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Pseudo o-Minimality for Double Stone Algebras

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Abstract

Pseudo o-minimality is a generalization of o-minimality of linear orders to partial orders. Recently Lei Chen, Niandong Shi and Guohua Wu provided pseudo o-minimality for the class of Stone algebras. In this note we use the quantifier elimination property to show that the class of double Stone algebras is pseudo o-minimal in their expanded language.

Keywords: Double Stone algebra, Definable set, Quantifier elimination, Pseudo o-minimality.

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1. Introduction

The o-minimal linear ordered structures, introduced by Van Den Dries in [1], have been extensively studied in the last four decades, see [2, 3] for more details and major results. In [4], Toffalori generalized the notion of o-minimality to partially ordered structures and classified o-minimal Boolean algebras. Then Chen, Shi and Wu using this generalization, introduced the notion of pseudo o-minimality in Stone algebras [5]. They investigated definable sets in Stone algebras using Schmitt's results in [6]. In fact, Schmitt obtained the model completion theory

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$\mathbf{STA}^*$ of the theory of Stone algebras, **STA**. He also showed that $\mathbf{STA}^*$ is substructure complete which follows that it has the quantifier elimination property. For definitions and proofs, see [6]. The quantifier elimination property of $\mathbf{STA}^*$ implies that $\mathbf{STA}^*$ is pseudo o-minimal. In this paper, we first show that the theory of double Stone algebras admits quantifier elimination. Then, we investigate the definable sets in double Stone algebras, and examine the o-minimality feature and some other model theoretic features for double Stone algebras. We prove that the theory of double Stone algebras is pseudo o-minimal.

2. Preliminaries

For convenience, we recall some definitions from the literature that we need to use here.

A partially ordered set $(P, \leq)$ is said to be a lattice if every subset $\{a, b\}$ of P has a least upper bound (which is denoted by $a \vee b$ and called the join of a and b) and a greatest lower bound (which is denoted by $a \wedge b$ and called the meet of a and b) in P. It is said to be bounded if there exist two elements 0 and 1 in P such that $0 \leq a \leq 1$ for every $a \in P$. Note that for every $a, b \in P$, $a \leq b$ if and only if $a \wedge b = a$, if and only if $a \vee b = b$. The class of bounded distributive lattices can be axiomatized by the axioms stating boundedness $(\forall a, 0 \leq a \leq 1)$, absorption laws $(\forall a \forall b, a \wedge (a \vee b) = a$ and $\forall a \forall b, a \vee (a \wedge b) = a)$ in addition to associativity, commutativity, and distributivity laws of operations $\wedge$ and $\vee$ in the first order language $\{\wedge, \vee, 0, 1\}$.

Let L be a bounded lattice with 0 and 1 as the least and greatest elements, respectively. Then, L is said to be pseudo complemented if for every element $u \in L$, the set $\{v \in L | u \wedge v = 0\}$ has a maximum, say u^*, as pseudo complement of u. A Stone algebra is a pseudo complemented distributive lattice S that holds $u^* \vee u^{**} = 1$ for every $u \in S$.

The order-minimal (abbreviated as o-minimal) structures are expansions of ordered structures whose unary definable sets are definable with quantifier-free formulas involving only the order relation. Our aim is to prove o-minimality or some fragments of it for some subclasses of the class of Stone algebras. So we give the definitions of algebras as first order structures in their appropriate languages.

Definition 2.1. A first order structure $\mathcal{S}_* = (S, \wedge, \vee, *, 0, 1. \leq)$ is called a *Stone algebra* if $(S, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and the operation $*$ of pseudo complementation satisfies:

- $\forall a\ (a \wedge a^* = 0)$,
- $\forall a\ \forall b\ (a \wedge b = 0 \to b \leq a^*)$,
- $\forall a\ (a^* \vee a^{**} = 1)$.

Similarly, a first order structure $\mathcal{S}_+ = (S, \wedge, \vee, +, 0, 1, \leq)$ is called a *dual Stone algebra* if $(S, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and the operation $+$ of dual pseudo complementation satisfies the axioms:

- $\forall a \ (a \vee a^+ = 1)$,
- $\forall a \ \forall b \ (a \vee b = 1 \to a^+ \leq b)$,
- $\forall a \ (a^+ \wedge a^{++} = 0)$.

