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Exact Solution of Schrödinger Equation for Pentaquark Systems

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Abstract

In this paper, we present an exact analytical solution for five interacting quarks. We solve the Schrödinger equation for pentaquarks in the framework of five-body and two-body problems. For this purpose, we utilize Yukawa potential in Jacobi coordinates. Also finding the relation between the reduced masses and coupling constants of pentaquarks, we obtain the coupling constant of Yukawa potential for pentaquark systems. We calculate the energy of these systems in their ground state. The results are well consistent with the theoretical results. Our procedure to obtain these results is appropriate for other potentials and n-body systems.

Keywords: Exact solution, Schrödinger equation, Pentaquark, Coupling constant, Yukawa potential.

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1. Introduction

Nucleons are the building blocks of nuclei and are composite extended objects. The internal structure of these particles at low energies may be attributed to their three bound quarks q^3. Nucleons and their excited states, i.e. baryons, are accommodated into flavor singlets, octets, and decuplets. The strangeness of

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baryons is zero (nucleon, Δ) or negative (Λ, Σ, Ξ and Ω). Baryons whose quantum numbers cannot be derived from triplets of quarks are known as exotic.

LEPS Collaboration discovered the first exotic baryon $\Theta(1540)$ with the positive strangeness of $S = +1$, and this encouraged further theoretical and experimental studies. The width of this state is below $20MeV$. The $NA49$ Collaboration [1] detected traces of the exotic baryon Ξ (1862) with an strangeness of $S = -2$. The Ξ^{--} and Θ^+ resonances are perceived as $q^4\overline{q}$ pentaquarks belonging to a flavor antidecuplet whose quark structures are $ddss\overline{u}$ and $uudd\overline{s}$, respectively. The heavy pentaquark Θ_c (3099) may be also observed. In this system, an anticharm quark replaces an antistrange quark in the Θ^+.

There are theoretical studies of the parities and spins of these states. Some of the studies include chiral constituent quark models [2], correlated quark (or cluster) models [3], and chiral soliton models [4] which stimulated experimental studies. Ref. [5] presents a review of the theoretical literature on pentaquark models. Ref. [6] holds that a pentaquark with hidden-charm occurring in weak decays of Λ_0^b emerges in proton-nucleus collisions without electroweak intermediaries. It studies the cross-section of production for several scenarios of internal structure and reports a sizable cross-section. Azizi et al. make use of QCD sum-rules analyses on hidden-charm pentaquark states with the spin parities $J^P = \frac{5}{2}^{\pm}$ and $J^P = \frac{3}{2}^{\pm}$ in order to calculate their residues and masses [7].

Ref. [8] investigated pentaquarks and tetraquarks in lattice QCD Monte Carlo simulations. It is inspired by the findings of multi-quark systems. They studied the multi-quark potential in lattice QCD, explained the inter-quark interactions that occur in multi-quark systems and dealt with the accurate calculations of masses of low-lying 5Q states. Ruilin Zhu et al. [9] studied the spectra of doubly-heavy tetraquarks and pentaquarks in the non-relativistic constituent quark model. To solve the Schrödinger equation, they used the model-independent variational method. The chosen test radial functions in this equation are symmetric for the light quarks. They classified tetraquarks and pentaquarks based on the heavy quark symmetry and investigated the respective decay attributes.

A system composed of interacting non-relativistic constituent quarks may be the simplest yet most realistic model of hadronic systems. Many-body Schrödinger equation must be solved to identify the wave function in the model. That is of course a challenging task as the quark-quark interaction is dependent upon color, flavor, and spin. A variety of approximate methods are traditionally employed to solve the equation [10–14]. In this work, we use an alternative method for determining the wave function of five interacting constituent quarks in the flavor-exotic multiquark hadronic sector.

It is worth mentioning that the central part of Yukawa potential was used in this paper, and we will consider the tensor part in our subsequent works too.

2. Formalism

The Schrödinger equation may be studied in arbitrary n-dimensional spaces. Then, it will be easy to calculate the radial wave function via arbitrary n-dimensional Schrödinger equation. Jacobi coordinates can be utilized to illustrate the relative motions of the constituents. The hyperspherical formalism can be used for submitting a method to solve Schrödinger equation [15].

