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S-Acts with Finitely Generated Universal Congruence

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Abstract

Universal left congruences on semigroups were studied in "Y. Dandan, V. Gould, T. Quinn-Gregson and R. Zenab, Semigroups with finitely generated universal left congruence, *Monat. Math.* **190** (2019) 689 – 724". We consider universal congruences on acts over monoids and extend the results from semigroups to acts. Among other things, for an S-act A_S with zero over a monoid S, we prove that being finitely generated of the universal congruence ω_A and being pseudofinite of A_S coincide.

Keywords: S-act, universal congruence, pseudofinite, finitely generated.

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1. Introduction

Finitary conditions of a class of algebras are of great significance to understand the structure and behavior of semigroups, groups, rings and many other types of algebras. Here we focus on two finitary conditions, which are the case where being an S-act A_S over a monoid S *pseudofinite* and the weaker condition that the *universal congruence* ω_A is finitely generated. Dandan et al. [1] investigated

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left universal congruences on semigroups. In this article, this concept is studied for S-acts, particularly, those ones whose universal congruences are finitely generated. Also we find some relationships between finite generatedness of ω_A and pseudofiniteness of A_S.

The results of this article are collected in Section 2. For an S-act A_S, first we give equivalent conditions under which ω_A to be finitely generated, and also for being pseudofinite. The equivalence of being finitely generated of A_S, being finitely generated of ω_A and being pseudofinite of A_S is proved in Theorem 2.7 where A_S has a zero element.

Throughout the paper, S stands for a monoid. A (*right*) *S-act* A_S, is a set A with an *S-action* $\lambda : A \times S \to A$, denoting $\lambda(a, s) := as$, such that $a(st) = (as)t$ and $a1 = a$, for all $a \in A$ and $s, t \in S$. In other words, an S-act, described above, is a universal algebra $(A, (\lambda_s)_{s \in S})$ where for each $s \in S, \lambda_s : A \to A$ is a unary operation on A_S such that $\lambda_s \circ \lambda_t = \lambda_{st}$, and $\lambda_1 = id_A$.

For an S-act A_S, a congruence ρ on A_S is an equivalence relation on A_S with the additional property that, if $a \rho a'$ then $(as)\rho(a's)$ for $a, a' \in A_S, s \in S$. The *universal congruence* $A^2 = A \times A$ is denoted by ω_A. Here $\rho(H)$ for $H \subseteq A \times A$ denotes the congruence generated by H (i.e. the smallest congruence on A_S containing H). For any $x, y \in H$, $x\rho(H)y$ if and only if $x = y$ or there is a sequence $x = p_1 s_1, q_1 s_1 = p_2 s_2, q_2 s_2 = p_3 s_3, \ldots, q_n s_n = y$ where for $i = 1, \ldots, n, (p_i, q_i) \in H \cup H^{-1}$ and $s_1, s_2, \ldots, s_n \in S$. The above sequence is called an *H-sequence of length n*. For more information and definitions concerning S-acts not mentioned here, see [3].

2. Results

We begin with the following definition:

Definition 2.1. Let A_S be an S-act with ω_A being generated by a finite subset $H \subseteq A^2$. Then A_S is called *pseudofinite relative to* H if there is $n \in \mathbb{N}$ such that any $a, b \in A$ are related with respect to an H-sequence of length at most n. Also if A_S is pseudofinite relative to $X^2 = X \times X$ for $X \subseteq A$, then it is simply said pseudofinite relative to X.

Clearly, if an S-act A_S is pseudofinite relative to H, then ω_A is finitely generated.

Theorem 2.2. *A group S is finitely generated* (*as group*) *if and only if ω_S is finitely generated.*

Proof. Let ω_S be generated by a finite set H. So for each $a \in S$, $(a, 1) \in \omega_S$ and then $a = 1$ or there exists a sequence

$$a = p_1 s_1,\ q_1 s_1 = p_2 s_2,\ q_2 s_2 = p_3 s_3,\ \ldots, q_n s_n = 1,$$

where for each $1 \leq i \leq n, (p_i, q_i) \in H \cup H^{-1}$ and $s_1, s_2, \ldots, s_n \in S$. So $a = p_1q_1^{-1}p_2q_2^{-1} \cdots p_nq_n^{-1}$ which implies that the finite set $\{pq^{-1} \mid (p,q) \in H \cup H^{-1}\}$ generates S. For the converse, let $X = \{p_1, p_2, \ldots, p_n\}$ be a generating set for S and $a, b \in S$. So from $ab^{-1} = q_1q_2 \cdots q_k$, in which for each $1 \leq i \leq k, q_i \in X \cup X^{-1}$, we have the sequence