Now by combining the above definitions, we have the definition of double Stone algebra in the language $L = \{\wedge, \vee, *, +, 0, 1, \leq\}$. A first order structure $\mathbb{S} = (S, \wedge, \vee, *, +, 0, 1, \leq)$ is called a *double Stone algebra* if $(S, \wedge, \vee, *, 0, 1, \leq)$ is a Stone algebra and $(S, \wedge, \vee, +, 0, 1, \leq)$ is a dual Stone algebra.

Two important structures are defined in a Stone algebra $\mathcal{S} = (S, \wedge, \vee, *, 0, 1)$; the skeleton $Sk(\mathcal{S}) = \{a^* | a \in S\} = \{a \in S | a = a^{**}\}$ of $\mathcal{S}$ is a Boolean subalgebra of $\mathcal{S}$, and the dense set $D(\mathcal{S}) = \{a \in S | a^* = 0\}$ which is a sublattice of $\mathcal{S}$ and a filter of $\mathcal{S}$. The structures $Sk(\mathcal{S})$ and $D(\mathcal{S})$ together with the structure morphism σ from $Sk(\mathcal{S})$ into the lattice of filters of $D(\mathcal{S})$ defined by $\sigma(a) = \{d \in D(\mathcal{S}) | \ a^* \leq d\}$ for every $a \in Sk(\mathcal{S})$, determines $\mathcal{S}$ upto isomorphism. For details, see [6].

Note that in the double Stone algebra $\mathbb{S}$, the Stone algebra and its dual have the same lattice structures.

Example 2.2.

1) Every Boolean algebra $\mathbf{B} = (B, \vee, \wedge, ', 0, 1)$ is a double Stone algebra. Both of pseudocomplement and dual pseudocomplement operations $*, +$ are the complement operation $'$.

2) Every bounded chain $\mathbf{C}$ (a linearly ordered set with the least and greatest elements) is a double Stone algebra. For $x \in \mathbf{C}$, if $x \neq 0, 1$ then $x^* = 0$ and $x^+ = 1$.

3) Given a Boolean algebra $\mathbf{B} = (B, \vee, \wedge, ', 0, 1)$, Moisil in [7] introduced n-valued Lukasiewicz-Moisil algebra $B^{[n]} = \{(b_1, ...b_n) \in B^n | b_1 \leq b_2 \leq ... \leq b_n\}$, where

$$\begin{aligned}(a_1, ..., a_n) \vee (b_1, ..., b_n) &= (a_1 \vee b_1, ..., a_n \vee b_n),\\ (a_1, ..., a_n) \wedge (b_1, ..., b_n) &= (a_1 \wedge b_1, ..., a_n \wedge b_n),\\ (b_1, ..., b_n)^* &= (b_n^{'}, ..., b_n^{'}),\\ (b_1, ..., b_n)^+ &= (b_1^{'}, ..., b_1^{'}).\end{aligned}$$

Also, the least element is $(0, ..., 0)$, and the greatest element is $(1, ..., 1)$.

It is easy to see that all the structures above are double Stone algebras. By the following fact, we can construct many other examples. It is also worth mentioning that every double Stone algebra can be embedded in a power of four elements Boolean algebra, see [8].

Fact 2.3. *Let $\mathbb{S}_1, ..., \mathbb{S}_n$ are double Stone algebras. Then their product $\mathbb{S}_1 \times ... \times \mathbb{S}_n$ is a double Stone algebra, where the operations of the product are defined as the following:*

$$(u_1, ..., u_n) \vee (v_1, ..., v_n) = (u_1 \vee v_1, ..., u_n \vee v_n),$$
$$(u_1, ..., u_n) \wedge (v_1, ..., v_n) = (u_1 \wedge v_1, ..., u_n \wedge v_n),$$
$$(u_1, ..., u_n)^* = (u_1^*, ..., u_n^*),$$
$$(v_1, ..., v_n)^+ = (v_1^+, ..., v_n^+).$$

Proof. It is straightforward. □

The following properties are immediately obtained from Definition 2.1. The proofs are straightforward and similar to those for Stone algebras exist in the literature, e.g. see ([9], chapter 7).

