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Sombor Index Under Some Graph Products

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Abstract

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The Sombor index of a graph G, $SO(G)$, is defined as $\sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}$, where d_u is the degree of vertex u in $V(G)$. In the present paper, we determine a lower bound for the Sombor index of edge corona, R-edge and R-vertex corona products of two graphs. We also compute the exact value for the Sombor index of the line graphs of subdivision of tadpole, ladder and wheel graphs.

Keywords: Sombor index, edge corona, R-vertex corona, line graphs, subdivision of tadpole graph.

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1. Introduction

Throughout this paper, we consider only graphs to be simple and finite. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote the degree of a vertex v in G by d_v or $d_G(v)$ which is the number of vertices adjacent to v. The Sombor index is a topological index based on vertex-degree in a graph G which is defined as [6]

$$SO(G) = \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}.$$

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This topological index was motivated by the geometric interpretation of the degree radius of an edge uv, which is the distance from the origin to the ordered pair (d_u, d_v), where $d_u \leq d_v$. The Sombor index was introduced in 2020, and was soon followed by a remarkably large number of studies. The chemical applicability of the Sombor index has been investigated in [11] and it has been shown that this index is useful for predicting entropy and enthalpy of vaporization of octane isomers. Mathematical properties of the Sombor index and its variants have been studied in [9]. They presented the bounds on Sombor index of graphs in terms of various structural parameters, as well as the relationships between Sombor index and other indices. In [3] the graphs extremal with respect to the Sombor index over the chemical graphs, chemical trees and hexagon systems has investigated.

Das et al., in [4], found lower and upper bounds on the Sombor index of graphs based on some graph parameters.

Bounds for the Sombor index of join and vertex corona product of two graphs are characterized in [2, 5, 10]. Moreover, the Sombor index of double graph and strong double graph are obtained in [10]. Furthermore, numerous applications of the Sombor index have been investigated (see e.g., [1, 14, 15]). In this paper, we determine the lower bound for the Sombor index of edge corona, R-edge and R-vertex corona products of two graphs. We also compute the exact value for the Sombor index of the line graphs of subdivision of tadpole, ladder and wheel graphs.

2. Edge Corona Product

The edge corona of two graphs G and H denoted by $G \diamond H$ is obtained by taking one copy of G and $|E(G)|$ copies of H and joining each end vertices of i-th edge of G to every vertex in the i-th copy of H [7, 12]. As an example, you can see Figure 1.

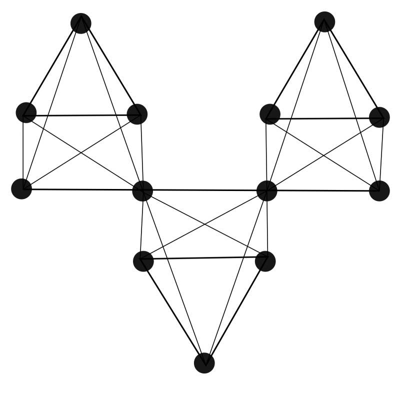


Figure 1: $P_4 \diamond C_3$.

Following Yan et al. [16], the graph $R(G)$ is obtained from G by replacing each edge of G by a triangle. Clearly, if G is a graph and H is a trivial graph, then $G \diamond H \cong R(G)$.

In what follows, the exact values of Sombor index of $R(G)$, when G is some certain graphs are computed. Moreover, we present a lower bound for the Sombor index of the edge corona product of two graphs. As a result we obtain the lower bound for the Sombor index of $R(G)$ when G is an arbitrary graph. We denote the path, complete and cycle graphs of order n by P_n, K_n and C_n, respectively. Our notations are standard and can be taken from the standard books on graph theory.

2.1 Sombor Index of $R(G) \cong G \diamond K_1$

In this section, we compute the Sombor index of $R(G)$, when G is P_n, C_n and K_n.

Theorem 2.1. *The values of Sombor index of $R(P_n)$, $R(C_n)$ and $R(K_n)$ are*

1) $SO(R(P_n)) = 4(n-2)\sqrt{2} + 4(n-1)\sqrt{5}$,
2) $SO(R(C_n)) = 4n(\sqrt{2} + \sqrt{5})$,
3) $SO(R(K_n)) = \sqrt{2}n(n-1)^2 + 2n(n-1)\sqrt{(n-1)^2+1}$.