$$ab^{-1} = q_1q_2 \cdots q_k, 1(q_2 \cdots q_k) = q_2(q_3 \cdots q_k), \ldots, 1q_k = q_k1, 11 = 1,$$

and hence

$$a = q_1(q_2 \cdots q_kb), 1(q_2 \cdots q_kb) = q_2(q_3 \cdots q_kb), \ldots, 1(q_kb) = q_kb, 1b = b,$$

which gives that $(a, b) \in \rho(X \times \{1\})$. □

Let ρ_1 and ρ_2 be two congruences on an S-act A_S. Then it is said that ρ_2 is a *principal extension* of ρ_1 if $\rho_2 = \rho(\rho_1 \cup \{(x, y)\})$ for some $x, y \in A$.

Lemma 2.3. *For an S-act A_S, the following are equivalent:*

(i) *ω_A is finitely generated.*

(ii) *A finite chain $\iota = \vartheta_0 \subset \vartheta_1 \subset \cdots \subset \vartheta_n = \omega_A$ of congruences on A_S exists in such a way that for all $1 \leq i \leq n$, each ϑ_i is a principal extension of ϑ_{i-1}.*

(iii) *ω_A is generated by X^2, for a finite subset X of A_S.*

(iv) *There is a finite subset X of A_S such that $\omega_A = \langle \{x\} \times X \rangle$, for any $x \in X$.*

(v) *For each $u \in A$, there is a finite subset X of A_S with $u \in X$ and $\omega_A = \langle \{u\} \times X \rangle$.*

Proof. (i) ⇒ (ii) and (v) ⇒ (i) are clear.

(ii) ⇒ (iii) By (ii), $\vartheta_n = \omega_A = \langle \{(a_1, b_1), \ldots, (a_n, b_n)\} \rangle$. Consider the set $X = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$, so $\{(a_1, b_1), \ldots, (a_n, b_n)\} \subseteq X \times X$ and hence we get $\omega_A \subseteq \rho(X \times X) \subseteq \omega_A$.

(iii) ⇒ (iv) Let $a, b \in A$ and $x \in X$. By (iii), there exist $s_1, \ldots, s_n \in S$ and $(p_1, q_1), \ldots, (p_n, q_n) \in X^2$ such that $a = p_1s_1, q_1s_1 = p_2s_2, \ldots, q_ns_n = b$ and hence

$$a = p_1s_1,\ xs_1 = xs_1,\ q_1s_1 = p_2s_2,\ xs_2 = xs_2, \ldots,\ q_ns_n = b,$$

in which $(p_i, x) \in X \times \{x\}$ and $(x, q_i) \in \{x\} \times X$.

(iv) ⇒ (v) Consider $X_1 = X \cup \{u\}$. By (iv), for each $a, b \in A$, there exist $s_1, \ldots, s_n \in S$ and $(p_1, q_1), \ldots, (p_n, q_n) \in (\{x\} \times X) \cup (X \times \{x\})$ such that $a = p_1s_1, q_1s_1 = p_2s_2, \ldots, q_ns_n = b$. So

$$a = p_1s_1,\ us_1 = us_1,\ q_1s_1 = p_2s_2,\ us_2 = us_2, \ldots,\ q_ns_n = b,$$

in which $(p_i, u) \in X_1 \times \{u\}$ and $(u, q_i) \in \{u\} \times X_1$. Thus $\omega_A = \langle \{u\} \times X_1 \rangle$. □

Lemma 2.4. *Let A_S be an S-act and $\omega_A = \langle H\rangle = \langle K_1\rangle$ for some $H, K_1 \subseteq A^2$ where H is finite. Then there exists a finite subset K_2 of K_1 for which $\omega_A = \langle K_2\rangle$. Further, if A_S is pseudofinite relative to H, then it is pseudofinite relative to K_2.*

Proof. For $(a, b) \in H$, there is a K_1-sequence of length $n := n(a, b)$ such as $a = p_1t_1, q_1t_1 = p_2t_2, \ldots, q_nt_n = b$, where $(p_i, q_i) \in K_1 \cup K_1^{-1}$ and $t_i \in S$. For each $(a, b) \in H$, consider

$$K_{(a,b)} = \{(p_1, q_1), \ldots, (p_{n(a,b)}, q_{n(a,b)}), (q_1, p_1), \ldots, (q_{n(a,b)}, p_{n(a,b)})\} \cap K_1.$$

