Mathematics Interdisciplinary Research **8** (1) (2023) 19 – 25

Original Scientific Paper

Dominating Set for Bipartite Graph $\Gamma(v, k, l, 2)$

Abolfazl Bahmani, Mojgan Emami and Ozra Naserian⋆

Abstract

A bipartite graph (X, Y) in which X and Y are, respectively, the set of all l-subsets and all k-subsets of a v-set V as vertices and two vertices being adjacent if they have i elements in common, is denoted by $\Gamma(v, k, l, i)$. In this paper, using the structure of Steiner triple systems, we give dominating sets for $\Gamma(v, k, l, 2)$, where $4 \leq k \leq 6$ and $3 \leq l \leq 5$.

Keywords: Dominating set, Bipartite graph, Steiner triple system.

2010 Mathematics Subject Classification: 05C30, 05C35.

How to cite this article
A. Bahmani, M. Emami and O. Naserian, Dominating set for bipartite graph $\Gamma(v, k, l, 2)$, *Math. Interdisc. Res.* **8** (1) (2023) 19-25.

1. Introduction

Let t, k, v and λ be positive integers such that $0 \leq t \leq k \leq v$. Moreover, let V be a v-set and for a positive integer i let $P_i(V)$ be the set of all i-subsets of V. The pair $D = (V, \beta)$, where β is a subset of $P_k(V)$ (blocks) is called a $t-(v, k, \lambda)$ design such that every t-subset of V appears in exactly λ blocks [1]. The number of blocks of D is denoted by b. A $2-(v, 3, 1)$ design is called a Steiner triple system and is denoted by $STS(v)$ [1]. Note that:

Theorem 1.1. [1] *An $STS(v)$ exists if and only if $v \overset{6}{\equiv} 1$ or 3.*

A modified Steiner triple system on V denoted by $MSTS(v)$ is a proper subset of $P_3(V)$ such that each pair of elements of V occurs exactly once except for pairs

⋆Corresponding author (E-mail: o.naserian@gmail.com)
Academic Editor: Reza Sharafdini
Received 21 June 2022, Accepted 22 December 2022
DOI: 10.22052/MIR.2022.246511.1360

$(1,2),(2,3),\cdots,(v-2,v-1),(v-1,1)$, which do not occur at all and we have $|MSTS(v)| = \frac{(v-1)(v-2)}{6}$. A graph is a pair $G=(V,E)$, where $E \subseteq P_2(V)$ in which V and E, respectively, are called the vertex and edge set of G. Two vertices u and v are adjacent if $\{u,v\} \in E$. We use the classic terminology given in [2]. A dominating set for a graph G is a subset $S \subseteq V(G)$ such that every vertex of G either is in S or is adjacent to at least one element of S. The domination number of G, is the minimum size of a dominating set for G and is denoted by $\gamma(G)$[2].

Theorem 1.2. [2] *Let G be an n-vertex graph with minimum degree δ, then*

$$\gamma(G) \leq \frac{n(1+ln(\delta+1))}{\delta+1}.$$

Let $v,k,l\,(k \neq l)$ be positive integers, i be a non-negative integer, $v \geq k,l$ and $k,l \geq i$. Define the bipartite graph $\Gamma(v,k,l,i)$[3] by $V(\Gamma(v,k,l,i)) = P_k(V) \cup P_l(V)$ such that

$$\{u,w\} \in E(\Gamma(v,k,l,i)) \Leftrightarrow |u \cap w| = i, u \in P_k(V), w \in P_l(V).$$

In this paper, using the structure of Steiner triple systems, we give dominating sets for $\Gamma(v,k,l,2)$, where $4 \leq k \leq 6$ and $3 \leq l \leq 5$.

2. Results

Let v,k,l be positive integers. Then in bipartite graph $G=\Gamma(v,k,l,2)$ we consider $X=P_l(V)$, $Y=P_k(V)$ and $C_i=\{a_{2i-1},a_{2i}\}$, where $V=\{a_1,a_2,\cdots,a_v\}$ is the assumed v-set. In general, our method is that first we give a subset of X which is a dominating set for Y and we give a subset of Y which is a dominating set for X. Then clearly the union of these sets is a dominating set for G. First note that:

Theorem 2.1. *Let $G=\Gamma(v,4,3,2)$ and $v \geq 12$. If $v=12m+i$, where $0 \leq i \leq 11$, then $\gamma(G) \leq 42m^2+65m+25$.*

Proof. We separate the proof in two cases:

Case i. Let v be an odd integer.

