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Barnes−Godunova−Levin Type Inequalities for Generalized Sugeno Integral

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Abstract

This article will prove the Barnes–Godunova–Levin (B-G-L) type inequalities for generalized Sugeno integrals. Also, we use some techniques and properties of concave functions to prove theorems and to obtain new results. We will present a more robust version of the B-G-L type inequality for the operator $\star$.

Keywords: Sugeno integral, Generalized Sugeno integral, Concave function, Barnes–Godunova–Levin type inequality.

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1. Introduction

A famous nonadditive integral was introduced in 1974 by Sugeno. The generalized Sugeno integral associates the value

$$\int_A \varphi d\mu := \sup_{y\in Y}[y\Delta\mu(A\cap\Phi_y)],$$

to each pair consisting of a monotone measure μ on a set V and $\varphi : V \to [0, +\infty]$. Also Δ is a nonnegative, extended real-valued function on $[0, +\infty[\times[0, +\infty]$

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that is non-decreasing in both arguments. The specific modes of the generalized Sugeno integral are: the Shilkret integral [1], the seminormed fuzzy integral [2, 3], the q-integral [4, 5] and the Sugeno integral [6]. In the Sugeno integral, the function Δ returns the minimum of its two arguments, if it returns the product, we obtain the Shilkret integral.

In [7], Agahi et al. proved B-G-L inequality for Sugeno integral. Also, Abbaszadeh et al. proved B-G-L inequality for pseudo-integrals [8]. In this paper, we are going to prove a generalization of the B-G-L type inequality for the generalized Sugeno integral. The definition of generalized Sugeno integral and its properties are given in Section 2. A generalization of B-G-L inequality for generalized Sugeno integral is presented in Section 3. Finally, a generalized version of B-G-L type inequality based on an operation $\star$, that is continuous and non-decreasing in both arguments, is presented in Section 4.

2. Preliminaries

In this section, we will introduce the definitions and required theorems that will be used to prove the main theorem and its related results.

Definition 2.1. ([9, 10]). Let Λ be a σ-algebra of subsets of V and let $\mu : \Lambda \to [0, \infty]$ be a nonnegative extended real-valued set function. We say that μ is a monotone measure if:

(FM1) $\mu(\emptyset) = 0$;

(FM2) $A, B \in \Lambda$ and $A \subseteq B$ imply $\mu(A) \leq \mu(B)$ (monotonicity);

(FM3) $\{A\}_{n=1}^{+\infty} \subseteq \Lambda, A_1 \subseteq A_2 \subseteq \dots$, imply $\lim_{n\to+\infty} \mu(A_n) = \mu(\bigcup\limits_{n=1}^{\infty} A_n)$ (continuity from below);

(FM4) $\{A\}_{n=1}^{+\infty} \subseteq \Lambda, ... \subseteq A_2 \subseteq A_1, \mu(A_1) < \infty$, imply $\lim_{n\to+\infty} \mu(A_n) = \mu(\bigcap\limits_{n=1}^{\infty} A_n)$ (continuity from above).

When μ is a monotone measure, the triple (V, Λ, μ) is called a monotone measure space.

Let (V, Λ, μ) be a monotone measure space and Φ_+ be the class of all nonnegative measurable functions defined on (V, Λ). For any given $\varphi \in \Phi_+$, we write $\Phi_\alpha = \{v \in V \mid \varphi(v) \geq \alpha\}$, with $\alpha \geq 0$.

Definition 2.2. Let (V, Λ, μ) be a monotone measure space. If $\varphi \in \Phi_+$ and $D \in \Lambda$, then we have:

i: The Shilkret integral of φ on D with respect to the monotone measure μ is defined by [1, 10]

$$(N)\int_D \varphi d\mu := \sup_{\beta\geq 0}(\beta\cdot\mu(D\cap\Phi_\beta)).$$

ii: The Sugeno integral of φ on D for the monotone measure μ is defined by [6, 10, 11]

$$⨍_D \varphi d\mu := \bigvee_{\beta\geq 0}(\beta\wedge\mu(D\cap\Phi_\beta)).$$

Where $\wedge$, $\vee$ denotes the operation inf and sup on $[0,\infty[$ respectively. Since $\Phi_\beta=\emptyset$ where $\beta=+\infty$, the equation

$$⨍_D \varphi d\mu := \bigvee_{\beta\in[0,+\infty]}(\beta\wedge\mu(D\cap\Phi_\beta)),$$

can be accepted as the definition of Sugeno's integral with the convention of condition $0.\infty=\infty.0=0$ if necessary.

Sogno's integral is a well-known nonlinear integral [7, 10, 12], i.e., the equality

$$⨍(a\varphi+b\psi)\,d\mu = a⨍\varphi\,d\mu + b⨍\psi\,d\mu,$$

does not hold. In the following theorem, most of the initial properties of Sugeno's integral are presented [10, 11, 13].

Theorem 2.3. Let (V,Λ,μ) be a monotone measure space with $E,F\in\Lambda$ and $\varphi,\psi\in\Phi_+$. Then

1. $⨍_E c d\mu = c\wedge\mu(E)$ for any constant $c\in[0,+\infty[$,
2. if $E\subset F$, then $⨍_E\varphi d\mu\leq ⨍_F\varphi d\mu$,
3. if $\mu(E)<\infty$, then $\mu(E\cap\Phi_\beta)\geq\beta\Leftrightarrow ⨍_E\varphi d\mu\geq\beta$,
4. if $\varphi\leq\psi$ on E, then $⨍_E\varphi d\mu\leq ⨍_E\psi d\mu$,
5. $⨍_E\varphi d\mu\leq\mu(E)$,
6. $\mu(E\cap\Phi_\beta)\leq\beta\Rightarrow ⨍_E\varphi d\mu\leq\beta$,
7. $⨍_E\varphi d\mu<\beta\Leftrightarrow$ there exists $\alpha<\beta$ such that $\mu(E\cap\Phi_\alpha)<\beta$,
8. $⨍_E\varphi d\mu>\beta\Leftrightarrow$ there exists $\alpha>\beta$ such that $\mu(E\cap\Phi_\alpha)>\beta$.