So $K_{(a,b)} \subseteq K_1$, $|K_{(a,b)}| < \infty$ and $(a, b) \in \langle K_{(a,b)}\rangle$. Let $K_2 := \bigcup_{(a,b)\in H} K_{(a,b)}$. Since H is finite, K_2 is a finite subset of K_1 and hence $H \subseteq \langle K_2\rangle$. So $\omega_A = \langle H\rangle \subseteq \langle K_2\rangle \subseteq \omega_A$. Moreover, let $x, y \in A$. Then there is an H-sequence

$$x = p_1s_1,\ q_1s_1 = p_2s_2, \ldots,\ q_ms_m = y,$$

where $(p_i, q_i) \in H \cup H^{-1}$ and $s_i \in S$. Using the first part of the proof, for each $(p_i, q_i) \in H$, there is a K_2-sequence of the length $n(p, q)$ such as

$$p_i = u_{i1}t_{i1},\ w_{i1}t_{i1} = u_{i2}t_{i2}, \ldots,\ w_{in(p_i,q_i)}t_{in(p_i,q_i)} = q_i,$$

where $(u_{ij}, w_{ij}) \in K_2 \cup (K_2)^{-1}$ and $t_{ij} \in S$. So one gets a K_2-sequence of length $n(p_i, q_i)$ connecting p_is_i to q_is_i. Consider $m' = m \max\{n(p_i, q_i) \mid (p_i, q_i) \in H\}$. Thus there is a K_2-sequence from x to y of length at most m'. □

Lemma 2.5. *Let A_S be a non-singleton S-act such that $\omega_A = \langle H\rangle$ for some $H \subseteq A^2$. Let $\mathcal{C}(H) = \{x \mid \exists y \in A, (x, y) \in H \cup H^{-1}\}$. Then*
(i) *there exists $X \subseteq A$ such that $\omega_A = \langle X^2\rangle$.*
(ii) *$A_S = \langle \mathcal{C}(H)\rangle$.*

Proof. (i) Since $H \subseteq \mathcal{C}(H)^2$, it is enough to consider $X = \mathcal{C}(H)$.

(ii) Let $a \in A$. Consider $b \in A$ with $a \neq b$. Since $a\rho(H)b$, there exist $p_1, \ldots, p_n, q_1, \ldots, q_n \in A$ and $w_1, \ldots, w_n \in S$, where for $1 \leq i \leq n, (p_i, q_i) \in H \cup H^{-1}$ and

$$a = p_1w_1,\ q_1w_1 = p_2w_2,\ q_2w_2 = p_3w_3, \ldots,\ q_nw_n = b.$$

Hence, $a = xw$ for some $w \in S$ and $x \in \mathcal{C}(H)$. □

Proposition 2.6. *Let A' be a subact of an S-act A_S. Then ω_A is finitely generated if and only if A_S has a finite generator X with $\omega_{A'} = \rho(X^2)|_{A'\times A'}$. Moreover, A is pseudofinite if and only if there is a positive integer n for which there exists an X^2-sequence of length at most n from a to b, for each $a, b \in A'$.*

Proof. Let ω_A be generated by a finite set H. So $X = \mathcal{C}(H)$ is a finite set in which X^2 generates ω_A by Lemma 2.5. Clearly, $\omega_{A'} = \rho(X^2)|_{A'\times A'}$ and it follows from Lemma 2.5 that $\mathcal{C}(H)$ is a generating subset of A_S. For the converse, consider some $u \in A'$ and $Y = X \cup \{u\}$. Then for any $a \in A$, there is $x \in X$ with $a = xt$ for some $t \in S$ which implies $a = xt\rho(Y^2)ut$. From the hypothesis $\omega_{A'} = \rho(X^2)|_{A' \times A'}$ we get $ut\rho(Y^2)u$. Hence, $a\rho(Y^2)u$, so that $\omega_A = \rho(Y^2)$. Using Lemma 2.4, the second assertion holds. □

As a corollary of Proposition 2.6, we have the following:

Theorem 2.7. *For an S-act A with zero, the following are equivalent:*
(i) *A is finitely generated.*
(ii) *ω_A is finitely generated.*
(iii) *A is pseudofinite.*

Proof. Clearly, $\{0\}$ forms a subact of A, and hence (i) and (ii) are equivalent by Proposition 2.6.

(i)$\Rightarrow$ (iii) Let X be a finite generating subset of A_S and $a, b \in A$. So there exist $x_1, x_2 \in X$ and $s_1, s_2 \in S$ such that $a = x_1 s_1$, $0s_1 = 0s_2$ and $b = x_2 s_2$. This implies that A_S is pseudofinite relative to the finite set $X \times \{0\}$.