In spherical coordinates with an r-dependent potential, nonrelativistic Schrödinger equation is as follows:

$$H\Psi_{n,m,l}(x,\Omega_D) = E\Psi_{n,m,l}(x,\Omega_D),$$

where

$$\Big(\frac{-\hbar^2}{2\mu}\nabla_D^2 + V(x) - E_{n,l}\Big)\psi_{n,m,l}(x,\Omega_D) = 0.$$

For few-body systems, Jakobian variable change is employed. If the quarks are assumed to lie in the points

$$\nabla_D^2 = \frac{\partial^2}{\partial x^2} + \frac{D-1}{x}\frac{\partial}{\partial x} - \frac{l(l+D-2)}{x^2},$$

then the few-body equation is converted into a one-variable equation by using the hyper-radius. The coordinates' origin is assumed to be on the mass center of the few-body system.

$$\begin{aligned}
\vec{\rho}_1 &= \sqrt{\frac{\mu_{1,2}}{\mu}}(\vec{r}_1 - \vec{r}_2),\\
\vec{\rho}_2 &= \sqrt{\frac{\mu_3}{\mu}}(\frac{\vec{r}_1 + \vec{r}_2}{2} - \vec{r}_3),\\
&\vdots\\
\overrightarrow{\rho_i} &= \sqrt{\frac{i}{i+1}}(\overrightarrow{r_{i+1}} - \frac{1}{i}\sum_{j=1}^{i}\overrightarrow{r_j}),
\end{aligned}$$

where

$$\mu_{1,2} = \frac{m_1 m_2}{m_1+m_2}, \quad \mu_3 = \frac{(m_1+m_2)m_3}{m_1+m_2+m_3}, \quad \mu = \frac{m_1+m_2+\cdots+m_{N+1}}{N+1},$$

and

$$\vec{R} = \frac{\sum m_i \vec{r_i}}{\sum m_i}.$$

With equal masses of quarks, these equations turn out as:

$$\vec{R} = \frac{\sum_i^{N+1}\vec{r}_i}{N+1}, \qquad r = (\sum_{i=1}^{N}\rho_i^2)^{\frac{1}{2}}.$$

The angular section of the wave function is excluded from our consideration.

$$\psi_{n,l,m}(x,\Omega_D) = R_{n,l}(x)Y_l^m(\Omega_D). \tag{1}$$

The radial Schrödinger equation is further simplified by applying the variable change (1):

$$\Big(\frac{d^2}{dx^2} + \frac{D-1}{x}\frac{d}{dx} - \frac{l(l+D-2)}{x^2}\Big)R_{n,l}(x) + \frac{2\mu}{\hbar^2}\Big(E - V(r)\Big)R_{n,l}(x) = 0.$$

3. Pentaquark systems composed of mesons and baryons

Multi-hadron systems have recently become important issues of interest in hadron physics. A common case is the atomic nucleon as the bound state of neutrons and protons. However, similar systems of baryon number one or zero have been rarely detected. Perhaps the typical example is $\Lambda(1405)$, a quasi-bound state of $\overline{K}N$ and $\pi\Sigma$ [16]. Also, the isoscalar meson could be among the quasi-bound states of two mesons [17]. These hadronic components result from hadron-hadron interactions and may exist in a variety of baryonic and mesonic systems. Hadron composites may also be utilized to investigate hadron dynamics in nuclear matter [18].

A structure in exotic channels containing only one heavy quark has been also suggested: $\overline{D}N$ with a quark content of $uudd\bar{c}$ [19]. It is the charm counterpart of pentaquark $\Theta^+ \sim uudd\bar{s}$. Charm pentaquarks of different forms have also been investigated [20, 21]. The one-pion exchange interaction was used to arrive at a bound system consisting of a nucleon and a $\overline{D}$ meson.