Remark 1. Let $\Phi(\beta) = \mu(E \cap \Phi_\beta)$. From statements (2) and (3) of the previous theorem, the following result is obtained, which is very important and widely used.

$$\Phi(\beta) = \beta \Rightarrow ⨍_E \varphi d\mu = \beta.$$

Therefore, the answer to the equation $\Phi(\beta) = \beta$ will be the same numerical value as Sugeno's integral.

Let $W = [0, \infty[$ (or $W = [0, \infty]$), $range(\mu) := \mu(\Lambda)$ and let $\star : W \times W \to W$ be an operation. We say that $\star$ is non-decreasing if $a \geq b$ and $c \geq d$ implay that $a \star c \geq b \star d$.

Definition 2.4. ([14]). For $\varphi : V \to W$, (μ -measurable function), we define the generalized Sugeno integral of φ on a set $E \in \Lambda$, by

$$\int_E \varphi \Delta\mu := \sup_{y \in W} [y \Delta \mu(E \cap \Phi_y)], \tag{1}$$

where μ is a monotone measure on Λ and $\Delta : W \times \mu(\Lambda) \to W$ is a non-decreasing operator.

Theorem 2.5. ([14]). A generalized Sugeno integral satisfies the indicator property (i.e., $\int \chi_E \Delta\mu = \mu(E)$) if and only if we have two conditions:

a: $\beta \Delta 0 = 0$ for all $\beta \in [0, +\infty[$;

b: $1 \Delta \alpha = \alpha$ for all $\alpha \in [0, +\infty]$.

Here we have a monotone integral and it is homogeneous if and only if the function Δ returns the product of its two arguments. That is, the Shilkret integral is the homogeneous version of the generalized Sugeno integral.

Lemma 2.6. Let (V, Λ, μ) be a monotone measure space, $E \in \Lambda$ and $\varphi, \psi : V \to W$ be two μ-measurable functions:

(1) If $\varphi \leq \psi$ on E, then $\int_E \varphi \Delta\mu \leq \int_E \psi \Delta\mu$.

Furthermore, if $\beta \Delta 0 = 0$ for any $\beta \in [0, +\infty[$, then

(2) $\int_E c \Delta\mu = c \Delta \mu(E)$ for any constant $c \in [0, +\infty[$.

Proof. (1) Since $\varphi \leq \psi$ on E, we have

$$\Phi_\beta = \{v \mid \varphi(v) \geq \beta\} \subseteq \{v \mid \psi(v) \geq \beta\} = \Psi_\beta,$$

therefore,

$$E \cap \Phi_\beta \subseteq E \cap \Psi_\beta,$$

and so, by the monotonicity of μ, we have

$$\mu(E \cap \Phi_\beta) \leq \mu(E \cap \Psi_\beta),$$

for any $\beta \in [0, +\infty[$. On the other hand, since Δ is non-decreasing in the second argument, we have

$$\beta\Delta\mu(E \cap \Phi_\beta) \leq \beta\Delta\mu(E \cap \Psi_\beta),$$

for any $\beta \in [0, +\infty[$. Hence,

$$\sup_{\beta \geq 0}(\beta\Delta\mu(E \cap \Phi_\beta)) \leq \sup_{\beta \geq 0}(\beta\Delta\mu(E \cap \Psi_\beta)),$$

i.e.,

$$\int_E \varphi\Delta\mu \leq \int_E \psi\Delta\mu.$$

(2) As

$$\Phi_\beta = \{v \in V \mid c \geq \beta\} = \begin{cases} V, & if\, \beta \leq c, \\ \emptyset, & if\, \beta > c, \end{cases}$$

it follows that

$$\begin{aligned} \int_E c\Delta\mu &= \sup_{\beta \geq 0}(\beta\Delta\mu(E \cap \Phi_\beta)) = \sup_{\beta \in [0,c]}(\beta\Delta\mu(E \cap \Phi_\beta)) \vee \sup_{\beta > c}(\beta\Delta\mu(E \cap \Phi_\beta)) \\ &= \sup_{\beta \in [0,c]}(\beta\Delta\mu(E)) \vee \sup_{\beta > c}(\beta\Delta 0) = (c\Delta\mu(E)) \vee 0 \\ &= c\Delta\mu(E). \end{aligned}$$

□

Definition 2.7. ([15, 16]). Functions $\varphi, \psi : V \to W$ are called comonotone if, for all $u, v \in V$,

$$(\varphi(u) - \varphi(v))(\psi(u) - \psi(v)) \geq 0.$$

Obviously, φ and ψ are comonotone if and only if for any two real numbers r and s, either $\Phi_r \subseteq \Psi_s$ or $\Psi_s \subseteq \Phi_r$.

Lemma 2.8. Let (V, Λ, μ) be a monotone measure space. If φ and ψ are comonotone, then for any real numbers r and s,

$$\mu(\Phi_r \cap \Psi_s) = \mu(\Phi_r) \wedge \mu(\Psi_s). \tag{2}$$

Proof. Since φ and ψ are comonotone, either $\Phi_r \subseteq \Psi_s$ or $\Psi_s \subseteq \Phi_r$. Therefore,

$$\mu(\Phi_r) \wedge \mu(\Psi_s) \leq \mu(\Phi_r \cap \Psi_s).$$

On the other hand, it is clear that $\mu(\Phi_r \cap \Psi_s) \leq \mu(\Phi_r) \wedge \mu(\Psi_s)$. So we have,

$$\mu(\Phi_r \cap \Psi_s) = \mu(\Phi_r) \wedge \mu(\Psi_s).$$

□

Definition 2.9. ([17, 18]). A function $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be t-norm, if satisfies the following conditions:

(L$_1$): $T(u,1) = T(1,u) = u$ for any $u \in [0,1]$.

(L$_2$): For any $u_1, u_2, v_1, v_2 \in [0,1]$ with $u_1 \leq u_2$ and $v_1 \leq v_2$, $T(u_1, v_1) \leq T(u_2, v_2)$.

(L$_3$): T is symmetry and $T(u,v) = T(v,u)$ for any $u, v \in [0,1]$.

(L$_4$): $T\big(T(u,v), w\big) = T\big(u, T(v,w)\big)$ for any $u, v, w \in [0,1]$.

Note that a function $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be t-seminorm [2], if it satisfies the conditions (L_1) and (L_2).

