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The Effect of the Caputo Fractional Derivative on Polynomiography

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Abstract

This paper presents the visualization process of finding the roots of a complex polynomial - which is called polynomiography - by the Caputo fractional derivative. In this work, we substitute the variable-order Caputo fractional derivative for classic derivative in Newton's iterative method. To investigate the proposed root-finding method, we apply it for two polynomials $p(z) = z^5 - 1$ and $p(z) = -2z^4 + z^3 + z^2 - 2z - 1$ on the complex plane and compute the MNI and CAI parameters. Presented examples show that through the expressed process, we can obtain very interesting fractal patterns. The obtained patterns show that the proposed method has potential artistic application.

Keywords: the Caputo fractional derivative, root-finding method, Newton's method, polynomiography.

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1. Introduction

One of the important goals in computer-aided design is to produce beautiful and new patterns, and this requires initiative and innovation. Nowadays, in carpet design and tapestry design, a designer has to be aware of new techniques to obtain

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interesting patterns [3]. Polynomials, one of the fundamental subjects of mathematics with diverse applications, along with the root-finding methods can be used for the generation of interesting patterns. According to the Fundamental Theorem of Algebra, any polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

of degree n, with real or complex coefficients, has n zeros. Polynomial root-finding is one of the oldest mathematical problems. Historical documents show that Sumerian (3000 B.C.) and Babylonians (2000 B.C.) dealt with it [3, 15]. The iterative process

$$z_{i+1} = N(z_i),$$

is a common process to find the roots of complex polynomial $P(z)$ and there are many iterative methods in the literature, see for example [4, 7, 18]. The best-known method for root-finding is Newtons iterative method

$$N(z_i) = z_i - \frac{P(z_i)}{P'(z_i)},$$

which was proposed in the 17th century. In 1879, Cayley investigated the behavior of Newtons method for equation $z^3 - 1 = 0$ in the complex plane, which is known as Cayleys problem. Finally, in 1919, Julia solved this problem.

Polynomiography, introduced by Kalantari, is an interesting subject in connection with polynomial root-finding. Polynomiography is defined to be "the art and science of visualization in approximation of the zeros of a complex polynomial, via fractal and non-fractal images, created using the mathematical convergence properties of iteration functions" [15, 16]. The created pattern through polynomiography, is called a polynomiograph. Polynomiography can be used for students to interest them in concepts such as polynomials, series, convergence and finding root with fractal images. Additionally, we can use it for the carpet industry and visual arts.

After introducing the polynomiography, the artistic applications of polynomiography attracted scientists' attention. In recent years, many scientists have proposed various iteration processes from fixed point theory to generate different kinds of patterns. Let us to recall below some known iteration processes:

1) Mann iteration [17]

$$z_{i+1} = \alpha N(z_i) + (1-\alpha) z_i, \qquad 0 < \alpha \leq 1,$$

2) Ishikawa iteration [14]

$$\begin{cases} v_i = \beta N(z_i) + (1-\beta) z_i \\ z_{i+1} = \alpha N(v_i) + (1-\alpha) z_i, \quad 0 < \alpha \leq 1, \quad 0 \leq \beta \leq 1, \end{cases}$$

3) $S-$ iteration [1]

$$\begin{cases} v_i = \beta N(z_i) + (1-\beta) z_i \\ z_{i+1} = \alpha N(v_i) + (1-\alpha) N(z_i), \quad 0 < \alpha \leq 1, \ \ 0 \leq \beta \leq 1. \end{cases}$$

In [9], eleven different iteration processes based on Newton calssic method were studied. Also a review of 18 different iteration processes can be observed in [10]. To create different and interesting patterns, it is sufficient to change some parameters. For instance, Gdawiec in [11] changed the usual convergence test and obtained very interesting and diverse patterns.

