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Topics on $(H, Poly(P))$-Hypergroups

Saeed Mirvakili, Somayeh Hosseini and Bijan Davvaz*

Abstract

In this paper, we construct a hypergroup by using a hypergroup $(H, \circ)$ and a polygroup $(P, \cdot)$, and call it $(H, Poly(P))$-hypergroup. The method of constructing hypergroups in this paper is not present in the established techniques of group theory. Moreover, we compare $(H, Poly(P))$-hypergroups with K_H-hypergroups, complete hypergroups and extensions of polygroups by polygroups.

Keywords: (H, G)-Hypergroup, $(H, Poly(P))$-Hypergroup, K_H-Hypergroup, Hypergroup, Polygroup.

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1. Introduction

The concept of a hypergroup was first introduced by Marty [1]. He introduced the (semi)hypergroup as a natural generalization of a (semi)group in where the combination of two elements becomes a non-empty set. Since then, many researches have been done in hypergroups and other hyperstructures and many valuable books have been written in this regard [2–8].

Corsini [9] introduced quasicanonical hypergroups and later, Bonansinga [10] and Massouros [11] studied them. Also, Comer [12] introduced polygroups independently. He explained how to make a polygroup by two polygroups. Recently, Mosayebi, Hamidi and Ameri have studied the auto-Engel polygroup [13].

*Corresponding author (E-mail: saeed_mirvakili@pnu.ac.ir)
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De Salvo introduced K_H-hypergroups [14, 15]. He also introduced (H, G)-hypergroups [16]. In addition, the construction of other hyperstructures by the method in [16] has been done by a number of researchers [17–20]. Extensions of polygroups by polygroups was also constructed by Stephen D. Comer [21]. The methods presented above for constructing supergroups are not applicable in group theory. It is an idea for us to look for other similar methods.

In this paper, we introduce an extension of a hypergroup. We construct a hypergroup by using a hypergroup $(H, \circ)$ and a polygroup $(P, \cdot)$. It is called an $(H, Poly(P))$-hypergroup. Through careful analysis, we have determined the requisite conditions for an $(H, Poly(P))$-hypergroup to exhibit a polygroup structure. Also, we explain some new results. Finally, we compare $(H, Poly(P))$-hypergroups with K_H-hypergroups, complete hypergroups and extension of polygroups by polygroups.

We mention the required definitions in hypergroups as follows (see [2, 3]).

Suppose that the set H contains at least one element and $P^*(H)$ is the power set of H except the empty set. A hyperoperation or hypercomposition on H is a map (binary hyperoperation) $\circ : H \times H \to P^*(H)$. Then, $(H, \circ)$ is called an algebraic hypercompositional structure or hypergroupoid. For two subsets A, B of H we define $A \circ B$ as $A \circ B = \bigcup\{u \circ u \mid u \in A, v \in B\}$.

(A)An (scalar) identity of the algebraic hypercompositional structure $(H, \circ)$ is an element e belong H such that $u = u \circ e = e \circ u)u \in u \circ e \cap e \circ u$, for every element u belong H.

If commutativity law holds, i.e. for all $u, v \in H$, we have $u \circ v = v \circ u$ then the algebraic hypercompositional structure $(H, \circ)$ is named a commutative algebraic hypercompositional structure.

If associativity law holds, i.e. for all $u, v, w \in H$, we have $u \circ (v \circ w) = (u \circ v) \circ w$ then the algebraic hypercompositional structure $(H, \circ)$ is named a semihypergroup.

If reproductivity law holds, i.e. for all $u \in H$, we have $u \circ H = H \circ u = H$ then the semihypergroup $(H, \circ)$ is named a hypergroup.

A reversible hypergroup is a hypergroup $(H, \circ)$ with identity element, and a unary operation $' : H \to H$ such that for all $u, v, w \in H$, $u \in v \circ w$ implies that $w \in v' \circ u$ and $v \in u \circ w'$, (In fact v' is an inverse of v, i. e. $e \in v' \circ v \cap v \circ v'$).

A semihypergroup $(P, \cdot)$ with scalar identity e is called a polygroup if for all u, v, w in P, $u \in v \cdot w$ implies $v \in u \cdot w^{-1}$ and $w \in v^{-1} \cdot u$, where $^{-1}$ is a unary operation on P. (In fact the reversible laws hold). A canonical hypergroup is a polygroup with the commutative law.

The (H, G)-hypergroup was defined by De Salvo as follows:

Definition 1.1. ([16]). Let $(G, \cdot)$ and $(H, \circ)$ be a group and a hypergroup, respectively. Consider the non-empty set K and let $\{A_g\}_{g \in G}$ be the family of non-empty subsets of K and a partition of K, where $A_e = H($ e is an identity element of $G)$. Now we can construct a new hyperoperation $*$ on K by:

$$x * y = \begin{cases} x \circ y, & \text{if } (x, y) \in H^2, \\ A_{a \cdot b}, & \text{if } (x, y) \in A_a \times A_b \neq H^2. \end{cases}$$

Then $(K, *)$ is a new hypergroup and it is called an (H, G)-hypergroup.

In Table 1, we can see the Cayley table of an (H, G)-hypergroup. When $K = \bigcup_{g_i \in G} A_i$, $A_i = \{x_{i1}, \ldots, x_{in_i}\}$, $i = 1, 2, \ldots, m$.

Table 1: The structure of (H, G)-hypergroup.