To simplify the Schrödinger equation we use the following variable change

$$R_{n,l}(x) = x^{-\frac{D+1}{2}}\phi_{n,l}(x).$$

In general, we considered the number of particles $A = N + 1$ and $D = 3N$. In this part, we analyze pentaquarks as systems consisting of mesons and baryons. Therefore $A = 2$, $N = 1$, $D = 3$, then

$$R(x) = x^{-1}\phi_{n,l}(x). \tag{2}$$

Using the variable change (2), the Schrödinger equation becomes:

$$\Big(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx}\Big)R_{n,l}(x) + \frac{2\mu}{\hbar^2}\Big(E - V(x)\Big)R_{n,l}(x) = 0. \tag{3}$$

Using Equations (2) and (3) for the ground state ($l = 0$) can be reformulated as this compact form:

$$\frac{2\mu}{\hbar^2}\Big(E - V(x)\Big)\phi(x) + \frac{d^2\phi(x)}{dx^2} = 0. \tag{4}$$

Yukawa potential was first introduced in the 1930s to study strong interaction of nucleons through meson exchange [22]. In solid-state physics and plasma physics, it is called Thomas-Fermi potential and Debye-Huckel potential, respectively [23]. Unlike Yukawa potential, the corresponding Schrödinger equation cannot be solved precisely and analytically. Various approximations to the problem have been already adopted [24, 25].

Gonul et al. solved the bound state problem for the potentials of Yukawa type within the framework of Riccati equation [26]. Karakoc and Boztosun also solved the radial Schrödinger equation for these potentials through asymptotic iteration technique [27]. Liverts et al. solved the Schrödinger equation with a Yukawa potential via the quasi-linearization method [22]. The central section of Yukawa potential is

$$V(r) = -g^2 \frac{e^{-kx}}{x}, \tag{5}$$

where $k = m_\pi$ and g^2 is the coupling constant. Using Yukawa potential extension, we arrive at

$$V(r) = -g^2(-a_1x^6 + b_1x^5 - c_1x^4 + d_1x^3 - e_1x^2 + f_1x - h_1 + \frac{L_1}{x}), \tag{6}$$

thus,

$$\left[\begin{array}{ll} a_1 = \frac{k^7}{7!}, & a = 2\mu g^2 a_1, \\ b_1 = \frac{k^6}{6!}, & b = 2\mu g^2 b_1, \\ c_1 = \frac{k^5}{5!}, & c = 2\mu g^2 c_1, \\ d_1 = \frac{k^4}{4!}, & d = 2\mu g^2 d_1, \\ e_1 = \frac{k^3}{3!}, & e = 2\mu g^2 e_1, \\ f_1 = \frac{k^2}{2!}, & f = 2\mu g^2 f_1, \\ h_1 = k, & h = 2\mu g^2 h_1, \\ L_1 = 1, & L = 2\mu g^2 L_1. \end{array}\right. \tag{7}$$

By introducing Equations (6) and (7) into Equation (2), the following equation is obtained

$$\frac{d^2}{dx^2}\phi + 2\mu g^2(-a_1x^6 + b_1x^5 - c_1x^4 + d_1x^3 - e_1x^2 + f_1rx - h_1 + \frac{L_1}{x}) + 2\mu E\phi = 0. \tag{8}$$

To solve this equation, we use the anastz method [28]. In this method the general form of the proposed wave fanction is as follows

$$\phi(x) = N_0 M(x), \tag{9}$$

where N_0 is the normalized constant and

$$M(x) = x^\gamma f_\nu(x) \exp[Z(x)],$$

$$f_\nu(x) = \begin{cases} \prod_{c=1}^{\nu}(x-\alpha_i^\nu), & \nu = 1,2,3, \\ 1, & \nu = 0. \end{cases}$$

In the proposed wave function, $Z(x)$ is analogous to Hermite Polynomial. For the ground state we suppose $v = 0$. The wave function that is proposed will be

$$Z(x) = -\frac{1}{4}\alpha x^4 - \frac{1}{3}\beta x^3 - \frac{1}{2}\eta x^2 - \tau x.$$