In most papers about polynomiography, the authors use constant parameters for simplicity. In [3], some modifications of the root-finding method, proposed in [4], were used with periodic parameters and authors investigated numerically some properties of the proposed methods by the mean number of iterations (MNI) and the convergence area index (CAI).

In [12], the classic derivative in the standard Newton method is replaced by the fractional Riemann-Liouville and the Caputo derivatives which are named the standard fractional Newton method and denoted by

$$z_{i+1} = z_i - \frac{P(z_i)}{D^\nu P(z_i)}. \tag{1}$$

Instead of the Picard fractional Newton method, the Mann, Ishikawa and S iterations versions were also investigated in [13]. In recent years, some fractional iterative methods have been proposed for solving nonlinear equation $f(x) = 0$. For instance, by using the Caputo and the Riemann-Liouville derivatives, two fractional Newton-type method

$$x_{i+1} = x_i - \Gamma(\nu+1) \frac{f(x_i)}{D^\nu f(x_i)},$$

$$x_{i+1} = x_i - \left(\Gamma(\nu+1) \frac{f(x_i)}{D^\nu f(x_i)} \right)^{\frac{1}{\nu}},$$

were introduced in [2, 6]. Also in [8], authors have proposed several Chebyshev-type schemes using the Caputo and the Riemann-Liouville derivatives. Candelarion et al. [5], presented the Conformable fractional Newton-type method for solving $f(x) = 0$, numerically.

Unlike the above works, in which fixed fractional derivatives have been used, we are going to use the variable fractional Caputo derivative, to obtain new fractal images. This paper is organized as follows: in Section 2, the basics of polynomiography are introduced. Next, in Section 3, some graphical examples and numerical results are presented. Finally, Section 4 ends this paper with some concluding remarks.

2. Polynomiography

To generate a polynomiograph, at first we consider polynomial $P(z)$, an area A in the complex plane ($A \subset \mathbb{C}$) and a root-finding method

$$z_{i+1} = N(z_i).$$

We can replace the above iteration process with other iteration processes e.g. Mann, Ishikawa and S iterations. Note that the aim of using other iterations in polynomiography, is to create new interesting images. Now, $z_0 \in A$ is selected and we proceed with the iteration process. The sequence $\{z_i\}_{i=0}^{\infty}$, which is named the orbit of the starting point z_0, either converges to a root of $P(z)$ or not. If $\{z_i\}_{i=0}^{\infty}$ converges to a root z^*, we say that z_0 is attracted to z^*. The basin of attraction of $z^*$is the following set

$$\{z_0 \in A \mid \lim_{i\to\infty} z_i = z^*\}.$$

Iteration process is finished when convergence test is satisfied or maximum number of iterations is reached. We can colour the starting point z_0 using two basic colouring functions:

- Iteration colouring: We assign a colour to z_0 according to the number of performed iterations (see Algorithm 1).

- Basin of attarction: Each root of polynomial $P(z)$ gets its own distinct colour and we assign a colour to z_0 according to the nearest root to the point at which we have stopped iterating (see Algorithm 2).

In above iterative processes, we need a stop criterion or a convergence test. For root finding methods the standard convergence tests are

$$|z_{i+1} - z_i| < \varepsilon, \qquad |P(z_{i+1})| < \varepsilon,$$

where $\varepsilon > 0$ is a given accuracy and $z_{i+1},\ z_i$ are two successive points in iterative process. In [11], Gdawiec proposed some different convergence tests, for instance

$$\begin{aligned}
&\left|0.01(z_{i+1} - z_i)| + |0.029|z_{i+1}|^2 - 0.03|z_i|^2\right| < \varepsilon, \\
&\left|\frac{0.05}{|z_{i+1}|^2} - \frac{0.045}{|z_i|^2}\right| < \varepsilon, \\
&\left|0.01 z_{i+1}^{10} - 0.001 z_i^{10}\right| < \varepsilon.
\end{aligned}$$

In [4], the convergence test

$$\left|\frac{0.05}{|z_{i+1}|^2} - \frac{0.05}{|z_i|^2}\right| < \varepsilon$$

was used for polynomiography.