Proof. 1) There are two edges with endpoints of degree 2 and $2n-2$ edges with endpoints of degrees 2 and 4. Also, there are $n-3$ edges with endpoints of degree 4. Therefore, we have $SO(R(P_n)) = 2\sqrt{4+4} + (2n-2)\sqrt{4+16} + (n-3)\sqrt{16+16}$ and the result is evidently true.

2) There are n edges with endpoints of degree 4 and $2n$ edges with endpoints of degrees 2 and 4. Therefore, we have $SO(R(C_n)) = n\sqrt{16+16} + 2n\sqrt{4+16}$ and the desired result now follows.

3) There are $\frac{n(n-1)}{2}$ edges with endpoints of degree $2(n-1)$. Also $2\frac{n(n-1)}{2}$ edges with endpoints of degrees 2 and $2(n-1)$. Therefore

$$SO(R(K_n)) = \frac{n(n-1)}{2}\sqrt{4(n-1)^2 + 4(n-1)^2} + 2\frac{n(n-1)}{2}\sqrt{4(n-1)^2 + 4},$$

which will complete the proof. □

In general we can find the exact value of the Sombor index of $R(G)$ when G is an r-regular graph.

Theorem 2.2. *Let $G = (V_G, E_G)$ be an r-regular graph with $|E_G| = m$. We have*

$$SO(R(G)) = 2m(\sqrt{2}r + 2\sqrt{1+r^2}).$$

Proof. Since G is r-regular graph of size m, there are $2m$ edges with endpoints of degrees 2 and $2r$. Also there are m edges with endpoints of degree $2r$. Therefore,

$$SO(R(G)) = m\sqrt{4r^2 + 4r^2} + 2m\sqrt{4 + 4r^2},$$

which will complete the proof. □

2.2 Sombor Index of $G \diamond H$

The irregularity of a graph G is defined as $irr(G) = \sum |d_u - d_v|$, where the summation embraces all pairs of adjacent vertices of G [13]. In this part, a lower bound for the Sombor index of edge corona of two graphs is presented. We begin with a crucial lemma, related to degree properties of the edge corona of two graphs.

Lemma 2.3. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs with $|V_H| = m$. Then*

$$d_{G\diamond H}(v) = \begin{cases} (m+1)d_G(v) & v \in V_G, \\ d_H(v) + 2 & v \in V_H. \end{cases}$$

Theorem 2.4. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs with $|E_G| = n$ and $|V_H| = m$. Then*

$$\begin{aligned} SO(G \diamond H) \geq\ & (m+1)SO(G) + \frac{\sqrt{2}\ n}{2}\ irr(H) \\ & + \sum_{\{e=uv | u\in V_G, v\in V_H\}} \frac{|(m+1)d_G(u) - d_H(v) - 2|}{\sqrt{2}}. \end{aligned}$$

Proof. By the definition of the edge corona of two graphs and the definition of Sombor index, we have

$$\begin{aligned} SO(G \diamond H) &= \sum_{uv\in E(G\diamond H)} \sqrt{d^2_{G\diamond H}(u) + d^2_{G\diamond H}(v)} \\ &= |E_G| \sum_{uv\in E_H} \sqrt{(d_H(u)+2)^2 + (d_H(v)+2)^2} \\ &+ \sum_{uv\in E_G} \sqrt{d^2_G(u)(m+1)^2 + d^2_G(v)(m+1)^2} \\ &+ \sum_{\{e=uv|u\in V_G, v\in V_H\}} \sqrt{d^2_G(u)(m+1)^2 + (d_H(v)+2)^2}. \end{aligned}$$

Based on the definition of the Sombor index and irregularity of graph and since $\sqrt{a^2+b^2} \geq \frac{|a-b|}{\sqrt{2}}$, we have the result. □

If $H \cong \overline{K_n}$, then there are no edges $uv \in E_H$. So we have the following corollary for the Sombor index of $G \diamond \overline{K_n}$ as a result of Theorem 2.2.

Corollary 2.5. *Let G and $H \cong \overline{K_n}$ be two graphs. We have*

$$SO(G \diamond \overline{K_n}) \geq (n+1)SO(G) + \sum_{\{e=uv | u \in V_G, v \in V_H\}} \frac{|(n+1)d_G(u) - 2|}{\sqrt{2}}.$$

A fan graph $F_{n,2}$ is isomorphic to $K_2 \diamond \overline{K_n}$. For the Sombor index of this graph by Theorem 2.2 and since $SO(K_2) = \sqrt{2}$, we have $SO(F_{n,2}) = \sqrt{2}(n+1) + 2n\sqrt{(n+1)^2 + 4}$.