Example 2.10. The following functions are t-norms and symmetry:

1. $M(u,v) = \min\{u,v\}$.
2. $\Pi(u,v) = u \cdot v$.
3. $O_L(u,v) = \max\{u + v - 1, 0\}$.

Definition 2.11. ([2]). Let T be a t-seminorm and (V, Λ, μ) be a monotone measure. If $\varphi \in \Phi_+$ and $D \in \Lambda$, then the seminormed Sugeno integral of φ on D with respect to the monotone measure μ is defined by

$$\int_{D,T} \varphi d\mu := \sup_{y \in [0,1]} T\,[y, \mu(D \cap \Phi_y)].$$

Definition 2.12. Let $\mathbb{R}$ be the set of real numbers. If J is a real interval and $\varphi : J \to \mathbb{R}$ is a function, then φ is said to be concave (on J) provided:

$$u, v \in J, \beta \in [0,1] \Rightarrow \varphi(\beta u + (1-\beta)v) \geq \beta\varphi(u) + (1-\beta)\varphi(v).$$

3. Results

The following inequality, known as the B-G-L inequality for classic measures, is a well-known integral inequality for concave functions. (see [19, 20]):

$$\left(\int_a^b \varphi^r(x)dx\right)^{\frac{1}{r}} \left(\int_a^b \psi^s(x)dx\right)^{\frac{1}{s}} \leq B(r,s) \int_a^b \varphi(x)\psi(x)dx, \tag{3}$$

where $r, s > 1, B(r,s) = \frac{6(b-a)^{\frac{1}{r}+\frac{1}{s}-1}}{(1+r)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}$ and φ, ψ are nonnegative concave functions on $[a,b]$.

According to the following example, in general, the classic B-G-L inequality is not true for generalized Sugeno integral.

Example 3.1. Let φ, ψ be two real-valued functions defined as $\varphi(u) = \psi(u) = \sqrt[4]{u}$ where $u \in [0, 100]$. Let m be the Lebesgue measure, $\Delta := \wedge$ and $r = s = 4$ in inequality (3). Obviously,

- $\int_0^{100} \varphi^4 \Delta m = \int_0^{100} \psi^4 \Delta m = \sup_{y\in[0,100]} [y \wedge (100 - y)] = 50,$
- $\int_0^{100} \varphi\psi \Delta m = \sup_{y\in[0,10]} [y \wedge (100 - y^2)] = 9.5125,$
- $B(4, 4) = 0.26833.$

Therefore:

$$\left(\int_0^{100} \varphi^4 \Delta m\right)^{\frac{1}{4}} \left(\int_0^{100} \psi^4 \Delta \mu\right)^{\frac{1}{4}} = 7.0711 \geq 2.5525 = B(4, 4) \int_0^{100} \varphi\psi \Delta m,$$

and, consequently, inequality (3) is not true for the generalized Sugeno integral.

We are trying to express this integral inequality for the generalized Sugeno integral and a nonlinear measure μ, when φ is a concave function. Throughout this article, we consider $\mathbb{B}(0, 1)$ as the Borel $\sigma-$algebra over $[0, 1]$.

Theorem 3.2. Let $r, s \in]0, +\infty[$. If $\varphi, \psi : [0, 1] \to W$ are two nonnegative measurable concave functions defined on $([0, 1], \mathbb{B}(0, 1))$ and μ is an arbitrary monotone measure such that both $(\int_0^1 \varphi^r \Delta\mu)^{\frac{1}{r}}$ and $(\int_0^1 \psi^s \Delta\mu)^{\frac{1}{s}}$ are finite, then

1. If $\varphi(0) < \varphi(1)$ and $\psi(0) < \psi(1)$, then

$$\left(\int_0^1 \varphi^r \Delta\mu\right)^{\frac{1}{r}} \left(\int_0^1 \psi^s \Delta\mu\right)^{\frac{1}{s}} \Delta \left(\mu\left[\frac{(\int_0^1 \varphi^r \Delta\mu)^{\frac{1}{r}} - \varphi(0)}{\varphi(1) - \varphi(0)}, 1\right] \wedge \mu\left[\frac{(\int_0^1 \psi^s \Delta\mu)^{\frac{1}{s}} - \psi(0)}{\psi(1) - \psi(0)}, 1\right]\right)$$
$$\leq \int_0^1 \varphi\psi \Delta\mu.$$

2. If $\varphi(0) = \varphi(1)$, $\psi(0) = \psi(1)$, $\beta\Delta 0 = 0$ for all $\beta \in [0, +\infty[$, $\int_0^1 \varphi^r \Delta\mu \leq \mu([0, 1])$, $\int_0^1 \psi^s \Delta\mu \leq \mu([0, 1])$ and $(\mu([0, 1]))^{\frac{1}{r}+\frac{1}{s}} \leq \mu([0, 1])$, then

$$\left(\int_0^1 \varphi^r \Delta\mu\right)^{\frac{1}{r}} \left(\int_0^1 \psi^s \Delta\mu\right)^{\frac{1}{s}} \Delta\varphi(0)\psi(0) \leq \int_0^1 \varphi\psi \Delta\mu.$$

3. If $\varphi(0) > \varphi(1)$ and $\psi(0) > \psi(1)$, then

$$\left(\int_0^1 \varphi^r \Delta\mu\right)^{\frac{1}{r}} \left(\int_0^1 \psi^s \Delta\mu\right)^{\frac{1}{s}} \Delta \left(\mu\left[0, \frac{(\int_0^1 \varphi^r \Delta\mu)^{\frac{1}{r}} - \varphi(0)}{\varphi(1) - \varphi(0)}\right] \wedge \mu\left[0, \frac{(\int_0^1 \psi^s \Delta\mu)^{\frac{1}{s}} - \psi(0)}{\psi(1) - \psi(0)}\right]\right)$$
$$\leq \int_0^1 \varphi\psi \Delta\mu.$$