Algorithm 1: Iteration colouring.

Input: $P(z) \in \mathbb{C}[Z]$ – polynomial, $A \subset \mathbb{C}$ – area, K – maximum number of iterations, I – iteration method, N – root finding method, T – convergence test.

Output: Polynomiograph for the $P(z)$ on the area A.

for $z_0 \in A$ **do**
  $i = 0$
  while $i \leq K$ **do**
    $z_{i+1} = I(N, P, z_i)$
    if $T(z_i, z_{i+1}) = true$ **then**
      break
    $i = i + 1$
  determine the colour for z_0 according to i.

Algorithm 2: Basin of attraction.

Input: $P(z) \in \mathbb{C}[Z]$ – polynomial, $A \subset \mathbb{C}$ – area, K – maximum number of iterations, I – iteration method, N – root finding method, T – convergence test, Each root of polynomial $P(z)$ gets its own distinc colour.

Output: Basin of attraction for the $P(z)$ and the area A.

for $z_0 \in A$ **do**
  $i = 0$
  while $i \leq K$ **do**
    $z_{i+1} = I(N, P, z_i)$
    if $T(z_i, z_{i+1}) = true$ **then**
      break
    $i = i + 1$
  determine the colour for z_0 according to the nearest root to the z_{i+1}.

In the rest of this paper, in order to obtain different interesting patterns, we will consider Equation (1) with the Caputo derivative of order $\nu \in (n-1,\ n]$, $\ n \in \mathbb{N}$. Recall that for a real-valued function $f(t)$, the Caputo derivative of order $\nu \in (n-1,\ n]$, $\ n \in \mathbb{N}$, is defined as

$$D_C^{\nu} f(t) = \begin{cases} \frac{1}{\Gamma(n-\nu)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\nu+1-n}} ds & \nu \in (n-1, n), \\ \frac{d^n}{dt^n} f(t) & \nu = n. \end{cases}$$

The Caputo fractional derivative operation is linear and has the following properties:

$$D_C^{\nu} t^m = \begin{cases} \frac{\Gamma(m+1)}{\Gamma(m+1-\nu)} t^{m-\nu} & m > n-1, \\ 0 & m \leq n-1, \end{cases}$$
$$D_C^{\nu} c = 0,$$

where c is constant. Fortunately, for analytic functions (e.g. polynomials, exponentials), the fractional Caputo derivative can be generalized to the complex plane. Thus, for $\nu \in (n-1, n)$ we have

$$D_C^{\nu} z^m = \frac{\Gamma(m+1)}{\Gamma(m+1-\nu)} z^{m-\nu},$$

where $z \in \mathbb{C} \setminus \{c \in \mathbb{C} : \Im(c) = 0 \wedge \Re(c) < 0\}$, $m \neq -1, -2, -3, \ldots$ and $m > n-1$ [12, 13].

3. Numerical Results

In this section, we perform a series of experiments for two polynomials. Algorithm 1 for the polynomiographs generation has been implemented in Matlab and all polynomiographs have been generated on a computer with the following specication: Intel i3-4130 (@3.4 GHz) processor, 4 GB RAM and Windows 8 (64-bit). Also we can compute the MNI and CAI parameters for these examples as followes:

$$CAI = \frac{N_c}{N},$$
$$MNI = \frac{S_c}{N},$$

where N_c is the number of convergent points in the polynomiograph and N is the number of all points in the polynomiograph, obviously $0 \leq CAI \leq 1$ and S_c is the sum of the needed iterations for convergent points.