Lemma 2.4. *The following properties are satisfied for every elements u and v in a Stone algebra $\mathbb{S}$:*

i) $u \leq u^{**}$, $\quad u \leq v \rightarrow v^* \leq u^*$, $\quad u = v \rightarrow u^* = v^*$,
$u^{++} \leq u$, $\quad u \leq v \rightarrow v^+ \leq u^+$, $\quad u = v \rightarrow u^+ = v^+$.

ii) $(u \vee v)^* = u^* \wedge v^*$, $\quad (u \wedge v)^* = u^* \vee v^*$,
$(u \vee v)^+ = u^+ \wedge v^+$, $\quad (u \wedge v)^+ = u^+ \vee v^+$.

iii) $u^{***} = u^*$,
$u^{+++} = u^+$.

iv) $0^* = 1$, $\quad 1^* = 0$,
$0^+ = 1$, $\quad 1^+ = 0$.

v) $(u \vee v)^{**} = u^{**} \vee v^{**}$, $\quad (u \wedge v)^{**} = u^{**} \wedge v^{**}$,
$(u \vee v)^{++} = u^{++} \vee v^{++}$, $\quad (u \wedge v)^{++} = u^{++} \wedge v^{++}$.

Lemma 2.5. *For any $u \in \mathbb{S}$:*

i) $u^{++} \leq u^{**}$,

ii) $u^{+*} = u^{++}$,

iii) $u^{*+} = u^{**}$,

iv) $u^* \leq u^+$.

Proof. i) By Lemma 2.4, $u^{++} \leq u$ and $u \leq u^{**}$; thus $u^{++} \leq u^{**}$.

ii) For every $u \in \mathbb{S}, u^+ \in Sk(\mathbb{S})$. Since $Sk(\mathbb{S})$ is a Boolean algebra, we have $u^+ \vee u^{+*} = u^+ \vee u^{++} = 1$ and $u^+ \wedge u^{+*} = u^+ \wedge u^{++} = 0$. Then $u^{+*} = u^{++}$ by uniqueness of complementation in $Sk(\mathbb{S})$.

iii) It is similar to (ii).

iv) From Lemma 2.4, we have $u^{++} \leq u$. So $u^* \leq u^{++*}$. On the other hand $u^{++*} = u^{+++} = u^+$ by (ii). Thus, $u^* \leq u^+$.

□

Let $\mathbb{S}$ be a double Stone algebra. Let

$$Sk^*(\mathbb{S}) = \{u^*|u \in \mathbb{S}\} = \{u \in \mathbb{S}|u = u^{**}\} \quad \text{and}$$
$$Sk^+(\mathbb{S}) = \{u^+|u \in \mathbb{S}\} = \{u \in \mathbb{S}|u = u^{++}\}.$$

These subalgebras of $\mathbb{S}$ are Boolean algebras and by Lemma 2.5, $Sk^*(\mathbb{S}) = Sk^+(\mathbb{S})$. The Boolean algebra $Sk(\mathbb{S}) := Sk^*(\mathbb{S})$ is named the skeletone of the Stone algebra $\mathbb{S}$. It plays an essential role in the study of double Stone algebras. Also two others sets are defined in the structure $\mathbb{S}$: a dense set of $\mathbb{S}$, $D^*(\mathbb{S}) = \{u \in S|u^* = 0\}$ which is a filter of $\mathbb{S}$, and a dual dense set of $\mathbb{S}$, $D^+(\mathbb{S}) = \{u \in S|u^+ = 1\}$ which is an ideal of $\mathbb{S}$. The set $DD(\mathbb{S}) = \{u \in S|u^* = 0, u^+ = 1\}$ is called the doubly dense set of $\mathbb{S}$. $DD(\mathbb{S})$ is nonempty and forms a relatively complemented sublattice of $\mathbb{S}$. Using the above lemmas, one can easily show that every element $u \in \mathbb{S}$ can be written as the meet of an element of $Sk(\mathbb{S})$ and an element of $D^*(\mathbb{S})$, namely $u = u^{**} \wedge (u \vee u^*)$; moreover, $u \in \mathbb{S}$ is the joint of $u^{++} \in Sk(\mathbb{S})$ and $(u \wedge u^+) \in D^+(\mathbb{S})$.

3. Quantifier elimination

In this section, we show that the theory of double Stone algebras admits the quantifier elimination property. This model theoretic property helps us to characterize the definable sets in models of the theory of double Stone algebras.