Corollary 2.6. *Let $G = (V_G, E_G)$ be a graph. We have*

$$SO(R(G)) \geq 2SO(G) + \sqrt{2} \sum_{u \in V_G} d_G(u)|d_G(u) - 1|.$$

2.3 Sombor Index of $G \ominus H$

The R-edge corona of G and H, denoted by $G \ominus H$, is the graph obtained from vertex disjoint $R(G)$ and $|I(G)|$ copies of H by joining the i-th vertex of $I(G)$ to every vertex in the i-th copy of H. Where $I(G)$ is the set of newly added vertices, i. e. $I(G) = V(R(G)) \setminus V(G)$ [8].

Lemma 2.7. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs with $|V_H| = m$. Then,*

$$d_{G \ominus H}(v) = \begin{cases} 2d_G(v) & v \in V_G, \\ d_H(v) + 1 & v \in V_H, \\ m + 2 & v \in I(G). \end{cases}$$

Now we determine the lower bound for the Sombor index of $G \ominus H$.

Theorem 2.8. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs with $|E_G| = n$ and $|V_H| = m$. Then,*

$$SO(G \ominus H) \geq 2SO(G) + \frac{\sqrt{2}\, n}{2}\, irr(H) + \sum_{\{e=uv | u \in V_G, v \in I(G)\}} \frac{|2d_G(u) - m - 2|}{\sqrt{2}} + \sum_{\{e=uv | u \in I(G), v \in V_H\}} \frac{|d_H(v) - m - 1|}{\sqrt{2}}.$$

Proof. By the definition of R-edge corona of two graphs and the definition of

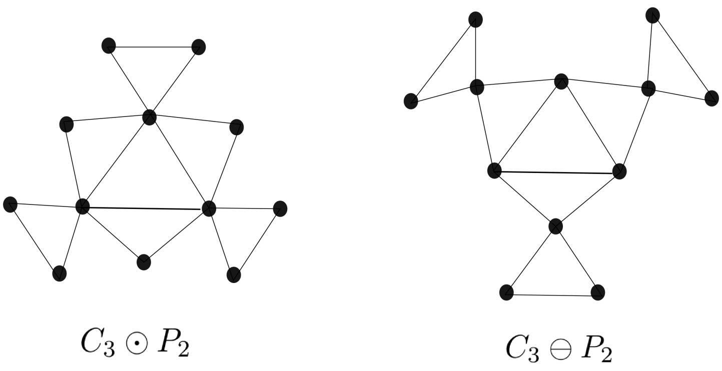


Figure 2: R-edge and R-vertex corona of C_3 and P_2.

Sombor index, we have

$$\begin{aligned}
SO(G \ominus H) &= \sum_{uv \in E(G \ominus H)} \sqrt{d^2_{G\ominus H}(u) + d^2_{G\ominus H}(v)} \\
&= |E_G| \sum_{uv \in E_H} \sqrt{(d_H(u)+1)^2 + (d_H(v)+1)^2} \\
&+ \sum_{uv \in E_G} \sqrt{4d^2_G(u) + 4d^2_G(v)} \\
&+ \sum_{\{e=uv | u \in V_G, v \in I(G)\}} \sqrt{4d^2_G(u) + (m+2)^2} \\
&+ \sum_{\{e=uv | u \in I(G), v \in V_H\}} \sqrt{(d_H(v)+1)^2 + (m+2)^2}.
\end{aligned}$$

By the definition of Sombor index and irregularity of graph and since $\sqrt{a^2+b^2} \geq \frac{|a-b|}{\sqrt{2}}$ we have the result. □

2.4 Sombor Index of $G \odot H$

The R-vertex corona of G and H, denoted by $G \odot H$, is the graph obtained from vertex disjoint $R(G)$ and $|V(G)|$ copies of H by joining the i-th vertex of $V(G)$ to every vertex in the i-th copy of H [8]. In Figure 2, you can see the R-edge and R-vertex corona of C_3 and P_2.

Lemma 2.9. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs with $|V_H| = m$.*

Then,

$$d_{G\odot H}(v) = \begin{cases} 2d_G(v) + m & v \in V_G, \\ d_H(v) + 1 & v \in V_H, \\ 2 & v \in I(G). \end{cases}$$

Now we determine the lower bound for the Sombor index of $G \odot H$.