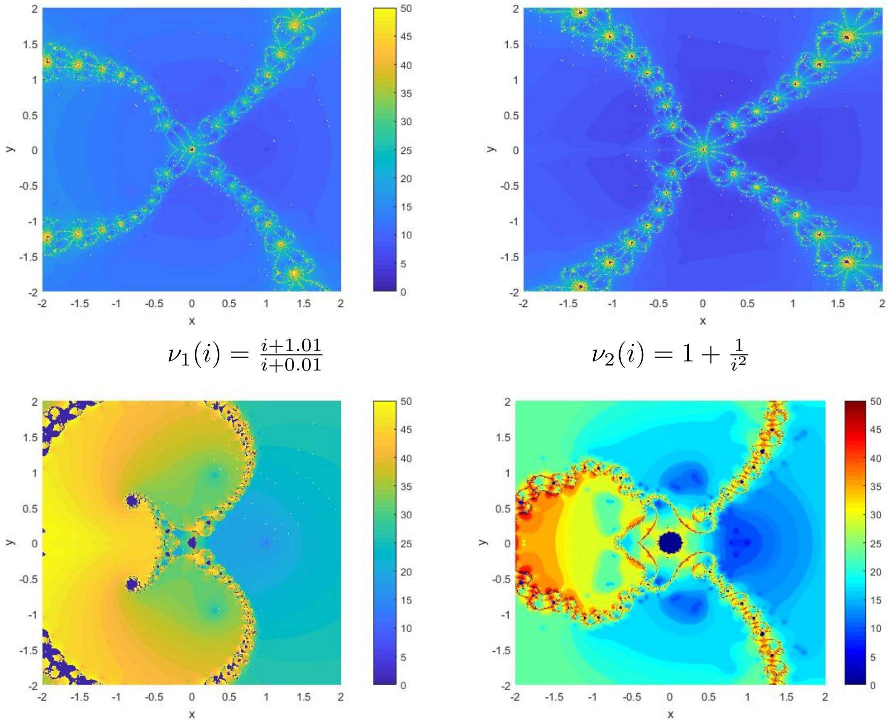
Example 3.1. Let us to consider $P(z) = z^5 - 1$, $k = 50$ and convergence test $|P(z_{i+1})| < 10^{-4}$. After considering area $A = [-2\ ,\ 2] \times [-2\ ,\ 2] \subset \mathbb{C}$, we apply the following fractional Caputo Newton method:

$$z_{i+1} = z_i - \frac{P(z_i)}{D_C^{\nu(i)}\ P(z_i)} \Rightarrow z_{i+1} = z_i - \frac{\Gamma(6-\nu(i))(z_i^5 - 1)}{\Gamma(6)\ z_i^{5-\nu(i)}}, \tag{2}$$

where $\nu(i) < 5$. To produce points, we use $\Delta x = \Delta y = 0.01$ for partitioning area A and the obtained grid points are selected as z_0. Examples of polynomiographs for some variable fractional Caputo derivatives, are presented in Figure 1 and the numerical results obtained for MNI and CAI are shown in Table 1.

Table 1: Results obtained for some variable fractional Caputo derivatives for Example 3.1.

	CAI	MNI
$\nu_1(i) = \frac{i+1.01}{i+0.01}$	0.9978	13.5392
$\nu_2(i) = 1 + \frac{1}{i^2}$	0.9964	10.8949
$\nu_3(i) = \frac{1.5i+1.01}{i+0.01}$	0.9650	34.3235
$\nu_4(i) = \frac{\sin(i)+7}{\sin(2i)+5}$	0.9899	21.2924



$\nu_1(i) = \frac{i+1.01}{i+0.01}$ $\nu_2(i) = 1 + \frac{1}{i^2}$

$\nu_3(i) = \frac{1.5i+1.01}{i+0.01}$ $\nu_4(i) = \frac{\sin(i)+7}{\sin(2i)+5}$

Figure 1: Polynomiographs for some variable fractional Caputo derivatives for Example 3.1.