Proof. Let $r, s \in]0, +\infty[$, $(\int_0^1 \varphi^r \Delta\mu)^{\frac{1}{r}} = \gamma_1$ and $(\int_0^1 \psi^s \Delta\mu)^{\frac{1}{s}} = \gamma_2$. Since $\varphi, \psi : [0,1] \to W$ are two concave functions, for $u \in [0,1]$ we have

$$\varphi(u) = \varphi((1-u).0 + u.1) \geq (1-u).\varphi(0) + u.\varphi(1) = l_1(u),$$

$$\psi(u) = \psi((1-u).0 + u.1) \geq (1-u).\psi(0) + u.\psi(1) = l_2(u).$$

(1) If $\varphi(0) < \varphi(1)$ and $\psi(0) < \psi(1)$, then l_1 and l_2 are comonotone. Therefore by Lemma 2.6 (1) and Lemma 2.8, we have

$$\begin{aligned}
\int_0^1 \varphi(u)\psi(u)\Delta\mu &\geq \int_0^1 l_1(u)l_2(u)\Delta\mu = \sup_{\beta\in W} (\beta\Delta\mu([0,1] \cap \{l_1(u)l_2(u) \geq \beta\})) \\
&\geq \gamma_1\gamma_2\Delta\mu([0,1] \cap \{l_1(u)l_2(u) \geq \gamma_1\gamma_2\}) \\
&\geq \gamma_1\gamma_2\Delta\mu([0,1] \cap \{l_1(u) \geq \gamma_1\} \cap \{l_2(u) \geq \gamma_2\}) \\
&= \gamma_1\gamma_2\Delta(\mu([0,1] \cap \{l_1(u) \geq \gamma_1\}) \wedge \mu([0,1] \cap \{l_2(u) \geq \gamma_2\})) \\
&= \gamma_1\gamma_2\Delta(\mu([0,1] \cap \{(1-u).\varphi(0) + u.\varphi(1) \geq \gamma_1\}) \\
&\wedge \mu([0,1] \cap \{(1-u).\psi(0) + u.\psi(1) \geq \gamma_2\})) \\
&= \gamma_1\gamma_2\Delta\left(\mu\left([0,1] \cap \left\{u \geq \frac{\gamma_1 - \varphi(0)}{\varphi(1) - \varphi(0)}\right\}\right)\right. \\
&\wedge \left.\mu\left([0,1] \cap \left\{u \geq \frac{\gamma_2 - \psi(0)}{\psi(1) - \psi(0)}\right\}\right)\right) \\
&= \gamma_1\gamma_2\Delta\left(\mu\left[\frac{\gamma_1 - \varphi(0)}{\varphi(1) - \varphi(0)}, 1\right] \wedge \mu\left[\frac{\gamma_2 - \psi(0)}{\psi(1) - \psi(0)}, 1\right]\right).
\end{aligned}$$

(2) If $\varphi(0) = \varphi(1)$, $\psi(0) = \psi(1)$, $\beta\Delta 0 = 0$ for all $\beta \in [0, +\infty[$ and $\int_0^1 \varphi^r \Delta\mu \leq \mu([0,1])$, $\int_0^1 \psi^s \Delta\mu \leq \mu([0,1])$ and $(\mu([0,1]))^{\frac{1}{r}+\frac{1}{s}} \leq \mu([0,1])$, then

$$\varphi(u) = \varphi((1-u).0 + u.1) \geq \varphi(0) = l_1(u),$$

$$\psi(u) = \psi((1-u).0 + u.1) \geq \psi(0) = l_2(u).$$

Thus, by Lemma 2.6 and since Δ is non-decreasing, we have

$$\begin{aligned}
\int_0^1 \varphi(u)\psi(u)\Delta\mu &\geq \int_0^1 l_1(u)l_2(u)\Delta\mu = \int_0^1 \varphi(0)\psi(0)\Delta\mu = \varphi(0)\psi(0)\Delta\mu([0,1]) \\
&\geq \varphi(0)\psi(0)\Delta(\mu([0,1]))^{\frac{1}{r}+\frac{1}{s}} \\
&= \varphi(0)\psi(0)\Delta((\mu([0,1]))^{\frac{1}{r}}.(\mu([0,1]))^{\frac{1}{s}}) \\
&\geq \varphi(0)\psi(0)\Delta\gamma_1\gamma_2.
\end{aligned}$$

(3) If $\varphi(0) > \varphi(1)$ and $\psi(0) > \psi(1)$, then l_1 and l_2 are comonotone. Therefore by Lemma 2.6 (1) and Lemma 2.8, we have

$$\begin{aligned}
\int_0^1 \varphi(u)\psi(u)\Delta\mu &\geq \int_0^1 l_1(u)l_2(u)\Delta\mu = \sup_{\beta\in W}(\beta\Delta\mu([0,1]\cap\{l_1(u)l_2(u)\geq\beta\}))\\
&\geq \gamma_1\gamma_2\Delta\mu([0,1]\cap\{l_1(u)l_2(u)\geq\gamma_1\gamma_2\})\\
&\geq \gamma_1\gamma_2\Delta\mu([0,1]\cap\{l_1(u)\geq\gamma_1\}\cap\{l_2(u)\geq\gamma_2\})\\
&= \gamma_1\gamma_2\Delta(\mu([0,1]\cap\{l_1(u)\geq\gamma_1\})\wedge\mu([0,1]\cap\{l_2(u)\geq\gamma_2\}))\\
&= \gamma_1\gamma_2\Delta(\mu([0,1]\cap\{(1-u).\varphi(0)+u.\varphi(1)\geq\gamma_1\})\\
&\wedge\ \mu([0,1]\cap\{(1-u).\psi(0)+u.\psi(1)\geq\gamma_2\}))\\
&= \gamma_1\gamma_2\Delta\left(\mu\left([0,1]\cap\left\{u\leq\frac{\gamma_1-\varphi(0)}{\varphi(1)-\varphi(0)}\right\}\right)\right.\\
&\wedge\ \left.\mu\left([0,1]\cap\left\{u\leq\frac{\gamma_2-\psi(0)}{\psi(1)-\psi(0)}\right\}\right)\right)\\
&= \gamma_1\gamma_2\Delta\left(\mu\left[0,\frac{\gamma_1-\varphi(0)}{\varphi(1)-\varphi(0)}\right]\wedge\mu\left[0,\frac{\gamma_2-\psi(0)}{\psi(1)-\psi(0)}\right]\right),
\end{aligned}$$

and this is the desired result. □

Remark 2. The above theorem is valid if we replace t-seminorm of T with Δ.