Definition 3.1.

i) Let **T** be a theory in a first-order language L. Then **T** is said to have the quantifier elimination property if for every formula ϕ in L, there exists a quantifier-free formula ψ in L which is equivalent to ϕ under **T**.

ii) Let **K** be an elementary class, i.e., **K** is the class of all models of a fixed first-order theory. For example the class of Stone algebras, as well as, the class of double Stone algebras are elementary classes in their languages. An elementary class **K** is said to have quantifier elimination property if its theory admits the quantifier elimination property.

iii) The theory T is said to have the amalgamation property if for any models $\mathbb{K}$, $\mathbb{L}$, $\mathbb{M}$ of T and elementary embeddings $i : \mathbb{K} \to \mathbb{L}$ and $j : \mathbb{K} \to \mathbb{M}$, there is a model $\mathbb{N}$ of T and embeddings $f : \mathbb{L} \to \mathbb{N}$ and $g : \mathbb{M} \to \mathbb{N}$ such that $f \circ i = g \circ j$.

iv) The theory T is said to be model-complete if every embedding between two models of T is elementary. It is easy to see that if T has quantifier elimination property then it is model-complete. The theory T' is the model-completion of T if T' is model-complete, every model of T can be extended to a model of T', and every model of T' can be extended to a model of T.

v) A model $\mathbb{M}$ of a theory T is said to be existentially closed if it satisfies every existential formula which is satisfied in a model $\mathbb{N}$ of T that is an extension of $\mathbb{M}$.

Peter H. Schmitt in [6] determined the model completion $\mathbf{STA}^*$ of the theory of Stone algebras, **STA**. Then, he showed that $\mathbf{STA}^*$ has quantifier elimination property. Lei Chen, Niandong Shi and Guohua Wu used this result to show that models of $\mathbf{STA}^*$ and so all Stone algebras are pseudo o-minimal.

Davey in [10] used duality theory and checked it on bounded distributive lattice ordered algebras and showed that every double Stone algebra is a Hom-set endowed with a Boolean topology and the continuous action of the endomorphism monoid of an appropriate algebra:

- Suppose $\mathbf{A} = ISP(A)$ is the class of intended algebras and $B \in \mathbf{A}$. In this case the Hom-set $\mathbf{A}(B, A)$ (the set of all morphisms from B to A for some given algebra A of the class $\mathbf{A}$) is a Boolean space (regarded as subspace of A^B). Denote the dual of algebra B by $B^* = \mathbf{A}(B, A)$.
- B^* is an object of category $\mathbf{Z}$ of partially ordered Boolean spaces whose $End(A)$ (the monoid $\mathbf{A}(A, A)$ of endomorphisms of A) act continuously on it.
- Therefore the functor $\mathfrak{D} : \mathbf{A} \to \mathbf{Z}$ with $\mathfrak{D}(B) = B^* = \mathbf{A}(B, A)$ is obtained.
- Then by defining $\mathbf{Z}$-structure on $\mathbf{A}$, the Hom-set $\mathbf{Z}(B^*, A)$ is called B^{**} and we have $\eta_B : B \to B^{**} = \mathbf{Z}(\mathbf{A}(B, A), A)$ as an isomorphism between bounded distributive lattice ordered algebras B and the Hom-set endowed with a Boolean topology and the continuous action of its endomorphism monoid $B^{**} = \mathfrak{D}(B^*)$.
- Then the functor $\mathfrak{E} : \mathbf{Z} \to \mathbf{A}$ with $\mathfrak{E}(\phi)\mu = \mu\phi$ maps each boolean space $\mathfrak{X}$ in $\mathbf{Z}$ to an algebra in $\mathbf{A}$, i.e $\mathfrak{E}(\mathfrak{X}) \in \mathbf{A}$.

Theorem 3.2. *(Clarc and Krauss [11])* $(\mathfrak{D}, \mathfrak{E})$ *is a full duality between* $ISP(\mathbb{S})$ *and* $IS_cP(\mathfrak{S})$*.*

Davey also proved that the class of double Stone algebras is an equational class whose members are given by $ISP\mathbf{4}$ where $\mathbf{4} = (\{0, a, b, 1\}, \wedge, \vee, *, +, 0, 1)$ is 4-chain. If $\mathbb{S} = \mathfrak{E}(\mathfrak{X})$ is a double Stone algebra, then:

$$Sk(\mathbb{S}) = \{\sigma \in S | \sigma^{**} = \sigma\} = \{\sigma \in S | \sigma^{-1}\{a, b\} = \emptyset\},$$

is the skeleton of $\mathbb{S}$, and

$$DD(\mathbb{S}) = \{\delta \in S | \delta^* = 0 \text{ and } \delta^+ = 0\} = \{\delta \in S | \delta^{-1}\{0, 1\} = \emptyset\},$$

is the sublattice of doubly dense elements of $\mathbb{S}$.
The following two theorems provides the tools to continue.