From Table 1 we see that the change of the $\nu_n(i)$ has a great influence on the MNI parameter, whereas has a small impact on CAI values. Moreover, the patterns obtained through $\nu_1(i)$ and $\nu_2(i)$ are more similar to non-fractional standard Newton's method and the patterns obtained from $\nu_3(i)$ and $\nu_4(i)$ are more irregular and intriguing.

If for $\nu_1(i) = \frac{i+1.01}{i+0.01}$ we use $A = [-60\ ,5] \times [-20\ ,\ 20]$, we will obtain the shown polynomiograph in Figure 2.

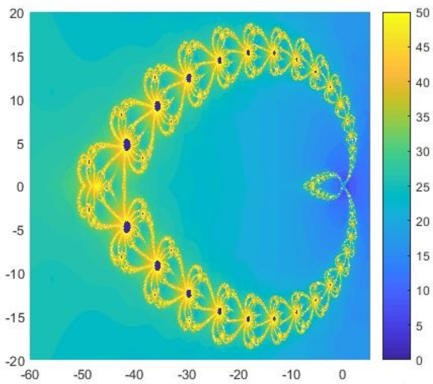


Figure 2: Polynomiograph obtained for $\nu_1(i)$ and $A = [-60\ ,5] \times [-20\ ,\ 20]$.

Now for $P(z)$, we apply $A = [0\ ,\ 60] \times [-20\ ,20]$ and $\nu_5(i) = \frac{i^3-4i^2+5}{25(i^2+1)} + \frac{1}{15}$. The correspondig polynomigraph and basin of attarction, are observed in Figure 3.

Note that in numerical results, if we reach the maximum number of iterations k, we assume that the generated sequence does not converge to any root of the polynomial $P(z)$. As seen in the polynomiographs, some points do not converge to any root.

Example 3.2. Consider $P(z) = -2z^4+z^3+z^2-2z-1$, $A = [1\ ,\ 2]\times[-0.5\ ,\ 0.5]$, $k = 15$, $\Delta x = \Delta y = 0.005$ and convergence test $|P(z_{i+1})| < 0.001$. Figure 4 presents the polynomiographs obtained by some variable fractional Caputo derivatives. The values of the MNI and CAI parameters, corresponding to this example are shown in Table 2. Unlike to previous example, change of the $\nu_n(i)$ has a great influence on the CAI measure. For $\nu_7(i)$ we have CAI= 1, thus for all starting points, the

Table 2: Results obtained for some variable fractional Caputo derivatives, for example 2.

	CAI	MNI
$\nu_1(i) = \frac{i+1.01}{i+0.01}$	0.8971	11.6678
$\nu_4(i) = \frac{\sin(i)+7}{\sin(2i)+5}$	0.5449	12.9274
$\nu_6(i) = 1+\sin(i)^2$	0.3576	13.0734
$\nu_7(i) = \frac{t^2+1}{t^2+8}$	1	10.8313

