frac{3}{16\sqrt{2}} \\ 0 & 0 & \frac{1}{8}-\frac{3}{16\sqrt{2}} & \frac{1}{8}+\frac{1}{16\sqrt{2}} \end{pmatrix}_{4\times 4},$$

$$\mathbf{P}=\begin{pmatrix} \frac{1}{18}-\frac{1}{32\sqrt{3}} & \frac{1}{18}+\frac{1}{8\sqrt{3}} & \frac{1}{18}+\frac{\sqrt{3}}{32} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{5}{36}-\frac{1}{4\sqrt{3}} & \frac{5}{36} & \frac{5}{36}+\frac{1}{4\sqrt{3}} & \frac{5}{18} & \frac{5}{18} & \frac{5}{18} \\ \frac{1}{18}-\frac{\sqrt{3}}{32} & \frac{1}{18}-\frac{1}{8\sqrt{3}} & \frac{1}{18}+\frac{1}{32\sqrt{3}} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ 0 & 0 & 0 & \frac{1}{18}-\frac{1}{32\sqrt{3}} & \frac{1}{18}+\frac{1}{8\sqrt{3}} & \frac{1}{18}+\frac{\sqrt{3}}{32} \\ 0 & 0 & 0 & \frac{5}{36}-\frac{1}{4\sqrt{3}} & \frac{5}{36} & \frac{5}{36}+\frac{1}{4\sqrt{3}} \\ 0 & 0 & 0 & \frac{1}{18}-\frac{\sqrt{3}}{32} & \frac{1}{18}-\frac{1}{8\sqrt{3}} & \frac{1}{18}+\frac{1}{32\sqrt{3}} \end{pmatrix}_{6\times 6}.$$

3. The Proposed Computational Scheme

In order to search a solution using the ChC wavelets, at first, we will explain the following theorem.

Theorem 3.1. *Suppose $\mathcal{T}:[0,1]\times[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ be a continuous real-valued operation and $Y^T\mathbf{\Phi}(t)$ is an estimation of the function $y(t)$ using the ChC wavelets. Then we have*

$$\mathcal{T}(z,t,y(t))\simeq\mathbf{\Phi}^T(z)Y\mathbf{\Phi}(t),$$

where $Y=[y_{pq}]$ and its elements are calculated as $y_{pq}=\mathcal{T}(z_p,t_q,y_q)$ for $p=1,2,\ldots,\hat{s}$ and $q=1,2,\ldots,\hat{s}$.

Proof. The evidence is clear via assuming Equation (8). □

The Theorem 3.1 and the outcome achieved for the ChC wavelets are applied in order to figure out the issue shown in Equation (2). For this aim, the target solution by the ChC wavelets with unknown coefficients is estimated. The cardinality of the stated basic functions as well as the resulted operational matrix are applied for converting the principal problem to an adaptive algebraic system of equations. An answer is approximately acquired by resolving the resulted system and computing the coefficients of the expansion by a sutiable approch. Due to that, consider this

$$y(z)\simeq Y^T\mathbf{\Phi}(z), \tag{10}$$

where Y is an $\hat{s}$-column vector which has the unknown coefficients and the vector $\mathbf{\Phi}(z)$ is supposed as in Equation (7).

The ChC wavelets are applied to state the function $g(z)$ as

$$g(z)\simeq G^T\mathbf{\Phi}(z), \tag{11}$$

whereas the coefficients of the ChC wavelets cover in the $\hat{s}$-column vector G.

Replacing Equations (10) and (11) into Equation (2), we have

$$Y^T\mathbf{\Phi}(z) \simeq G^T\mathbf{\Phi}(z) + \int_0^z K(z,t,Y^T\mathbf{\Phi}(t))\mathrm{d}t. \tag{12}$$

In addition, using Theorem 3.1, leads to

$$K(z,t,Y^T\mathbf{\Phi}(t)) \simeq \mathbf{\Phi^T}(z)\mathbf{K}\mathbf{\Phi}(t), \tag{13}$$

whereas

$$\mathbf{K} = [k_{pq}] = K(z_p, t_q, y_q), \qquad p = 1,2,\ldots,\hat{s}, \quad q = 1,2,\ldots,\hat{s}.$$