Corollary 3.3. Let $r, s \in]0,+\infty[$. If $\varphi,\psi : [0,1] \to W$ are two nonnegative measurable concave functions and μ is the Lebesgue measure on $\mathbb{R}$ such that both $(\int_0^1 \varphi^r\Delta\mu)^{\frac{1}{r}}$ and $(\int_0^1 \psi^s\,\Delta\mu)^{\frac{1}{s}}$ are finite, then

1. If $\varphi(0) < \varphi(1)$ and $\psi(0) < \psi(1)$, then

   $$\left(\int_0^1 \varphi^r\Delta\mu\right)^{\frac{1}{r}}\left(\int_0^1 \psi^s\Delta\mu\right)^{\frac{1}{s}}\Delta\left(\left(1-\frac{(\int_0^1\varphi^r\Delta\mu)^{\frac{1}{r}}-\varphi(0)}{\varphi(1)-\varphi(0)}\right)\wedge\left(1-\frac{(\int_0^1\psi^s\Delta\mu)^{\frac{1}{s}}-\psi(0)}{\psi(1)-\psi(0)}\right)\right)$$

   $\leq \int_0^1 \varphi\psi\Delta\mu.$

2. If $\varphi(0) = \varphi(1)$, $\psi(0) = \psi(1)$, $\beta\Delta 0 = 0$ for all $\beta \in [0,+\infty[$, $\int_0^1 \varphi^r\Delta\mu \leq 1$ and $\int_0^1 \psi^s\Delta\mu \leq 1$, then

   $$\left(\int_0^1 \varphi^r\,\Delta\mu\right)^{\frac{1}{r}}\left(\int_0^1 \psi^s\,\Delta\mu\right)^{\frac{1}{s}}\Delta\varphi(0)\psi(0) \leq \int_0^1 \varphi\psi\Delta\mu.$$

3. If $\varphi(0) > \varphi(1)$ and $\psi(0) > \psi(1)$, then

   $$\left(\int_0^1 \varphi^r\Delta\mu\right)^{\frac{1}{r}}\left(\int_0^1 \psi^s\Delta\mu\right)^{\frac{1}{s}}\Delta\left(\left(\frac{(\int_0^1\varphi^r\Delta\mu)^{\frac{1}{r}}-\varphi(0)}{\varphi(1)-\varphi(0)}\right)\wedge\left(\frac{(\int_0^1\psi^s\Delta\mu)^{\frac{1}{s}}-\psi(0)}{\psi(1)-\psi(0)}\right)\right)$$

   $\leq \int_0^1 \varphi\psi\Delta\mu.$

In the following, we will present a more general case of Theorem 3.2 and we will also prove it.

Theorem 3.4. Let $r, s \in]0, +\infty[$. If $\varphi, \psi : [c, d] \to W$ are two nonnegative measurable concave functions defined on $([c, d], \mathbb{B}(c, d))$ and μ is an arbitrary monotone measure such that both $(\int_c^d \varphi^r \Delta\mu)^{\frac{1}{r}}$ and $(\int_c^d \psi^s \Delta\mu)^{\frac{1}{s}}$ are finite, then

1. If $\varphi(c) < \varphi(d)$ and $\psi(c) < \psi(d)$, then

$$\begin{aligned}
& \left(\int_c^d \varphi^r \Delta\mu\right)^{\frac{1}{r}} \left(\int_c^d \psi^s \Delta\mu\right)^{\frac{1}{s}} \\
\Delta \quad & \left(\mu\left[\frac{(\int_c^d \varphi^r \Delta\mu)^{\frac{1}{r}}(d-c) + c\varphi(d) - d\varphi(c)}{\varphi(d) - \varphi(c)}, d\right]\right. \\
\wedge \quad & \left.\left[\frac{(\int_c^d \psi^s \Delta\mu)^{\frac{1}{s}}(d-c) + c\psi(d) - d\psi(c)}{\psi(d) - \psi(c)}, d\right]\right) \\
\leq \quad & \int_c^d \varphi\psi\Delta\mu.
\end{aligned}$$

2. If $\varphi(c) = \varphi(d)$, $\psi(c) = \psi(d)$, $\beta\Delta 0 = 0$ for all $\beta \in [0, +\infty[$, $\int_c^d \varphi^r \Delta\mu \leq \mu([c, d])$ and $\int_c^d \psi^s \Delta\mu \leq \mu([c, d])$, then

$$\left(\int_c^d \varphi^r \Delta\mu\right)^{\frac{s}{r+s}} \left(\int_c^d \psi^s \Delta\mu\right)^{\frac{r}{r+s}} \Delta\varphi(c)\psi(c) \leq \int_c^d \varphi\psi\Delta\mu.$$

3. If $\varphi(c) > \varphi(d)$ and $\psi(c) > \psi(d)$, then

$$\begin{aligned}
& \left(\int_c^d \varphi^r \Delta\mu\right)^{\frac{1}{r}} \left(\int_c^d \psi^s \Delta\mu\right)^{\frac{1}{s}} \\
\Delta \quad & \left(\mu\left[c, \frac{(\int_c^d \varphi^r \Delta\mu)^{\frac{1}{r}}(d-c) + c\varphi(d) - d\varphi(c)}{\varphi(d) - \varphi(c)}\right]\right. \\
\wedge \quad & \left.\mu\left[c, \frac{(\int_c^d \psi^s \Delta\mu)^{\frac{1}{s}}(d-c) + c\psi(d) - d\psi(c)}{\psi(d) - \psi(c)}\right]\right) \\
\leq \quad & \int_c^d \varphi\psi\Delta\mu.
\end{aligned}$$

Proof. Let $r, s \in]0, +\infty[$, $(\int_c^d \varphi^r \Delta\mu)^{\frac{1}{r}} = \gamma_1$ and $(\int_c^d \psi^s \Delta\mu)^{\frac{1}{s}} = \gamma_2$. Since $\varphi, \psi : [c, d] \to W$ are two concave functions, for $u \in [c, d]$ we have

$$\varphi(u) = \varphi\left((1-\frac{u-c}{d-c}).c + (\frac{u-c}{d-c}).d\right) \geq \left(1-\frac{u-c}{d-c}\right).\varphi(c)+\left(\frac{u-c}{d-c}\right).\varphi(d) = l_1(u),$$

$$\psi(u) = \psi\left((1-\frac{u-c}{d-c}).c + (\frac{u-c}{d-c}).d\right) \geq \left(1-\frac{u-c}{d-c}\right).\psi(c)+\left(\frac{u-c}{d-c}\right).\psi(d) = l_1(u).$$