1) If $v \overset{6}{\equiv} 1$ or 5, then any $MSTS(v) \subset X$ is a dominating set for Y, since any vertex in Y such as $B=\{a_1,a_2,a_3,a_4\}$ contains at least one non-consecutive pair, therefore B is dominated by a block of $MSTS(v)$.

2) If $v \overset{6}{\equiv} 3$, then any $STS(v)$ is a dominating set for Y, since any vertex in Y such as $B=\{a_1,a_2,a_3,a_4\}$ is dominated by a block of $STS(v)$ containing exactly two points in common with B. This dominating set is of size $24m^2+38m+15$.

Now we give a subset of Y as a dominating set for X. Let $C=\{C_1,C_2,\cdots,C_{\frac{v-1}{2}}\}$. Then the set $P_2(C) \subset Y$ is a dominating set for X of size $18m^2+27m+10$. Hence the cardinality of a dominating set for G is $(24m^2+38m+15)+(18m^2+27m+10) = 42m^2 + 65m + 25$. In this case $\gamma(G) \leq 42m^2 + 65m + 25$.

Case ii. Let v be an even integer. Let $V^{'} = V \cup \{x\}$, where $x \notin V$. Similar to the Case i on $V^{'}$ we may consider either $STS(v + 1)$ or $MSTS(v + 1)$ and then remove the blocks containing x. The remaining blocks dominate Y. Then we give a subset of Y as the dominating set for X. Let $C = \{C_1, C_2, \cdots , C_j\}$ and $C^{'} = \{C_{j+1}, \cdots , C_{\frac{v}{2}}\}$, where we consider $j = \frac{v}{4}$ if $v \overset{4}{\equiv} 0$, else we consider $j = \frac{v-2}{4}$. The set $P_2(C) \cup P_2(C^{'})$ is a dominating set for X and its cardinality is $33m^2 + 50m + 19$.

□

Theorem 2.2. *Let* $G = \Gamma(v, 5, 3, 2)$ *and* $v \geq 11$. *If* $v = 12m+i$, *where* $0 \leq i \leq 11$, *then*

$$\gamma(G) \leq 48m^2 + 64m + 26.$$

Proof. Suppose that A and B be two subsets of V such that $V = A\cup B$, $|A\cap B| = 0$ or 1, $|A| \overset{6}{\equiv} 1$ or 3 and $|B| \overset{6}{\equiv} 1$ or 3. We may choose these two sets in a lot of ways by considering $v(mod\, 12)$. By Theorem 1.1 there exist $STS(|A|)$ and $STS(|B|)$ and the set of all blocks of these designs is a dominating set for Y of size $12m^2 + 22m + 13$. Note that the maximum size of this dominating set occurs when $v = 12m+11 = (6m+3)+(6m+9)+(-1)$ and so A and B have a common element. To give a dominating set for X, we may consider two cases:

Case i. Let v be an odd integer. Let $C = \{C_1, C_2, \cdots , C_{\frac{v-3}{2}}\}$. We add a_{v-1} to all elements of $P_2(C)$ to get a set of five tuples over V. We do the same procedure with a_{v-2} to get a similar set of five tuples. Now by adding $\{a_1, a_2, a_3, a_{v-2}, a_{v-1}\}$ to the union of these two later sets we have a dominating set for X of size $36m^2+42m+13$.

Case ii. Let v be an even integer. Let $C = \{C_1, C_2, \cdots , C_{\frac{v-2}{2}}\}$. We add a_v to all elements of $P_2(C)$ to get a set of five tuples over V. We do the same procedure with a_{v-1} to get a similar set of five tuples. The union of these two later sets is a dominating set for X of size $36m^2 + 42m + 12$. □