Equation (13) is replaced in Equation (12) which results in

$$Y^T\mathbf{\Phi}(z) \simeq G^T\mathbf{\Phi}(z) + \int_0^z \mathbf{\Phi^T}(z)\mathbf{K}\mathbf{\Phi}(t)\mathrm{d}t, \tag{14}$$

or the following equivalent

$$Y^T\mathbf{\Phi}(z) \simeq G^T\mathbf{\Phi}(z) + \mathbf{\Phi^T}(z)\mathbf{K}\int_0^z \mathbf{\Phi}(t)\mathrm{d}t. \tag{15}$$

By means of operational matrix, the mentioned relation is potentially shown as

$$Y^T\mathbf{\Phi}(z) \simeq G^T\mathbf{\Phi}(z) + \mathbf{\Phi^T}(z)\mathbf{K}\mathbf{P}\mathbf{\Phi}(z). \tag{16}$$

Also, using cardinality of the basic function, this equation is achieved as

$$\mathbf{\Phi^T}(z)\mathbf{K}\mathbf{P}\mathbf{\Phi}(z) \simeq diag(\mathbf{K}\mathbf{P})\mathbf{\Phi}(z) = \Gamma^T\mathbf{\Phi}(z). \tag{17}$$

Hence, Equations (10)-(17) give

$$Y^T\mathbf{\Phi}(z) \simeq G^T\mathbf{\Phi}(z) + \Gamma^T\mathbf{\Phi}(z),$$

and we attain the below system of nonlinear algebraic equations by employing the orthogonal property of the ChC wavelets:

$$Y^T - G^T - \Gamma^T = 0.$$

The resulting system can be solved with a suitable method for determining Y. At last, the estimated solution is gained by placing Y in Equation (2).

4. Convergence Analysis

Here, the convergence of the proposed approach is investigated. For this purpose, we analyze some of the mathematical prerequisites required in this unit.

Definition 4.1. [12] Consider that the weight function w is on the interval (ε, δ) and $\bar{s} \geq 0$ is assumed to be a nonnegative integer. The Sobolev space $\mathbb{S}_w^{\bar{s}}(\varepsilon, \delta)$ is illustrated as follows

$$\mathbb{S}_w^{\bar{s}}(\varepsilon, \delta) = \{y \in L_w^2(\varepsilon, \delta) : \; for \quad p = 0, 1, \ldots, \bar{s}, \quad y^{(p)} \in L_w^2(\varepsilon, \delta)\}.$$

Remark 3. The weighted inner product, introdued here, is mostly utilized in handling the Sobolev space mentioned above

$$< y, v >_{\bar{s},w} = \sum_{p=0}^{\bar{s}} \int_{\varepsilon}^{\delta} y^{(p)}(t) v^{(p)}(t) w(t) \mathrm{d}t.$$

Then, $\mathbb{S}_w^{\bar{s}}(\varepsilon, \delta)$ with the equipped norm

$$\| \, y \, \|_{\mathbb{S}_w^{\bar{s}}(\varepsilon,\delta)} = \Big(\sum_{p=0}^{\bar{s}} \|y^{(p)}\|^2_{L_w^2(\varepsilon,\delta)} \Big)^{1/2},$$

produces a Hilbert space.

Remark 4. The semi-norm in relation with the mentioned norm is described as

$$|y|_{\mathbb{S}_w^{\bar{s};S}(\varepsilon,\delta)} = \Big(\sum_{p=\min(\bar{s},S+1)}^{\bar{s}} \|y^{(p)}\|^2_{L_w^2(\varepsilon,\delta)} \Big)^{1/2}.$$

Remark 5. Recall that $|y|_{\mathbb{S}_w^{\bar{s};S}(\varepsilon,\delta)} \leq \| \, y \, \|_{\mathbb{S}_w^{\bar{s}}(\varepsilon,\delta)}$. Whenever $\bar{s} \leq S+1$, there is

$$|y|_{\mathbb{S}_w^{\bar{s};S}(\varepsilon,\delta)} = \| \, y^{(\bar{s})} \, \|^2_{L_w^2(\varepsilon,\delta)} = |y|_{\mathbb{S}_w^{\bar{s}}(\varepsilon,\delta)}.$$