Theorem 3.3. *(Hodges, 1993[12]) Let L be a first-order language and* $\mathbf{T}$ *a theory in L. The following are equivalent.*

i) $\mathbf{T}$ *has the quantifier elimination property.*

ii) $\mathbf{T}$ *is model-complete and* $\mathbf{T}_\forall$ *has the amalgamation property.*

Theorem 3.4. *(Clark [13]) For a double Stone algebra $\mathbb{S} = E(\mathfrak{X})$, the following are equivalent:*

i) $\mathbb{S}$ *is existentially closed.*

ii) $\mathbb{S}$ *satisfies the following $\forall\exists$-axioms:*
(DS1) $DD(\mathbb{S})$ is nonempty and form a relatively complemented sublattice of $\mathbb{S}$.
(DS2) For every $\gamma, \delta \in DD(\mathbb{S})$, there is an element $\sigma \in \mathbb{S}$ such that $(\gamma \wedge \delta) \vee (\delta \wedge \sigma^*) = \gamma \vee \delta$.
(DS3) $DD(\mathbb{S})$ contains no covers.
(DS4) If $\delta^* = 0$ and $\delta < 1$, then there is an element $\gamma > \delta$ such that $\gamma^+ = \delta^+$.
(DS5) If $\delta^+ = 1$ and $\delta > 0$, then there is an element $\gamma < \delta$ such that $\gamma^* = \delta^*$.

We denote the theory of double Stone algebras by **DBS**. This theory is the complete theory of a double Stone algebra $\mathbb{S}$, i.e. the set of all sentences true in $\mathbb{S}$, together with the additional axioms described by $DS1, ..., DS5$, in Theorem 3.4.

Theorem 3.5. *The theory* **DBS** *admits quantifier elimination.*

Proof. Since every model of **DBS** is existentially closed, the theory **DBS** is model complete and has the amalgamation property by [14]. Hence, by Theorem 3.3, **DBS** has the quantifier elimination property. □

4. Pseudo o-minimality

In [4], Carlo Toffalori generalized the notion of order-minimality of linear orders to the partially ordered structures. Let $\mathcal{A} = (A, \leq, ...)$ be a structure partially ordered by $\leq$. $\mathcal{A}$ is said to be quasi o-minimal if and only if the only subsets of A definable in $\mathcal{A}$ are the finite Boolean combinations of sets defined by formulas $a \leq \nu$ or $\nu \leq b$ with a and b in A. By Theorem 2.3 of [4], every Boolean algebra with only finitely many atoms is quasi o-minimal.

Using the idea of [4], Lei Chen, Niandong Shi and Guohua Wu in [5] defined pseudo o-minimality in lattice-ordered structures. Here, we define pseudo o-minimality in lattice-ordered structures using their comparability graphs.

Let $\mathcal{A} = (A, \leq, \wedge, \vee, ...)$ be a lattice-ordered structure. The comparability graph of $\mathcal{A}$ is a undirected graph $\mathcal{G} = (A, E)$ such that an undirected edge $u - v$ is in E if and only if u and v are different and comparable, i.e. $u \neq v$ and $u \leq v$ or $v \leq u$. A set $X \subseteq A$ is said to be connected if for every different elements

$u, v \in X$ there exists a path in the corresponding graph $\mathcal{G}$ connecting u and v. A set $Y \subseteq A$ is said to be strongly connected if for any $u, v \in Y$, either $u \wedge v \in Y$ or $u \vee v \in Y$. It is clear that every strongly connected set is connected too. If $\mathcal{A}$ is a Stone algebra, then $Sk(\mathcal{A})$ and $D(\mathcal{A})$ are strongly connected.