(1) If $\varphi(c) < \varphi(d)$ and $\psi(c) < \psi(d)$, then l_1 and l_2 are comonotone. Therefore by Lemma 2.6 (1) and Lemma 2.8, we have

$$\begin{aligned}
\int_c^d \varphi(u)\psi(u)\Delta\mu &\geq \int_c^d l_1(u)l_2(u)\Delta\mu = \sup_{\beta\in W}(\beta\Delta\mu([c,d]\cap\{l_1(u)l_2(u)\geq\beta\})) \\
&\geq \gamma_1\gamma_2\Delta\mu([c,d]\cap\{l_1(u)l_2(u)\geq\gamma_1\gamma_2\}) \\
&\geq \gamma_1\gamma_2\Delta\mu([c,d]\cap\{l_1(u)\geq\gamma_1\}\cap\{l_2(u)\geq\gamma_2\}) \\
&= \gamma_1\gamma_2\Delta(\mu([c,d]\cap\{l_1(u)\geq\gamma_1\})\wedge\mu([c,d]\cap\{l_2(u)\geq\gamma_2\})) \\
&= \gamma_1\gamma_2\Delta\left(\mu\left([c,d]\cap\{(1-\frac{u-c}{d-c}).\varphi(c)+(\frac{u-c}{d-c}).\varphi(d)\geq\gamma_1\}\right)\right. \\
&\wedge \left.\mu\left([c,d]\cap\{(1-\frac{u-c}{d-c}).\psi(c)+(\frac{u-c}{d-c}).\psi(d)\geq\gamma_2\}\right)\right) \\
&= \gamma_1\gamma_2\Delta\left(\mu\left([c,d]\cap\left\{u\geq\frac{\gamma_1(d-c)+c\varphi(d)-d\varphi(c)}{\varphi(d)-\varphi(c)}\right\}\right)\right. \\
&\wedge \left.\mu\left([c,d]\cap\left\{u\geq\frac{\gamma_2(d-c)+c\psi(d)-d\psi(c)}{\psi(d)-\psi(c)}\right\}\right)\right) \\
&= \gamma_1\gamma_2\Delta\left(\mu\left[\frac{\gamma_1(d-c)+c\varphi(d)-d\varphi(c)}{\varphi(d)-\varphi(c)},d\right]\right. \\
&\wedge \left.\mu\left[\frac{\gamma_2(d-c)+c\psi(d)-d\psi(c)}{\psi(d)-\psi(c)},d\right]\right).
\end{aligned}$$

(2) If $\varphi(c) = \varphi(d)$, $\psi(c) = \psi(d)$, $\beta\Delta 0 = 0$ for all $\beta\in[0,+\infty[$ and $\int_c^d \varphi^r\Delta\mu \leq \mu([c,d])$ and $\int_c^d \psi^s\Delta\mu \leq \mu([c,d])$, then

$$\varphi(u) = \varphi((1-\frac{u-c}{d-c}).c + (\frac{u-c}{d-c}).d) \geq \varphi(c) = l_1(u),$$
$$\psi(u) = \psi((1-\frac{u-c}{d-c}).c + (\frac{u-c}{d-c}).d) \geq \psi(c) = l_2(u).$$

Thus, by Lemma 2.6 (1), (2) and since Δ is non-decreasing, we have

$$\begin{aligned}
\int_c^d \varphi(u)\psi(u)\Delta\mu &\geq \int_c^d l_1(u)l_2(u)\Delta\mu = \int_c^d \varphi(c)\psi(c)\Delta\mu = \varphi(c)\psi(c)\Delta\mu([c,d]) \\
&\geq \varphi(c)\psi(c)\Delta(\gamma_1\gamma_2)^{\frac{rs}{r+s}} \\
&= \varphi(c)\psi(c)\Delta\left(\int_c^d \varphi^r\Delta\mu\right)^{\frac{s}{r+s}}\left(\int_c^d \psi^s\Delta\mu\right)^{\frac{r}{r+s}}.
\end{aligned}$$

(3) If $\varphi(c) > \varphi(d)$ and $\psi(c) > \psi(d)$, then l_1 and l_2 are comonotone. Therefore by Lemma 2.6 (1) and Lemma 2.8, we have

$$\begin{aligned}
\int_c^d \varphi(u)\psi(u)\Delta\mu \quad &\geq \quad \int_c^d l_1(u)l_2(u)\Delta\mu = \sup_{\beta\in W}(\beta\Delta\mu([c,d]\cap\{l_1(u)l_2(u)\geq\beta\})) \\
&\geq \quad \gamma_1\gamma_2\Delta\mu([c,d]\cap\{l_1(u)l_2(u)\geq\gamma_1\gamma_2\}) \\
&\geq \quad \gamma_1\gamma_2\Delta\mu([c,d]\cap\{l_1(u)\geq\gamma_1\}\cap\{l_2(u)\geq\gamma_2\}) \\
&= \quad \gamma_1\gamma_2\Delta(\mu([c,d]\cap\{l_1(u)\geq\gamma_1\})\wedge\mu([c,d]\cap\{l_2(u)\geq\gamma_2\})) \\
&= \quad \gamma_1\gamma_2\Delta\left(\mu\left([c,d]\cap\left\{(1-\frac{u-c}{d-c}).\varphi(c)+(\frac{u-c}{d-c}).\varphi(d)\geq\gamma_1\right\}\right)\right. \\
&\wedge \quad \left.\mu\left([c,d]\cap\left\{(1-\frac{u-c}{d-c}).\psi(c)+(\frac{u-c}{d-c}).\psi(d)\geq\gamma_2\right\}\right)\right) \\
&= \quad \gamma_1\gamma_2\Delta\left(\mu\left([c,d]\cap\left\{u\leq\frac{\gamma_1(d-c)+c\varphi(d)-d\varphi(c)}{\varphi(d)-\varphi(c)}\right\}\right)\right. \\
&\wedge \quad \left.\mu\left([c,d]\cap\left\{u\leq\frac{\gamma_2(d-c)+c\psi(d)-d\psi(c)}{\psi(d)-\psi(c)}\right\}\right)\right) \\
&= \quad \gamma_1\gamma_2\Delta\left(\mu\left[c,\frac{\gamma_1(d-c)+c\varphi(d)-d\varphi(c)}{\varphi(d)-\varphi(c)}\right]\right. \\
&\wedge \quad \left.\mu\left[c,\frac{\gamma_2(d-c)+c\psi(d)-d\psi(c)}{\psi(d)-\psi(c)}\right]\right),
\end{aligned}$$

and this is the desired result. □

Corollary 3.5. Let $r,s\in]0,+\infty[$. If $\varphi,\psi:[c,d]\to W$ are two nonnegative measurable concave functions and μ is the Lebesgue measure on $\mathbb{R}$ such that both $(\int_c^d\varphi^r\Delta\mu)^{\frac{1}{r}}$ and $(\int_c^d\psi^s\Delta\mu)^{\frac{1}{s}}$ are finite, then