Theorem 2.10. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs with $|V_G| = n$ and $|V_H| = m$. Then,*

$$SO(G \odot H) \geq \sqrt{2}\ irr(G) + \frac{\sqrt{2}\ n}{2}\ irr(H) + \sum_{\{e=uv|u\in V_G, v\in I(G)\}} \frac{|2 - 2d_G(u) - m|}{\sqrt{2}}$$
$$+ \sum_{\{e=uv|u\in V_G, v\in V_H\}} \frac{|2d_G(u) - d_H(v) + m - 1|}{\sqrt{2}}.$$

Proof. By the definition of R-vertex corona of two graphs and the definition of the Sombor index, we have

$$\begin{aligned} SO(G \odot H) &= \sum_{uv\in E(G\odot H)} \sqrt{d^2_{G\odot H}(u) + d^2_{G\odot H}(v)} \\ &= |V_G| \sum_{uv\in E_H} \sqrt{(d_H(u)+1)^2 + (d_H(v)+1)^2} \\ &+ \sum_{uv\in E_G} \sqrt{(2d_G(u)+m)^2 + (2d_G(v)+m)^2} \\ &+ \sum_{\{e=uv|u\in V_G, v\in I(G)\}} \sqrt{4 + (2d_G(u)+m)^2} \\ &+ \sum_{\{e=uv|u\in V_G, v\in V_H\}} \sqrt{(2d_G(u)+m)^2 + (d_H(v)+1)^2}. \end{aligned}$$

Now based on the definition of irregularity of graph and the fact that $\sqrt{a^2+b^2} \geq \frac{|a-b|}{\sqrt{2}}$, we have the result. □

3. Sombor Index of the Line Graphs of Subdivision Graphs

In this section, we derive an expression for the Sombor index of the line graph of the subdivision graph of tadpole graphs, the wheel graphs and the ladder graphs.

The (m, n)-tadpole graph, $T_{m,n}$ is a graph consisting of a cycle graph on m (at least 3) vertices and a path graph on n vertices, connected with a bridge.

Theorem 3.1. *Let $T_{m,n}$ be a Tadpole graph. We have*

$$SO(T_{m,n}) = (m + n - 4)\sqrt{8} + 3\sqrt{13} + \sqrt{5}.$$

Proof. The graph $T_{m,n}$ contains $m+n$ edges, out of which 3 edges are with end-points of degree 3 and 2, one edge with endpoints of degrees 1 and 2 and the remaining $m+n-4$ edges are with endpoints of degree 2. Therefore, $SO(T_{m,n}) = (m+n-4)\sqrt{4+4} + 3\sqrt{4+9} + \sqrt{1+4}$ and the result follows. □

For any $k \in \mathbb{N}$, $G^{\frac{1}{k}}$ is a simple graph which is obtained from replacing each edge of G with a path of length k. This operation is called k-subdivision of G. In particular, when $k = 2$, 2-subdivision graph of G denoted by $S(G)$ and called subdivision graph of G. The following theorem is about the Sombor index of k-subdivision of graph G.

Theorem 3.2. [5] *Let $G = (V_G, E_G)$ be a graph with $|V_G| = n$ and $|E_G| = m$. For every $k \geq 2$, we have*

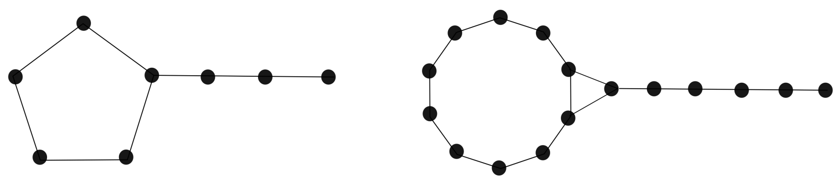
$$SO(G^{\frac{1}{k}}) = 2m(k-2)\sqrt{2} + \sum_{u \in V_G} d_u \sqrt{d_u^2 + 4}.$$

By the above theorem, we can calculate the Sombor index of subdivision of Tadpole graph as follows.

Corollary 3.3. *Let $S(T_{m,n})$ be subdivision of a Tadpole graph. We have*

$$SO(S(T_{m,n})) = 4\sqrt{2}(n+m-2) + \sqrt{5} + 3\sqrt{13}.$$

The line graph of the graph G, written $L(G)$, is the simple graph whose vertices are the edges of G, with $uv \in E(L(G))$ when u and v have a common end point in G. Now, we derive an expression for the Sombor index of $L(S(T_{m,n}))$. As an example, you can see Figure 3.