Theorem 2.3. *Let* $G = \Gamma(v, 6, 3, 2)$ *and* $v \geq 9$. *If* $v = 6m + i$, *where* $0 \leq i \leq 5$, *then*

$$\gamma(G) \leq \frac{21m^2 + 19m + 6}{2}.$$

Proof. Let v be an odd integer. If $v \overset{6}{\equiv} 1$ or 5, then $MSTS(v)$ is a dominating set for Y and if $v \overset{6}{\equiv} 3$ then $STS(v)$ is a dominating set for Y. If v is an even integer, consider $V' = V - \{a_v\}$, then $|V'|$ is an odd integer and as above we have a dominating set for Y. This dominating set for Y is of size $6m^2 +7m+2$. Now we give a subset of Y as a dominating set of X. If v is even let $C = \{C_2, C_3, \cdots , C_{\frac{v}{2}}\}$. Then we add C_1 to all elements of $P_2(C)$ to get a set of six tuples over V. The union of this set with $A = \{C_2C_3C_4, C_5C_6C_7, \cdots\}$ is a dominating set for X. Note that if $|C|$ is not a multiple of 3, then the last triple in A, in the above arrangement, may build with any other one or any two other elements of C. If v is an odd integer, let $V' = V - \{a_v\}$ then $|V'|$ is even and as above we have a dominating set for X which is of size $\frac{21m^2+19m+6}{2}$. □

Theorem 2.4. *Let* $G = \Gamma(v,5,4,2)$ *and* $v \geq 32$. *If* $v = 24m+i$, *where* $0 \leq i \leq 23$, *then*

$$\gamma(G) \leq 162m^2 + 255m + 102.$$

Proof. We consider two cases:

Case i. Let v be an odd integer. Let $C = \{C_1, C_2, \cdots, C_{\frac{v-1}{2}}\}$. We add a_v to all elements of $P_2(C)$ to get a set of five tuples over V. This set is a dominating set for X. Now we give a dominating set for Y. Consider the set C as above and partition it into three subsets as A_1, A_2 and A_3 such that one of the following conditions hold:

1) If $\frac{v-1}{2} \overset{3}{\equiv} 0$, we may consider $|A_1| = |A_2| = |A_3|$.
2) If $\frac{v-1}{2} \overset{3}{\equiv} 1$, we may consider $|A_1| = |A_2| = |A_3| - 1$.
3) If $\frac{v-1}{2} \overset{3}{\equiv} 2$, we may consider $|A_1| = |A_2| - 1 = |A_3| - 1$.

The set $\bigcup_{i=1}^{3} P_2(A_i)$ is a dominating set for Y. The union of these dominating sets for X and Y is a dominating set for G of size $96m^2 + 164m + 70$.

Case ii. Let v be an even integer. Let $C = \{C_1, C_2, \cdots, C_{\frac{v-2}{2}}\}$. We add a_v to all elements of $P_2(C)$ to get a set of five tuples over V. We do the same procedure with a_{v-1} to get a similar set of five tuples. The union of these two later sets and $\{\{a_1, a_2, a_{v-2}, a_{v-1}, a_v\}, \{a_3, a_4, a_{v-4}, a_{v-1}, a_v\}\}$ give a dominating set for X. Now we give a dominating set for Y. Let $C' = C \cup \{C_{\frac{v}{2}}\}$. We partition the set C' into four subsets as A_1, A_2, A_3 and A_4 such that:

1) If $\frac{v}{2} \overset{4}{\equiv} 0$, we may consider $|A_1| = |A_2| = |A_3| = |A_4|$,
2) If $\frac{v}{2} \overset{4}{\equiv} 1$, we may consider $|A_1| = |A_2| = |A_3| = |A_4| - 1$,
3) If $\frac{v}{2} \overset{4}{\equiv} 2$, we may consider $|A_1| = |A_2| = |A_3| - 1 = |A_4| - 1$,
4) If $\frac{v}{2} \overset{4}{\equiv} 3$, we may consider $|A_1| = |A_2| - 1 = |A_3| - 1 = |A_4| - 1$.