(iii) $\Rightarrow$ (ii) This is obvious. □

Proposition 2.8. *Let B be a homomorphic image of an S-act A_S. If ω_A is finitely generated, then so is ω_B. Moreover, if A_S is pseudofinite, then so is B.*

Proof. Suppose that $\omega_A = \rho(X)$ for some finite subset X of A^2 and $\varphi : A \to B$ is an epimorphism. For any $b, b' \in B$, there exists $a, a' \in A$ such that $\varphi(a) = b$ and $\varphi(a') = b'$. Since $a\rho(X)a'$, one gets $a = a'$ or there exists a sequence

$$a = p_1 w_1,\ q_1 w_1 = p_2 w_2, \ldots,\ q_n w_n = a',$$

where $w_i \in S$ and $(p_i, q_i) \in X \cup X^{-1}$ for all $1 \leq i \leq n$. So

$$b = \varphi(a) = \varphi(p_1)w_1,\ \varphi(q_1)w_1 = \varphi(p_2)w_2, \ldots,\ \varphi(q_n)w_n = \varphi(a') = b',$$

which means that $\omega_B = \rho(\varphi(X))$ where $\varphi(X) = \{(\varphi(a), \varphi(a')) \mid (a, a') \in X\}$. Clearly, if A is pseudofinite relative to X, then B is pseudofinite relative to $\varphi(X)$. □

Corollary 2.9. *Let A and B be S-acts. If $\omega_{A \times B}$ is finitely generated (pseudofinite), then both ω_A and ω_B are finitely generated (pseudofinite).*

Proof. It follows from Proposition 2.8 by applying the projection morphisms. □

Now let A be an S-act and B be a T-act. Then $A \times B$ is an $S \times T$-act by the action

$$\begin{gathered} \mu : (A \times B) \times (S \times T) \longrightarrow A \times B \\ \mu((a, b), (s, t)) = (as, bt). \end{gathered}$$

Proposition 2.10. *Let A be an S-act and B be a T-act. If ω_A and ω_B are finitely generated (pseudofinite), then $\omega_{A\times B}$ is finitely generated ($A \times B$ is a pseudofinite $S \times T$-act).*

Proof. Let $\omega_A = \rho(X^2)$ and $\omega_B = \rho(Y^2)$ for some finite subsets $X \subseteq A$ and $Y \subseteq B$, respectively. For any $(a, b), (a', b') \in A \times B$ we have $a = a'$ or $a = p_1w_1, q_1w_1 = p_2w_2, \ldots, q_mw_m = a'$ where $m \in \mathbb{N}, w_i \in S$ and $(p_i, q_i) \in X^2$ for all $1 \le i \le m$, and $b = b'$ or $b = p'_1w'_1, q'_1w'_1 = p'_2w'_2, \ldots, q'_nw'_n = b'$ where $n \in \mathbb{N}^0, w'_i \in S$ and $(p'_i, q'_i) \in Y^2$ for all $1 \le i \le n$. If $n \ge m$, consider $w_{m+1} = w_{m+2} = \cdots = w_n = w_m$ and $p_{m+1} = p_{m+2} = \cdots = p_n = q_m$ and $q_{m+1} = q_{m+2} = \cdots = q_n = q_m$. Then

$$(a, a') = (p_1w_1, p'_1w'_1), (q_1w_1, q'_1w'_1) = (p_2w_2, p'_2w'_2), \ldots, (q_nw_n, q'_nw'_n) = (b, b'),$$

so that

$$(a, a') = (p_1, p'_1)(w_1, w'_1), (q_1, q'_1)(w_1, w'_1) = (p_2, p'_2)(w_2, w'_2), \ldots,$$
$$(q_n, q'_n)(w_n, w'_n) = (b, b').$$

A similar argument holds for the case $n < m$. Thus $\omega_{A\times B} = \rho(X \times Y)^2$. The statement on pseudofinite also holds, for, $\max\{m, n\}$ is no less than the length of the $(X \times Y)^2$-sequence. □

Suppose that K and L are non-empty sets and P is a matrix of order $|K| \times |L|$ with entries p_{ij} taken from a semigroup S. The *Rees matrix semigroup* $\bar{S} = \mathcal{N}[S; K, L; P]$ is the set $(K \times S \times L)$ with the binary operation $(i, s, j)(k, t, l) = (i, sp_{jk}t, l)$. Now for an S-act A, the set $\mathcal{A} = K \times A \times L$ is an $\bar{S}$-act by the action $(i, a, j)(k, s, l) = (i, ap_{jk}s, l)$ and we call it the *Rees matrix induced action*. Under these notations, we have the following:

Theorem 2.11. *$\omega_{\mathcal{A}}$ is finitely generated if and only if K and L are finite and there exists a finite set $V \subseteq A$ such that for each $a \in A$ there is $v \in V$ such that $a\rho(H)v$ in which*

$$H = \{(ap_{j\mu}, bp_{ji}) \mid j \in L, i, \mu \in K, a, b \in V\}.$$

Proof. Let $\omega_{\mathcal{A}}$ be finitely generated. Using Lemma 2.3, suppose that $\omega_{\mathcal{A}} = \langle U^2\rangle$ where $U \subseteq \mathcal{A}$ is finite and the projection images $K' = \pi_I(U), L' = \pi_J(U)$ and $V = \pi_A(U)$ are finite subsets of K, A and L, respectively. If $\mathcal{A}$ is finite, then one can take $\mathcal{A} = U$ and $V = A$ and the result is complete. Otherwise, consider $(i, a, j), (t, b, z) \in \mathcal{A}$ be distinct. Then we have the chain

$$(i, a, j) = (i_1, a_1, j_1)(\alpha_1, s_1, \beta_1), (t_1, b_1, z_1)(\alpha_1, s_1, \beta_1) = (i_2, a_2, j_2)(\alpha_2, s_2, \beta_2), \ldots,$$
$$(t_n, b_n, z_n)(\alpha_n, s_n, \beta_n) = (t, b, z),$$

where $n \in \mathbb{N}$ and $w_i = (\alpha_i, s_i, \beta_i) \in \bar{S}$ and $((i_k, a_k, j_k), (t_k, b_k, z_k)) \in U^2$ for all $1 \le k \le n$. Clearly, $i = i_1 \in K'$, so that $K \subseteq K'$ and hence $K = K'$ is finite.

Also if L is infinite, then one can consider distinct elements j and z of $L\backslash L'$. Then we have $j = \beta_1, \beta_1 = \beta_2, \ldots, \beta_n = z$, and so $j = z$, which is impossible. Hence, L is finite. On the other hand, consider $i = t, j = z$ and $b \in V$. Then there exists an H-sequence from an arbitrary $a \in A$ to an element $b' \in V$. Indeed, for each $1 \leq k \leq n$,, if $w_k \in \bar{S}$, so $b = b'$, or the index k be a least with $w_k = 1$, $a\rho(H)a_k = b' \in V$.

Conversely, let $W = \{a \in A \mid (a, b) \in H \text{ for some } b \in A\}$. We show that $\omega_{\mathcal{A}} = \langle \mathcal{R}^2 \rangle$ where $\mathcal{R} = K \times (V \cup W) \times L$, which is finite. Let $(i, a, j), (i', a', j') \in \mathcal{A}$. Since for each $x, y \in \mathcal{R}, (x, y) \in \mathcal{R}^2$, it suffices to show that each element of $\mathcal{A}$ is $\mathcal{R}^2$-related to an element of $\mathcal{R}$. If $(i, a, j) \in \mathcal{R}$, then we are done and otherwise, the element a is connected to $b \in V$ via an H-sequence

$$a = a_1 w_1, b_1 w_1 = a_2 w_2, \ldots, b_n w_n = b.$$

Consider the notation as $(a_l, b_l) = (u_l p_{\mu_l j_l}, v_l p_{i_l j_l}) \in H$ for all $1 \leq l \leq n$. If each w_l belongs to $\bar{S}$, then there is an $\mathcal{R}^2$-sequence

$$\begin{aligned}&(i, a, j) = (i, u_1, j_1)(\mu_1, t_1, j), (i, v_1, j_1)(\mu_1, t_1, j) = (i, u_2, j_2)(\mu_2, t_2, j), \ldots, \\ &(i, v_n, j_n)(\mu_n, t_n, j) = (i, b, j),\end{aligned}$$

where $(u_i, v_i) \in V^2$ and $t_i \in S^1$.

Also, if there is $1 \leq k \leq n$ with $w_k = 1$, there exists an $\mathcal{R}^2$-sequence

$$\begin{aligned}&(i, a, j) = (i, u_1, j_1)(\mu_1, t_1, j), (i, v_1, j_1)(\mu_1, t_1, j) = (i, u_2, j_2)(\mu_2, t_2, j), \ldots, \\ &(i, v_{k-1}, j_{k-1})(\mu_{k-1}, t_{k-1}, j) = (i, u_k, j),\end{aligned}$$

in which $u_k \in W$. In both cases, (i, a, j) is $\mathcal{R}^2$-related to an element of $\mathcal{R}$. Similarly, (i', a', j') is also $\mathcal{R}^2$-related to an element of $\mathcal{R}$, which the proof is complete. □

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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