Definition 4.1. A lattice-ordered structure $\mathcal{A} = (A, \leq, \wedge, \vee, ...)$ is said to be pseudo o-minimal if every definable subset of A is a Boolean combination of finitely many strongly connected subsets of A. A theory $\mathbf{T}$ is called pseudo o-minimal if every model of $\mathbf{T}$ is pseudo o-minimal.

We use the quantifier elimination provided in the last section to show that the theory **DBS** of double Stone algebras is pseudo o-minimal. In the following, by a term or formula of **DBS**, we mean a term or formula in the first order language of the theory **DBS**. Also equivalence between formulas is under the theory **DBS**.

Lemma 4.2. *Every atomic formula of* **DBS** *is equivalent to a conjunction of the formulas* $\tau \leq \nu$*, where* $\tau \in \{a, u \wedge a, u^* \wedge a, u^+ \wedge a, u^{**} \wedge a, u^{++} \wedge a, u \wedge u^+ \wedge a\}$ *and* $\nu \in \{b, u \vee b, u^* \vee b, u^+ \vee b, u^{**} \vee b, u^{++} \vee b, u \vee u^* \vee b\}$*, where* a, b *are terms of* **DBS** *and do not contain variable* u*.*

Proof. By Lemma 2.4, Lemma 2.5, and distributivity, every atomic formula is equivalent to a conjunction of formulas $\wedge_{j=1}^{m} \tau_j \leq \vee_{k=1}^{n} \nu_k$ such that τ_js and ν_ks are one of $u_j, u_j, u_j^{**}, a, u_k, u_k^{+}, u_k^{++}, b$, where a and b are constants. □

Corollary 4.3. *Each atomic formula of* **DBS** *is equivalent to a finite conjunction of the following formulas:*

$a \leq b$	$a \leq u \vee b$	$a \leq u^* \vee b$
$a \leq u^+ \vee b$	$a \leq u^{**} \vee b$	$a \leq u^{++} \vee b$
$a \leq u \vee u^* \vee b$	$u \wedge a \leq b$	$u \wedge a \leq u \vee b$
$u \wedge a \leq u^* \vee b$	$u \wedge a \leq u^+ \vee b$	$u \wedge a \leq u^{**} \vee b$
$u \wedge a \leq u^{++} \vee b$	$u \wedge a \leq u \vee u^* \vee b$	$u^* \wedge a \leq b$
$u^* \wedge a \leq u \vee b$	$u^* \wedge a \leq u^* \vee b$	$u^* \wedge a \leq u^+ b$
$u^* \wedge a \leq u^{**} \vee b$	$u^* \wedge a \leq u^{++} \vee b$	$u^* \wedge a \leq u \vee u^* \vee b$
$u^+ \wedge a \leq b$	$u^+ \wedge a \leq u \vee b$	$u^+ \wedge a \leq u^* \vee b$
$u^+ \wedge a \leq u^+ \vee b$	$u^+ \wedge a \leq u^{**} \vee b$	$u^+ \wedge a \leq u^{++} \vee b$
$u^+ \wedge a \leq u \vee u^* \vee b$	$u^{**} \wedge a \leq b$	$u^{**} \wedge a \leq u \vee b$
$u^{**} \wedge a \leq u^* \vee b$	$u^{**} \wedge a \leq u^+ \vee b$	$u^{**} \wedge a \leq u^{**} \vee b$
$u^{**} \wedge a \leq u^{++} \vee b$	$u^{**} \wedge a \leq u \vee u^* \vee b$	$u^{++} \wedge a \leq b$
$u^{++} \wedge a \leq u \vee b$	$u^{++} \wedge a \leq u^* \vee b$	$u^{++} \wedge a \leq u^+ \vee b$
$u^{++} \wedge a \leq u^{**} \vee b$	$u^{++} \wedge a \leq u^{++} \vee b$	$u^{++} \wedge a \leq u \vee u^* \vee b$
$u \wedge u^+ \wedge a \leq b$	$u \wedge u^+ \wedge a \leq u \vee b$	$u \wedge u^+ \wedge a \leq u^* \vee b$
$u \wedge u^+ \wedge a \leq u^+ \vee b$	$u \wedge u^+ \wedge a \leq u^{**} \vee b$	$u \wedge u^+ \wedge a \leq u^{++} \vee b$
$u \wedge u^+ \wedge a \leq u \vee u^* \vee b$		

Theorem 4.4. *All the sets defined by formulas in Corollary 4.3, are strongly connected sets in a double Stone algebra.*