1. If $\varphi(c)<\varphi(d)$ and $\psi(c)<\psi(d)$, then

$$\begin{aligned}
&\left(\int_c^d\varphi^r\Delta\mu\right)^{\frac{1}{r}}\left(\int_c^d\psi^s\Delta\mu\right)^{\frac{1}{s}} \\
\Delta\quad&\left(\left(d-\frac{(\int_c^d\varphi^r\Delta\mu)^{\frac{1}{r}}(d-c)+c\varphi(d)-d\varphi(c)}{\varphi(d)-\varphi(c)}\right)\right. \\
\wedge\quad&\left.\left(d-\frac{(\int_c^d\psi^s\Delta\mu)^{\frac{1}{s}}(d-c)+c\psi(d)-d\psi(c)}{\psi(d)-\psi(c)}\right)\right) \\
\leq\quad&\int_c^d\varphi\psi\Delta\mu.
\end{aligned}$$

2. If $\varphi(c)=\varphi(d)$, $\psi(c)=\psi(d)$, $\beta\Delta 0=0$ for all $\beta\in[0,+\infty[$, $\int_c^d\varphi^r\Delta\mu\leq 1$ and $\int_c^d\psi^s\Delta\mu\leq 1$, then

$$\left(\int_c^d\varphi^r\,\Delta\mu\right)^{\frac{s}{r+s}}\left(\int_c^d\psi^s\,\Delta\mu\right)^{\frac{r}{r+s}}\Delta\varphi(c)\psi(c)\leq\int_c^d\varphi\psi\Delta\mu.$$

3. If $\varphi(c)>\varphi(d)$ and $\psi(c)>\psi(d)$, then

$$\begin{aligned}&\left(\int_c^d\varphi^r\Delta\mu\right)^{\frac{1}{r}}\left(\int_c^d\psi^s\Delta\mu\right)^{\frac{1}{s}}\\ \Delta&\left(\left(\frac{(\int_c^d\varphi^r\Delta\mu)^{\frac{1}{r}}(d-c)+c\varphi(d)-d\varphi(c)}{\varphi(d)-\varphi(c)}-c\right)\right.\\ \wedge&\left.\left(\frac{(\int_c^d\psi^s\Delta\mu)^{\frac{1}{s}}(d-c)+c\psi(d)-d\psi(c)}{\psi(d)-\psi(c)}-c\right)\right)\\ \leq&\int_c^d\varphi\psi\Delta\mu.\end{aligned}$$

By putting $W=[0,+\infty[$ and $\Delta:=M$ in Corollaries 3.3 and 3.5, we get the results of Agahi et al. [7].

Remark 3. By replacing Π with Δ in Theorems 3.2 and 3.4 one can obtain many other B-G-L type inequalities. Examples are:

- If $\varphi,\psi:[0,1]\to[0,+\infty[$ are two nonnegative measurable concave functions and μ is the Lebesgue measure on $\mathbb{R}$ such that both $\left((N)\int_0^1\varphi^r dx\right)^{\frac{1}{r}}$ and $\left((N)\int_0^1\psi^s\,dx\right)^{\frac{1}{s}}$ are finite, then

  1. If $\varphi(0)<\varphi(1)$ and $\psi(0)<\psi(1)$, then

$$\begin{aligned}\left((N)\int_0^1\varphi^r dx\right)^{\frac{1}{r}}\left((N)\int_0^1\psi^s dx\right)^{\frac{1}{s}}\quad\cdot&\left(\left(1-\frac{\left((N)\int_0^1\varphi^r dx\right)^{\frac{1}{r}}-\varphi(0)}{\varphi(1)-\varphi(0)}\right)\right.\\ \wedge&\left.\left(1-\frac{\left((N)\int_0^1\psi^s dx\right)^{\frac{1}{s}}-\psi(0)}{\psi(1)-\psi(0)}\right)\right)\\ \leq&\ (N)\int_0^1\varphi\psi dx.\end{aligned}$$

2. If $\varphi(0) = \varphi(1)$, $\psi(0) = \psi(1)$, $(N)\int_0^1 \varphi^r dx \leq 1$ and $(N)\int_0^1 \psi^s dx \leq 1$, then

$$\left((N)\int_0^1 \varphi^r\, dx\right)^{\frac{1}{r}}\left((N)\int_0^1 \psi^s\, dx\right)^{\frac{1}{s}}.\varphi(0)\psi(0)(N) \leq \int_0^1 \varphi\psi dx.$$

3. If $\varphi(0) > \varphi(1)$ and $\psi(0) > \psi(1)$, then

$$\begin{aligned}\left((N)\int_0^1 \varphi^r dx\right)^{\frac{1}{r}}\left((N)\int_0^1 \psi^s dx\right)^{\frac{1}{s}} \quad & . \quad \left(\left(\frac{((N)\int_0^1 \varphi^r dx)^{\frac{1}{r}} - \varphi(0)}{\varphi(1)-\varphi(0)}\right)\right.\\ & \wedge \quad \left.\left(\frac{((N)\int_0^1 \psi^s dx)^{\frac{1}{s}} - \psi(0)}{\psi(1)-\psi(0)}\right)\right)\\ & \leq \quad (N)\int_0^1 \varphi\psi\Delta\mu.\end{aligned}$$