$T_{5,3}$ $L(S(T_{5,3}))$

Figure 3: $T_{5,3}$ and the line graph of $S(T_{5,3})$.

Theorem 3.4. *If G is $L(S(T_{m,n}))$, then*

$$SO(G) = \sqrt{5} + 3\sqrt{13} + (4m + 4n - 3)\sqrt{2}.$$

Proof. The line graph $L(S(T_{m,n}))$ contains a path of length $2n-1$ and let V_1 be the vertex of degree 3 which is attached to this path. Hence, with respect to the path $\sum\sqrt{d_u^2+d_v^2}$ is $\sqrt{5}+\sqrt{13}+\sqrt{2}(4n-6)$. Let $V_{1'}$ and $V_{2'}$ be the neighbors of V_1 which are of degree 3 in the line graph $L(S(T_{m,n}))$. There are 3 edges $V_1V_{1'}$, $V_1V_{2'}$ and $V_{1'}V_{2'}$ with endpoints of degree 3. With respect to these edges $\sum\sqrt{d_u^2+d_v^2}$ is $9\sqrt{2}$. Moreover, $L(S(T_{m,n}))$ contains $L(S(C_m))$ as a subgraph which has $2m$ edges. The edge $V_{1'}V_{2'}$ belongs to these $2m$ edges. Among the remaining $2m-1$ edges of $L(S(C_m))$, $2m-3$ edges have endpoints of degree 2 and two edges have endpoints of degrees 3 and 2. So, $\sum\sqrt{d_u^2+d_v^2}$ with respect to $2m-1$ edges is $(2m-3)\sqrt{8}+2\sqrt{13}$. Hence the Sombor index is

$$SO(L(S(T_{m,n}))) = \sqrt{5}+3\sqrt{13}+(4m+4n-3)\sqrt{2}.$$

□

The ladder graph L_n is given by $L_n = K_2\square P_n$. Now, we derive an expression for the Sombor index of $L(S(L_n))$. When $n=1$, $L(S(L_1))$ is the path P_2. When $n=2$, $L(S(L_2))$ is the cycle C_8 for which the calculation of Sombor indices is trivial. Hence we study the index of $L(S(L_n))$, when $n\geq 3$. As an example, you can see Figure 4.

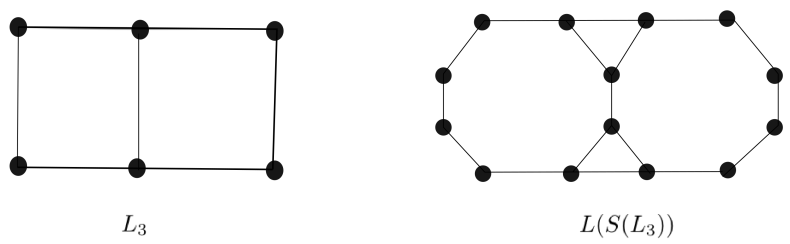


Figure 4: An example of a ladder graph with its line graph of subdivision.

Theorem 3.5. *If G is $L(S(L_n))$ and $n\geq 3$, then*

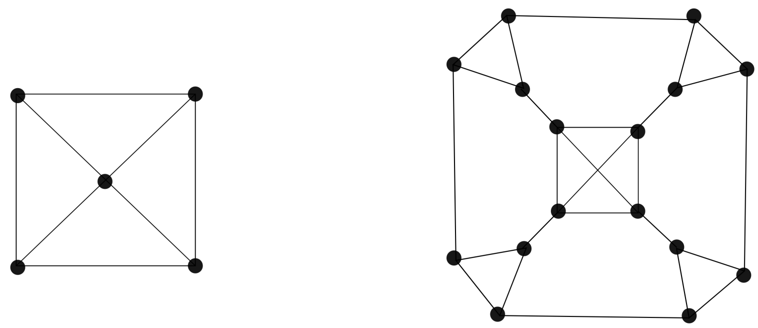
$$SO(G) = 4\sqrt{13}+(27n-48)\sqrt{2}.$$

Proof. The number of edges in the line graph $L(S(L_n))$ is $9n-10$ among which 6 edges have endpoints of degree 2 and also 4 edges with endpoints of degrees 2 and 3. The remaining $9n-20$ edges have endpoints of degree 3. Hence by summation the value of $\sqrt{d_u^2+d_v^2}$ for all $9n-10$ edges, we have $SO(G) = 4\sqrt{13}+(27n-48)\sqrt{2}$. □

A wheel graph W_{n+1}, $n\geq 3$, is a graph formed by connecting a single universal vertex to all vertices of a cycle C_n. Now, we derive an expression for the Sombor index of $L(S(W_{n+1}))$. As an example, you can see Figure 5.