Theorem 4.2. (Truncation error of the ChC wavelets expansion [22]) *Assume that $y \in \mathbb{S}_{w_r}^{\bar{s}}(0,1)$, $w_r(t)$ as in Equation (4) is the weight function, $P_{k,S}y(t) = \sum_{r=1}^{2^k} \sum_{s=1}^{S} c_{rs}\varphi_{rs}$, where the functions $\varphi_{rs}(t)$ are illustrated in Equation (5), and $c_{rs} = y(\gamma_{rs})$. Hence, the truncation error $y - P_{k,S}y$ complies*

$$\| \, y - P_{k,S}y \, \|_{L_w^2(0,1)} \leq \bar{C}_{\bar{s}}(S-1)^{-\bar{s}} \Big(\sum_{r=1}^{2^k} \sum_{q=\min(\bar{s},S)}^{\bar{s}} \big(\frac{1}{2^{k+1}}\big)^{2q} \| \, y^{(q)} \, \|^2_{L_{w_r}^2(I_{kr})} \Big)^{1/2},$$

where $I_{kr} = \left(\frac{r-1}{2^k}, \frac{r}{2^k}\right)$. Additionally, we have

$$\| \, y - P_{k,S}y \, \|_{L^\infty(0,1)} \leq$$
$$\hat{C}_{\bar{s}}(S-1)^{\frac{1}{2}-\bar{s}} 2^{(k+1)/2} \max_{r=1,2,\ldots,2^k} \Big(\sum_{q=\min(\bar{s},S)}^{\bar{s}} \big(\frac{1}{2^{k+1}}\big)^{2q} \| \, y^{(q)} \, \|^2_{L_{w_r}^2(I_{kr})} \Big)^{1/2}, \qquad (18)$$

at the maximum norm, $\hat{C}_{\bar{s}}$ is a positive constant that depends on $\bar{s}$ and is independent of k and S.

Definition 4.3. (Lipschitz continuous function [25]). Suppose $K \in \mathcal{C}([0,1] \times [0,1] \times \mathbb{R}, \mathbb{R})$. This function is named Lipschitz continuous functions considering their third arguments if there exist real and non-negative constant μ, where

$$|K(z,t,y) - K(z,t,v)| \leq \mu |y - v|, \qquad \forall y, v \in \mathbb{R}. \tag{19}$$

Theorem 4.4. *Consider that $g \in \mathbb{S}^{\bar{s_1}}_{w_r}(0,1)$, K complies the conditions described in Equation (19) and $y \in \mathbb{S}^{\bar{s_2}}_{w_r}(0,1)$ be the exact solution of the issue given in Equation (2). If $g_{\hat{s}}$ be the ChC wavelets expansion of g where $\hat{s} = 2^k S$, and $y_{\hat{s}}$ be the estimated solution of Equation (2) with the usage of the presented method, then we have*

$$\|y - y_{\hat{s}}\|_{L^\infty(0,1)} \lesssim \nu_1 + \mu \nu_2,$$

where

$$\nu_1 = \hat{C}_{\bar{s_1}} (S-1)^{\frac{1}{2} - \bar{s_1}} 2^{(k+1)/2} \max_{r=1,2,\ldots,2^k} \Big(\sum_{q=\min(\bar{s_1},S)}^{\bar{s_1}} \big(\frac{1}{2^{k+1}}\big)^{2q} \| g^{(q)} \|^2_{L^2_{w_r}(I_{kr})} \Big)^{1/2},$$

$$\nu_2 = \hat{C}_{\bar{s_2}} (S-1)^{\frac{1}{2} - \bar{s_2}} 2^{(k+1)/2} \max_{r=1,2,\ldots,2^k} \Big(\sum_{q=\min(\bar{s_2},S)}^{\bar{s_2}} \big(\frac{1}{2^{k+1}}\big)^{2q} \| y^{(q)} \|^2_{L^2_{w_r}(I_{kr})} \Big)^{1/2},$$

and $\hat{C}_{\bar{s_1}}$ as well as $\hat{C}_{\bar{s_2}}$ are positive constants that depend on $\bar{s_1}$ and $\bar{s_2}$ and are independent of $\hat{s}$.