Proof. We can easily prove all cases by computational operations and by using the properties listed in Lemma 2.4. For instance, we examine the set defined by formula $u \wedge u^+ \wedge a \leq u^{**} \vee b$. Suppose $\mathbb{S}$ is a double Stone algebra and $S_1 = \{u | u \wedge u^+ \wedge a \leq u^{**} \vee b\} \subseteq S$. Let $u, v \in S_1$:

$$\begin{aligned}
(u \vee v) \wedge (u \vee v)^+ \wedge a &= (u \vee v) \wedge ((u \vee v)^+ \wedge a) \\
&= (u \vee v) \wedge ((u^+ \wedge v^+) \wedge a) \\
&= (u \wedge u^+ \wedge v^+ \wedge a) \vee (v \wedge u^+ \wedge v^+ \wedge a) \\
&= ((u \wedge u^+ \wedge a) \wedge v^+) \vee ((v \wedge v^+ \wedge a) \wedge u^+) \\
&\leq (u \wedge u^+ \wedge a) \vee (v \wedge v^+ \wedge a) \\
&\leq (u^{**} \vee b) \vee (v^{**} \vee b) \\
&= (u^{**} \vee v^{**}) \vee b = (u \vee v)^{**} \vee b.
\end{aligned}$$

Thus, $u \vee v \in S_1$. Therefore, S_1 is a strongly connected subset in $\mathbb{S}$. □

Theorem 4.5. *The theory* **DBS** *is a pseudo o-minimal theory.*

Proof. Let $\mathbb{S} = (S, \wedge, \vee *, +, 0, 1)$ be a double Stone algebra and a model of **DBS**. Since **DBS** has the quantifier elimination property by Theorem 3.5, every formula in **DBS** is equivalent to a Boolean combination of atomic formulas in **DBS**. But, by Corollary 4.3, every atomic formula in **DBS** is equivalent to a conjunction of forty nine formulas listed there. Thus by Theorem 4.4, every unary definable set in every model of **DBS** is a Boolean combination of strongly connected sets. □

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References

[1] L. V. den Dries, Remarks on Tarski's problem concerning (R,+,*, exp), *Stud. Logic Found. Math.* **112** (1984) 97 – 121, https://doi.org/10.1016/S0049-237X(08)71811-1.

[2] J. F. Knight, A. Pillay and C. Steinhorn, Definable sets in ordered structures II, *Trans. Amer. Math. Soc.* **295** (1986) 593 – 605.

[3] A. Pillay and C. Steinhorn, Definable sets in ordered structures I, *Trans. Amer. Math. Soc.* **295** (1986) 565 – 592.

[4] C. Toffalori, Lattice ordered o-minimal structures, *Notre Dame J. Formal Logic* **39** (1998) 447 – 463, https://doi.org/10.1305/ndjfl/1039118862.

[5] L. Chen, N. Shi and G. Wu, Definable sets in Stone algebras, *Arch. Math. Logic* **55** (2016) 749 – 757, https://doi.org/10.1007/s00153-016-0491-x.

[6] P. H. Schmitt, The model-completion of Stone algebras, *Ann. Sci. Univ. Clermont No. 60 Math.* **60** (1976) 135 – 155.

[7] V. Boicescu, A. Filipoiu, G. Georgescu and S. Rudeanu, *Lukasiewicz-Moisil Algebras*, North-Holland, (1991).

[8] R. Balbes and P. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, (1974).

[9] T. S. Blyth, *Lattices and Ordered Algebraic Structures*, Springer-Verlag, (2005).

[10] B. A. Davey, Dualities for Stone algebras, double Stone algebras and relative Stone algebras, *Colloq. Math.* **46** (1982) 1 – 14.

[11] D. M. Clark and P. H. Krauss, Topological quasi varieties, *Acta Sci. Math. (Szeged).* **47** (1984) 3 – 39.

[12] W. Hodges, *Model Theory*, Cambridge University Press, (1993).

[13] D. M. Clark, The structure of algebraically and existentially closed Stone and double Stone algebras, *J. Symbolic Logic* **54** (1989) 363 – 375.

[14] V. Koubek and J. Sichler, Amalgamation in varieties of distributive double p-algebras, *Algebra Universalis* **32** (1994) 407 – 438, https://doi.org/10.1007/BF01195722.

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