The set $\bigcup_{i=1}^{4} P_2(A_i)$ is a dominating set for Y. The union of these dominating sets for X and Y is a dominating set for G of size $162m^2 + 255m + 102$. □

Theorem 2.5. *Let* $G = \Gamma(v,6,4,2)$ *and* $v \geq 41$. *If* $v = 8m+i$, *where* i *is odd and* $1 \leq i \leq 7$, *then*

$$\gamma(G) \leq 6m^2 + m + 4.$$

Proof. Since v is an odd integer we may consider $C = \{C_1, C_2, \cdots, C_{\frac{v-1}{2}}\}$ and partition it into four subsets as A_1, A_2, A_3 and A_4 such that $\max|(|A_i| - |A_j|)| \leqslant 1$, where $1 \leq i, j \leq 4$. The set $\bigcup_{i=1}^{4} P_2(A_i)$ is a dominating set for Y. Then we give a dominating set for X. Let $C' = C - \{C_{\frac{v-1}{2}}\}$ and partition it into two subsets as A_1 and A_2 such that either $|A_1| = |A_2|$ or $|A_1| = |A_2| - 1$. We add two elements a_{v-1} and a_v to all blocks of $\bigcup_{i=1}^{2} P_2(A_i)$. The union of this set and the following set is a dominating set for X:
$\{\{a_v, a_{v-2}, a_1, a_2, a_3, a_4\}, \{a_{v-1}, a_{v-2}, a_1, a_2, a_3, a_4\}, \{a_v, a_{v-2}, a_5, a_6, a_7, a_8\},$
$\{a_{v-1}, a_{v-2}, a_5, a_6, a_7, a_8\}\}$. □

Theorem 2.6. *Let $G = \Gamma(v, 6, 4, 2)$ and $v \geq 41$. If $v = 30m + i$, where i is even and $1 \leq i \leq 29$, then*

$$\gamma(G) \leq 60m^2 + 92m + 35.$$

Proof. Since v is an even integer we may consider $C = \{C_1, C_2, \cdots, C_{\frac{v}{2}}\}$ and partition it into five subsets as A_1, A_2, A_3, A_4 and A_5, such that $\max|(|A_i| - |A_j|)| \leqslant 1$, where $1 \leq i, j \leq 5$. The set $\bigcup_{i=1}^{5} P_2(A_i)$ is a dominating set for Y. Then we give a dominating set for X. We partition the set $C \setminus \{C_{\frac{v}{2}}\}$ into three subsets as A_1, A_2 and A_3 such that $\max|(|A_i| - |A_j|)| \leqslant 1$, where $1 \leq i, j \leq 3$. We add two elements a_{v-1} and a_v to all elements of $\bigcup_{i=1}^{3} P_2(A_i)$. This set is a dominating set for X. □

Theorem 2.7. *Let $G = \Gamma(v, 6, 5, 2)$ and $v \geq 41$. If $v = 30m + i$, $1 \leq i \leq 29$, then*

$$\gamma(G) \leq 12m^2 + 14m + 9.$$

Proof. We consider two cases:

Case i.
Let v be an odd integer. Let $C = \{C_1, C_2, \cdots, C_{\frac{v-1}{2}}\}$ and partition it into three subsets as A_1, A_2 and A_3 such that $\max|(|A_i| - |A_j|)| \leqslant 1$, where $1 \leq i, j \leq 3$. We add a_v to all elements of $P_2(A_i)$ for $1 \leq i \leq 3$. This set is a dominating set for Y. As shown in the following, we present a dominating set for X:

1) If $v \overset{12}{\equiv} 1, 5$, we add a point as x to V and consider $C_{\frac{v+1}{2}} = \{a_v, x\}$. Let $C = \{C_1, C_2, \cdots, C_{\frac{v-1}{2}}, C_{\frac{v+1}{2}}\}$. Since $\frac{v+1}{2} \overset{6}{\equiv} 1$ or 3, consider an STS on C and remove the blocks containing $C_{\frac{v+1}{2}}$. The set of remaining blocks is a dominating set for X.

2) If $v \overset{12}{\equiv} 3, 7, 11$, let $C = \{C_1, C_2, \cdots, C_{\frac{v-1}{2}}\}$. If $\frac{v-1}{2} \overset{6}{\equiv} 3$, consider an STS on C and if $\frac{v-1}{2} \overset{6}{\equiv} 1, 5$, consider an MSTS on C. In either case we have a dominating

set for X.