By inserting wave function into Equation (9), the wave function coefficients are obtained as

$$\alpha = \sqrt{a}, \quad \beta = -\frac{b}{2\alpha}, \quad \eta = \frac{c-\beta^2}{2\alpha}, \quad \tau = \frac{-d+2\beta}{2\alpha}.$$

E is obtained through

$$E = -\frac{1}{2\mu}[h - \tau^2 - \eta(1+2\gamma)].$$

Through solving the Schrödinger equation, we also arrived at the following equation to calculate the coupling constant for these systems

$$g = -\frac{\gamma\tau}{\mu}.$$

It must be notified that we arrived at the equations based on the proposed wave function with order-4 approximation. As the approximations change, so do the equations.

Table 1: Binding energies and masses of heavy pentaquarks (MeV).

Pentaquark	Meson + Baryon	g^2	E_b	Mass	Mass in Ref
$\theta_c(uudd\bar{c})$	$uud + d\bar{c}$	0.163	-8.22	2903.27	2985 ±50 [29]
$\theta_b(uudd\bar{b})$	$uud + d\bar{b}$	0.13	-6.7	6245.3	6391 ±50 [29]

4. Pentaquark systems composed of five quarks

For the pentaquark systems composed of five quarks $A = 5$, $N = 4$ and $D = 12$. The five components' relative motion is described against the Jacobi coordinates.

$$\rho_1 = \frac{\vec{r_1} - \vec{r_2}}{\sqrt{2}},$$

$$\rho_2 = \frac{\vec{r_1} + \vec{r_2} - 2\vec{r_3}}{\sqrt{6}},$$

$$\rho_3 = \frac{\vec{r_1} + \vec{r_2} + \vec{r_3} - 3\vec{r_4}}{\sqrt{12}},$$

$$\rho_4 = \frac{\vec{r_1} + \vec{r_2} + \vec{r_3} + \vec{r_4} - 4\vec{r_5}}{\sqrt{20}}.$$

In this section, the ground state energy and coupling constant of pentaquarks Θ_c and Θ_b are identified and shown in Table 2. We calculate pentaquark masses by

$$M = \sum_i m_i + E.$$

Our results presented in Table 1 show a good agreement with other theoretical

Table 2: Heavy meson masses and binding energies (MeV).

Pentaquark	g^2	E_b	Mass
$\theta_c(uudd\bar{c})$	37.15	-530.09	2259.9
$\theta_b(uudd\bar{b})$	35.97	-510	5589.2

findings in [29]. By fitting the masses in the pentaquarks' constituent mesons and baryons, we can come up with appropriate findings on pentaquark masses.

In Ref. [30], the pentaquark Θ_c is considered as consisting of two different mesons and baryons so that $\Theta_c(uudd\bar{c})$ is composed of a meson $m_{ud} = 72MeV$ and a baryon whose mass is $m_{ud\bar{c}} = 2430MeV$ and Θ_c mass is $m_{\Theta_c} = 2985 \pm 50MeV$. With this new combination of meson and baryon, we obtain the pentaquark mass with the order-2 approximation of the wave function as $m_{\Theta_c} = 3139.85MeV$ and $E = -10.143MeV$.

Ref. [29] also considers m_{Θ_b} as composed of ud and udb in which the triquark effective mass is $m_{ud\bar{c}} = 5770MeV$. The reduced mass is taken as $\mu = 4640MeV$, and then $m_{\Theta_b} = 6398 \pm 50MeV$ is derived.

5. Conclusion

The N-body potential problem is used in both quantum mechanics and classical. The interacting potential in hypercentral formalism depends just on the hyper-radius in Jacobi relative coordinates. In this paper, we suggested an exact analytical solution for five quarks systems. Also we solved the Schrödinger equation for pentaquark systems in the framework of two five-body and two-body problems.

We used five-body Yukawa potential and an ansatz for the eigen-function to solve Schrödinger equation accurately for pentaquark quark system in three dimensions. Our results comply well with other theoretical works. The same procedure could be followed with some other potentials for interacting quarks and other states of energy. We will obtain the energy spectra and eigen-functions in our next studies.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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