Proof. Due to the presented process, we get

$$y_{\hat{s}}(z) \simeq g_{\hat{s}}(z) + \int_0^z K(z,t,y_{\hat{s}}(t)) \mathrm{d}t. \tag{20}$$

Subducting Equation (20) from Equation (2) there is

$$y(z) - y_{\hat{s}}(z) \simeq g(z) - g_{\hat{s}}(z) + \int_0^z K(z,t,y(t)) - \int_0^z K(z,t,y_{\hat{s}}(t)) \mathrm{d}t. \tag{21}$$

Equation (21) can be rewritten equivalently as

$$y(z) - y_{\hat{s}}(z) \simeq g(z) - g_{\hat{s}}(z) + \int_0^z [K(z,t,y(t)) - K(z,t,y_{\hat{s}}(t))] \mathrm{d}t.$$

Then, we achieve

$$|y(z) - y_{\hat{s}}(z)| \lesssim |g(z) - g_{\hat{s}}(z)| + \int_0^z |K(z,t,y(t)) - K(z,t,y_{\hat{s}}(t))| \mathrm{d}t.$$

Utilizing Equation (19), leads to

$$|y(z) - y_{\hat{s}}(z)| \lesssim |g(z) - g_{\hat{s}}(z)| + \mu \int_0^z |y(t) - y_{\hat{s}}(t)| \mathrm{d}t.$$

So we have

$$\|y - y_{\hat{s}}\|_{L^\infty(0,1)} \lesssim \|g - g_{\hat{s}}\|_{L^\infty(0,1)} + \mu\|y - y_{\hat{s}}\|_{L^\infty(0,1)}. \tag{22}$$

At last, replacing Equation (22) and Equation (18) in Theorem 4.2 leads to the proof. □

5. Numerical Examples

In this part, we propose several instances to estimate the solution of Volterra IEs of the second kind using the numerical method described in the previous sections. In order to demonstrate the performance of the method and clarify the efficiency and the accuracy of the presented method, we compare the results of our method with the results of some previous methods. The results are reported in Tables 1, 2, 3 and 4. Where $y(t)$ and $Y^T\mathbf{\Phi}(t)$ are the exact solution and the calculated solution by the proposed method, respectively. The numerical experiments are implemented in the software Mathematica 7.

Example 5.1. Consider the nonlinear VIE (see [33, 46])

$$y(z) = \frac{\sin(z)}{\sin(z)+\cos(z)} + \int_0^z \frac{2te^{t-z}\cos y(t)}{\sin(z)+\cos(z)}\mathrm{d}t, \qquad z \in [0,1].$$

The exact solution is $y(z) = z$.

Table 1: Absolute errors for Example 5.1.

z	Proposed method with $S=7, k=3$	Proposed method with $S=9, k=3$	Method in [33] with $m=5$, $N=10$ and $p=4$
0.2	$1.16573E-16$	$8.32667E-17$	$6.6761E-10$
0.3	$1.47105E-15$	$5.55112E-17$	$8.7809E-10$
0.4	$2.88658E-15$	$1.38778E-16$	$9.3511E-10$
0.6	$3.77476E-15$	$5.55112E-17$	$2.0606E-08$
0.7	$2.83107E-15$	$2.08167E-16$	$1.1895E-07$
0.8	$2.91434E-15$	$6.93889E-16$	$4.7716E-07$
1	$7.63278E-15$	$2.22045E-16$	$1.2962E-06$

Example 5.2. Consider the nonlinear VIE (see [10]):

$$y(z) = \frac{1}{2}z(2+z) - 2z\arctan(z) + \ln(1+z^2) + \int_0^z (-y(t) + 2\arctan(y(t))\mathrm{d}t, \quad 0 \le z \le 1.$$

The analytical solution of this example is $y(z) = z$.