3) If $v \overset{12}{\equiv} 9$, let $C = \{C_1, C_2, \cdots, C_{\frac{v-3}{2}}\}$. Since $\frac{v-3}{2} \overset{6}{\equiv} 3$, we may consider an STS on C. The union of all blocks of this STS and the following set is a dominating set for X:

$$\{\{a_1, a_2, a_3, a_{v-2}, a_{v-1}, a_v\}, \{a_5, a_6, a_7, a_{v-2}, a_{v-1}, a_v\},$$

$$\{a_9, a_{10}, a_{11}, a_{v-2}, a_{v-1}, a_v\}, \{a_{13}, a_{14}, a_{15}, a_{v-2}, a_{v-1}, a_v\}\}.$$

To sum up, we have a dominating set for G of size $12m^2 + 14m + 4$.

Case ii. Let v be an even integer. Consider $C = \{C_1, C_2, \cdots, C_{\frac{v-2}{2}}\}$ and partition it into three subsets A_1, A_2 and A_3, such that $\max|(|A_i| - |A_j|)| \leqslant 1$ for $1 \leq i, j \leq 3$. Let $A_1 = \{C_1, C_2, C_3, \cdots\}$, $A_2 = \{C_l, C_{l+1}, C_{l+2}, \cdots\}$ and $A_3 = \{C_s, C_{s+1}, C_{s+2}, \cdots\}$.

We add a_v to all elements of $\bigcup_{i=1}^{3} P_2(A_i)$. The union of this set and the following set gives a dominating set for Y:

$$\{\{a_1, a_2, a_3, a_{v-1}, a_v\}, \{a_5, a_6, a_7, a_{v-1}, a_v\}, \{a_{2l}, a_{2l+1}, a_{2l+2}, a_{v-1}, a_v\},$$

$$\{a_{2l+5}, a_{2l+6}, a_{2l+7}, a_{v-1}, a_v\}, \{a_{2s+1}, a_{2s+2}, a_{2s+3}, a_{v-1}, a_v\}\}.$$

In the following we present a dominating set for X:

1) If $v \overset{12}{\equiv} 0, 4$, we add two points as x and y to V and consider $C_{\frac{v+2}{2}} = \{x, y\}$. Let $C = \{C_1, C_2, \cdots, C_{\frac{v+2}{2}}\}$. Since that $\frac{v+2}{2} \overset{6}{\equiv} 1, 3$, we may consider an STS on C and remove the blocks containing $C_{\frac{v+2}{2}}$. The set of remaining blocks is a dominating set for X.

2) If $v \overset{12}{\equiv} 2, 10$, let $C = \{C_1, C_2, \cdots, C_{\frac{v}{2}}\}$. Since $\frac{v}{2} \overset{6}{\equiv} 1, 5$, we may consider an MSTS on C. This set is a dominating set for X .

3) If $v \overset{12}{\equiv} 6$, let $C = \{C_1, C_2, \cdots, C_{\frac{v}{2}}\}$. Given that $\frac{v}{2} \overset{6}{\equiv} 3$, we can conclude that the set of all blocks in an STS on C forms a dominating set for X.

4) If $v \overset{12}{\equiv} 8$, let $C = \{C_1, C_2, \cdots, C_{\frac{v-2}{2}}\}$. Since $\frac{v-2}{2} \overset{6}{\equiv} 3$, then an STS on C is a dominating set for X.

In conclusion, we have determined a dominating set for G with a size of $12m^2 + 14m + 9$. □

We should note that the bounds given in this paper for $\gamma(G)$ are sharper than the bound given in Theorem 1.2.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

Acknowledgments. The authors thank the referee for his/her valuable comments.

References

[1] I. Anderson, *Combinatorial Designs and Tournaments*, Oxford University Press, 1997.

[2] D. B. West, *Introduction to Graph Theory*, 2nd Edition, Prentice Hall, 2001.

[3] W. Fish, N. B. Mumba, E. Mwambene and B. G. Rodrigues, Binary codes and partial permutation decoding sets from biadjacency matrices of the bipartite graphs $\Gamma(2k+1, k, k+2, 1)$, *Graphs Combin.* 33 (2) (2017) 357-368, https://doi.org/10.1007/s00373-017-1765-8.

Abolfazl Bahmani
Department of Mathematics,
University of Zanjan,
Zanjan, Iran
abolbahmani@yahoo.com

Mojgan Emami
Department of Mathematics,
University of Zanjan,
Zanjan, Iran
emami@znu.ac.ir

Ozra Naserian
Department of Mathematics,
Islamic Azad University,
Zanjan, Iran
o.naserian@gmail.com