Theorem 3.6. *If G is $L(S(W_{n+1}))$, then*

$$SO(G) = n(12\sqrt{2}+\sqrt{9+n^2}+\frac{\sqrt{2}}{2}(n^2-1)).$$



W_5 $L(S(W_5))$

Figure 5: An example of wheel graph with its line graph of subdivision.

Proof. The graph $L(S(W_{n+1}))$ contains $5n + \frac{n(n-1)}{2}$ edges of which $4n$ edges have endpoints of degree 3. Hence $\sqrt{d_u^2 + d_v^2}$ with respect to these $4n$ edges is $12\sqrt{2}n$. Also there are n edges with endpoints of degrees n and 3. So $\sqrt{d_u^2 + d_v^2}$ with respect to these n edges is $n\sqrt{9 + n^2}$. The remaining $\frac{n(n-1)}{2}$ edges have endpoints of degree n which result in $\frac{\sqrt{2}}{2}(n^3 - n)$ value as $\sqrt{d_u^2 + d_v^2}$ with respect to these edges. Hence by summation the value of $\sqrt{d_u^2 + d_v^2}$ for all edges, we have the result. □

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Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] S. Amin, A. U. R. Virk, M. A. Rehman and N. A. Shah, Analysis of dendrimer generation by Sombor indices, *J. Chem.* **2021** (2021) 9930645.

[2] N. E. Arif, A. H. Karim and R. Hasni, Sombor index of some graph operations, *Int. J. Nonlinear Anal. Appl.* **13** (1) (2022) 2561 – 2571.

[3] R. Cruz, I. Gutman and J. Rada, Sombor index of chemical graphs, *Appl. Math. Comput.* **399** (2021) 126018.

[4] K. C. Das, A. S. Çevik, I. N. Cangul and Y. Shang, On Sombor index, *Symmetry* **13** (1) (2021) 140.

[5] N. Ghanbari and S. Alikhani, Sombor index of certain graphs, *Iranian J. Math. Chem.* **12** (1) (2021) 27 − 37.

[6] I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, *MATCH Commun. Math. Comput. Chem.* **86** (1) (2021) 11 − 16.

[7] Y. Hou and W-C. Shiu, The spectrum of the edge corona of two graphs, *Electron. J. Linear Algebra* **20** (2010) 586 − 594.

[8] J. Lan and B. Zhou, Spectra of graph operations based on R-graph, *Linear Multilinear Algebra* **63** (7) (2015) 1401 − 1422.

[9] H. Liu, I. Gutman, L. You and Y. Huang, Sombor index: Review of extermal results and bounds, *J. Math. Chem.* **66** (2022) 771 − 798.

[10] H. S. Ramane, I. Gutman, K. Bhajantri and D. V. Kitturmath, Sombor index of some graph transformations, *Commun. Combin. Optim.* **8** (1) (2023) 193 − 205.

[11] I. Redžepović, Chemical applicability of Sombor indices, *J. Serbian Chem. Soc.* **86** (5) (2021) 445 − 457.

[12] I. Rezaee Abdolhosseinzadeh, F. Rahbarnia, M. Tavakoli and A. R. Ashrafi, Some vertex-degree-based topological indices under edge corona product, *Ital. J. Pure Appl. Math.* **38** (2017) 81 − 91.

[13] M. Tavakoli, F. Rahbarnia, M. Mirzavaziri, A. R. Ashrafi and I. Gutman, Extremely irregular graphs, *Kragujevac J. Math.* **37** (1) (2013) 135 − 139.

[14] S. O. Ünal, An application of Sombor index over a special class of semigroup graph, *J. Math.* **2021** (2021) 3273117.

[15] S. O. Ünal, Sombor index over the tensor and Cartesian product of monogenic semigroup graphs, *Symmetry* **14** (5) (2022) 1071.

[16] W. Yan, B. -Y. Yang and Y. -N. Yeh, The behavior of Wiener indices and polynomials of graphs under five graph decorations, *Appl. Math. Lett.* **20** (2007) 290 − 295.

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