- If $\varphi, \psi : [c,d] \to W$ are two nonnegative measurable concave functions and μ is the Lebesgue measure on $\mathbb{R}$ such that both $\left((N)\int_c^d \varphi^r dx\right)^{\frac{1}{r}}$ and $\left((N)\int_c^d \psi^s\, dx\right)^{\frac{1}{s}}$ are finite, then

1. If $\varphi(c) < \varphi(d)$ and $\psi(c) < \psi(d)$, then

$$\begin{aligned}& \left((N)\int_c^d \varphi^r dx\right)^{\frac{1}{r}}\left((N)\int_c^d \psi^s dx\right)^{\frac{1}{s}}\\ . \quad & \left(\left(d - \frac{\left((N)\int_c^d \varphi^r dx\right)^{\frac{1}{r}}(d-c) + c\varphi(d) - d\varphi(c)}{\varphi(d)-\varphi(c)}\right)\right.\\ \wedge \quad & \left.\left(d - \frac{\left((N)\int_c^d \psi^s dx\right)^{\frac{1}{s}}(d-c) + c\psi(d) - d\psi(c)}{\psi(d)-\psi(c)}\right)\right)\\ \leq \quad & (N)\int_c^d \varphi\psi dx.\end{aligned}$$

2. If $\varphi(c) = \varphi(d)$, $\psi(c) = \psi(d)$, $(N)\int_c^d \varphi^r dx \leq 1$ and $(N)\int_c^d \psi^s dx \leq 1$, then

$$\left((N)\int_c^d \varphi^r\, dx\right)^{\frac{s}{r+s}}\left((N)\int_c^d \psi^s\, dx\right)^{\frac{r}{r+s}}.\varphi(c)\psi(c) \leq (N)\int_c^d \varphi\psi dx.$$

3. If $\varphi(c) > \varphi(d)$ and $\psi(c) > \psi(d)$, then

$$\begin{aligned}
& \left((N)\int_c^d \varphi^r dx\right)^{\frac{1}{r}} \left((N)\int_c^d \psi^s dx\right)^{\frac{1}{s}} \\
\cdot \quad & \left(\left(\frac{((N)\int_c^d \varphi^r dx)^{\frac{1}{r}}(d-c)+c\varphi(d)-d\varphi(c)}{\varphi(d)-\varphi(c)}-c\right)\right. \\
\wedge \quad & \left.\left(\frac{((N)\int_c^d \psi^s dx)^{\frac{1}{s}}(d-c)+c\psi(d)-d\psi(c)}{\psi(d)-\psi(c)}-c\right)\right) \\
\leq \quad & (N)\int_c^d \varphi\psi dx.
\end{aligned}$$

4. An inequality associated with B-G-L

We supply a generalized version of the B-G-L type inequality for generalized Sugeno integrals.

Theorem 4.1. Let $r, s \in]0, +\infty[$. If $\varphi, \psi : [c, d] \to W$ are two nonnegative measurable concave functions defined on $([c, d], \mathbb{B}(c, d))$, μ is an arbitrary monotone measure such that both $(\int_c^d \varphi^r \Delta\mu)^{\frac{1}{r}}$ and $(\int_c^d \psi^s \Delta\mu)^{\frac{1}{s}}$ are finite and if there is an operation $\star : W^2 \to W$ such that $\star$ is non-decreasing and continuous in both arguments, then

1. If $\varphi(c) < \varphi(d)$ and $\psi(c) < \psi(d)$, then

$$\begin{aligned}
& \left(\left(\int_c^d \varphi^r \Delta\mu\right)^{\frac{1}{r}} \star \left(\int_c^d \psi^s \Delta\mu\right)^{\frac{1}{s}}\right) \\
\Delta \quad & \left(\mu\left[\frac{(\int_c^d \varphi^r \Delta\mu)^{\frac{1}{r}}(d-c)+c\varphi(d)-d\varphi(c)}{\varphi(d)-\varphi(c)}, d\right]\right. \\
\wedge \quad & \left.\mu\left[\frac{(\int_c^d \psi^s \Delta\mu)^{\frac{1}{s}}(d-c)+c\psi(d)-d\psi(c)}{\psi(d)-\psi(c)}, d\right]\right) \\
\leq \quad & \int_c^d \varphi \star \psi \Delta\mu.
\end{aligned}$$

2. If $\varphi(c) = \varphi(d)$, $\psi(c) = \psi(d)$, $\beta\Delta 0 = 0$ for all $\beta \in [0, +\infty[$, $\int_c^d \varphi^r \Delta\mu \leq$

$\mu([c,d])$ and $\int_c^d \psi^s \Delta\mu \leq \mu([c,d])$, then

$$\left(\left(\int_c^d \varphi^r\,\Delta\mu\right)^{\frac{s}{r+s}} \star \left(\int_c^d \psi^s\,\Delta\mu\right)^{\frac{r}{r+s}}\right) \Delta\varphi(c)\psi(c) \leq \int_c^d \varphi \star \psi \Delta\mu.$$

3. If $\varphi(c) > \varphi(d)$ and $\psi(c) > \psi(d)$, then

$$\begin{aligned} & \left(\left(\int_c^d \varphi^r \Delta\mu\right)^{\frac{1}{r}} \star \left(\int_c^d \psi^s \Delta\mu\right)^{\frac{1}{s}}\right) \\ \Delta \quad & \left(\mu\left[c, \frac{(\int_c^d \varphi^r \Delta\mu)^{\frac{1}{r}}(d-c) + c\varphi(d) - d\varphi(c)}{\varphi(d)-\varphi(c)}\right]\right. \\ \wedge \quad & \left.\mu\left[c, \frac{(\int_c^d \psi^s \Delta\mu)^{\frac{1}{s}}(d-c) + c\psi(d) - d\psi(c)}{\psi(d)-\psi(c)}\right]\right) \\ \leq \quad & \int_c^d \varphi \star \psi \Delta\mu. \end{aligned}$$

Proof. The proof is routine and goes verbatim as in Theorem 3.4. □

By putting $W = [0, +\infty[$ and $\Delta := M$ in Theorem 4.1, we get the result of Agahi and et al. [7].

5. Conclusions

This paper presented the B-G-L type inequalities for the generalized Sugeno integral when the function under consideration is concave. For further investigation, we will investigate the B-G-L type inequalities for the generalized Sugeno integral when the function under consideration is $(\beta - m)$-concave.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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