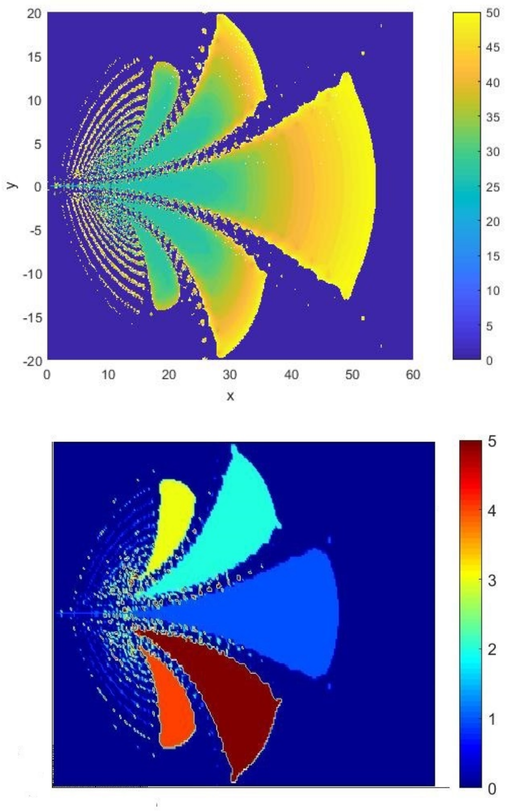
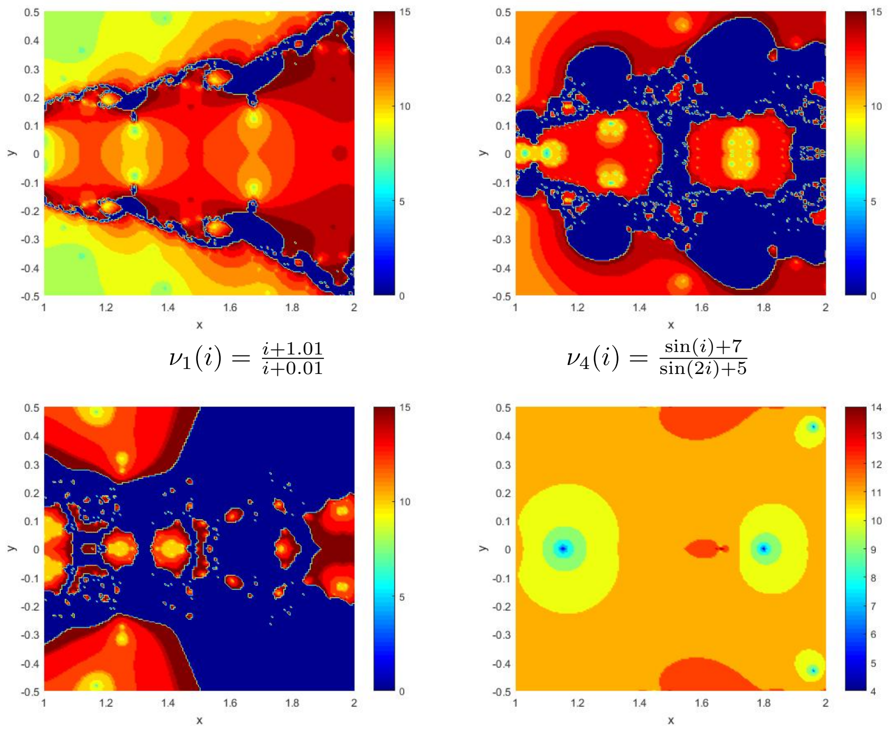


Figure 3: Polynomiograph (up) and basin of attraction (down) obtained for $P(z) = z^5 - 1,\ \nu_5(i)$ and $A = [0\ ,60] \times [-20\ ,\ 20]$.

method has converged to the roots and as seen, the polynomiography for $\nu_7(i)$ is regular. But for $\nu_6(i)$, CAI$=$ 0.3576, so in the considered area, there are many starting points that do not converge to the roots.

4. Conclusion

In this study, based on Picard fractional Newton's method with variable Caputo derivative, we obtained some polynomiographs. Via MNI and CAI parameters, we investigated the effect of the Caputo fractional derivative on the rate of convergence of points. We can use other types of iteration processes, e.g. Mann, Ishikawa, $S-$iterations and also other types of fractional derivatives. In the future work, we



$\nu_1(i) = \frac{i+1.01}{i+0.01}$ $\nu_4(i) = \frac{\sin(i)+7}{\sin(2i)+5}$

$\nu_6(i) = 1 + \sin(i)^2$ $\nu_7(i) = \frac{t^2+1}{t^2+8}$

Figure 4: Polynomiographs for some variable fractional Caputo derivatives, for Example 3.2.

will attempt to use Riemann-Liouville fractional derivatives in a iterative method with periodic parameters.

Conflicts of Interest. The author declares that there are no conflicts of interest regarding the publication of this article.

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