Table 2: Absolute errors for Example 5.2.

z	Proposed method with $S = 7, k = 3$	Proposed method with $S = 9, k = 3$	Method in [10] with $j = 129$
0.125	$2.97540E - 14$	$4.16334E - 17$	$1.17E - 7$
0.250	$5.52058E - 14$	$5.55112E - 17$	$6.74E - 8$
0.375	$7.69385E - 14$	$2.22045E - 16$	$4.52E - 6$
0.5	$5.38944E - 14$	$9.02056E - 17$	$2.16E - 5$
0.625	$2.46192E - 14$	$2.49800E - 16$	$1.68E - 5$
0.750	$6.73073E - 15$	$4.85723E - 16$	$3.89E - 5$
0.875	$7.07767E - 16$	$2.63678E - 16$	$2.77E - 5$
1	$4.16334E - 16$	$5.27356E - 16$	$9.51E - 5$

Example 5.3. Consider the VIE (see [10]):

$$y(z) = \frac{1}{3} z \cos(z^3) + z^3 - \frac{z}{3} + \int_0^z z t^2 \sin(y(t)) \mathrm{d}t, \quad 0 \le z \le 1,$$

has the exact solution $y(z) = z^3$.

Table 3: Absolute errors for Example 5.3.

z	Proposed method with $S = 7,\ k = 3$	Proposed method with $S = 9,\ k = 3$	Method in [10] with $j = 129$
0.125	$4.90927E - 15$	$2.27682E - 17$	$1.43E - 6$
0.250	$6.69846E - 14$	$8.50015E - 17$	$4.74E - 6$
0.375	$3.65291E - 13$	$2.15106E - 16$	$7.99E - 6$
0.5	$1.21364E - 12$	$2.68882E - 16$	$6.57E - 5$
0.625	$2.59269E - 12$	$1.09981E - 15$	$2.53E - 5$
0.750	$2.00723E - 12$	$6.43929E - 15$	$6.41E - 5$
0.875	$1.05443E - 11$	$1.73819E - 14$	$5.21E - 5$
1	$1.57594E - 11$	$5.19029E - 15$	$3.35E - 4$

Example 5.4. Consider the VIE (see [31, 36]):

$$y(z) = \frac{3}{2} - \frac{1}{2} e^{-2z} - \int_0^z (y(t)^2 + y(t)) \mathrm{d}t, \quad 0 \le z \le 1.$$

The exact solution of this equation is $y(z) = e^{-z}$.

6. Conclusion

An efficient approach for finding an estimated solution for Volterra integral equations of the second kind has been implemented in this paper using the Chebyshev

Table 4: Absolute errors for Example 5.4.

z	Proposed method with $S = 7$, $k = 3$	Proposed method with $S = 9$, $k = 3$	Method in [36] with $M = 3$, $N = 6$
0	$2.22045E-16$	0.00000000000	0.00000000000
0.1	$1.75415E-14$	$3.33067E-16$	$1.626820E-4$
0.2	$1.38778E-14$	$5.55112E-16$	$2.438211E-4$
0.3	$4.88498E-15$	$2.22045E-16$	$1.275379E-4$
0.4	$3.80251E-15$	$2.63678E-16$	$2.858734E-4$
0.5	$3.16414E-15$	$1.42247E-16$	$3.999309E-4$
0.6	$8.82627E-15$	$2.22045E-16$	$2.250428E-4$
0.7	$7.71605E-15$	$1.80411E-16$	$3.594621E-4$
0.8	$3.66374E-15$	$6.93889E-17$	$1.043774E-4$
0.9	$2.9976E-15$	$1.97758E-16$	$2.968310E-4$

cardinal wavelets. A nonlinear system of algebraic equations using the exceptional properties of the Chebyshev cardinal wavelets is obtained from the mentioned problem. The numerical instances showed the good standing and relevance of our method. In order to illustrate the applicability of the present method, the corresponding convergence was analytically reviewed in the Sobolev space. Research to find more applications of these orthogonal basis functions and the method of using these bases is one of the goals of our research group. It can also be mentioned that our method has the potential to be easily extended and implemented to the nonlinear Volterra and Fredholm integral equations of the first kind and nonlinear systems of integral Volterra equations and solve nonlinear constrained optimal control problems.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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