$*$	x_{11}	$\cdots$	x_{1n_1}	x_{21}	$\cdots$	x_{2n_2}	$\cdots$	x_{m1}	$\cdots$	x_{mn_m}
x_{11}	$x_{11} \circ x_{11}$	$\cdots$	$x_{11} \circ x_{1n_1}$	A_2	$\cdots$	A_2	$\cdots$	A_m	$\cdots$	A_m
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
x_{1n_1}	$x_{11} \circ x_{11}$	$\cdots$	$x_{11} \circ x_{1n_1}$	A_2	$\cdots$	A_2	$\cdots$	A_m	$\cdots$	A_m
x_{21}	A_2	$\cdots$	A_2	$A_{2\cdot 2}$	$\cdots$	$A_{2\cdot 2}$	$\cdots$	$A_{2\cdot m}$	$\cdots$	$A_{2\cdot m}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
x_{2n_2}	A_2	$\cdots$	A_2	$A_{2\cdot 2}$	$\cdots$	$A_{2\cdot 2}$	$\cdots$	$A_{2\cdot m}$	$\cdots$	$A_{2\cdot m}$
x_{m1}	A_m	$\cdots$	A_m	$A_{m\cdot 2}$	$\cdots$	$A_{m\cdot 2}$	$\cdots$	$A_{m\cdot m}$	$\cdots$	$A_{m\cdot m}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
x_{mn_m}	A_m	$\cdots$	A_m	$A_{m\cdot 2}$	$\cdots$	$A_{m\cdot 2}$	$\cdots$	$A_{m\cdot m}$	$\cdots$	$A_{m\cdot m}$

Example 1.2. Let $(G = \{e, a, b\}, \cdot) \cong (\mathbb{Z}_3, +)$ and $(H = \{0, 1\}, \circ)$ with $0 \circ 0 = 0, 0 \circ 1 = 1 \circ 0 = 1$ and $1 \circ 1 = H$. Set $K = \{0, 1, 2, 3, 4, 5\}$ with $A_e = H = \{0, 1\}, A_a = \{2\}$ and $A_b = \{3, 4, 5\}$. Then $(K, *)$ is an (H, G)-hypergroup. In Table 2, we can see the Cayley table of the (H, G)-hypergroup $(K, *)$.

Table 2: (H, G)-hypergroup of order 5.

$*$	0	1	2	3	4	5
0	{0}	{1 }	{2	{ 3,4,5}	{3,4,5 }	{3,4,5}
1	{1}	{0,1}	{2}	{3,4,5 }	{3,4,5}	{3,4,5}
2	{2}	{2 }	{3,4,5}	{0,1}	{0,1}	{0,1}
3	{3,4,5}	{3,4,5}	{0,1}	{2 }	{2}	{2}
4	{3,4,5}	{3,4,5}	{0,1}	{2 }	{2}	{2}
5	{3,4,5}	{3,4,5}	{0,1}	{2 }	{2}	{2}

2. $(H, Poly(P))$-hypergroup

In this section, we provide the structure of an $(H, Poly(P))$-hypergroup. Then we present the structure and properties of this new kind of hyperstructures.

Definition 2.1. Let $(P, \cdot)$ and $(H, \circ)$ be a polygroup and a hypergroup, respectively. Consider the non-empty set K and let $\{A_p\}_{p \in P}$ be the family of non-empty subsets of K and a partition of K, where $A_e = H$(e is an identity element of polygroup P).

Now we can construct a new hyperoperation $\star$ on K by:

$$x \star y = \begin{cases} x \circ y, & \text{if } (x,y) \in H^2, \\ \bigcup_{p \in a \cdot b} A_p, & \text{if } (x,y) \in A_a \times A_b \neq H^2. \end{cases}$$

The new hyperstructure $(K, \star)$ is named $(H, Poly(P))$-hypergroupoid.

Theorem 2.2. *The hyperstructure $(K, \star)$ is a quasihypergroup.*

Proof. We have $K \star x \subseteq K, x \star K \subseteq K$, for every $x \in K$. We show that for every $y \in K$, $y \subseteq x \star K$. We have four cases:

(1) If $x \in H$ and $y \in H$ then $y \in x \circ H \subseteq x \star K$.

(2) If $x \in H = A_e$ and $y \in A_b$, where $b \neq e$, then $y \in x \star y \subseteq x \star K$.

(3) If $x \in A_a$, where $a \neq e$, and $y \in H = A_e$ then $y \in x \star y \subseteq x \star K$.

(4) If $x \in A_a$, and $y \in A_b$, where $a \neq e \neq b$, then there is an element $c \in P$ such that $b \in a \cdot c$. So $y \in A_b \subseteq \bigcup_{p \in a \cdot c} A_p \subseteq x \star K$.

Therefore, we conclude that $y \in x \star K$. The similar arguments imply that for every $y \in K$, $y \subseteq K \star x$. This yields that $K = K \star x$, $K \star x = K$ and $(K, \star)$ is a quasihypergroup. □

Theorem 2.3. *The hyperstructure $(K, \star)$ is a semihypergroup.*

Proof. We show that $\star$ is associative. We have four cases:

(1) If $x, y, z \in H$ then $(x \star y) \star z = (x \circ y) \circ z = x \circ (y \circ z) = x \star (y \star z)$.

(2) If $x, y \in H$ and $z \in A_c$, where $c \neq e$, then $x \star y = x \circ y \in H = A_e$ and $(x \star y) \star z = (x \circ y) \star z = \bigcup_{p \in e \cdot c} A_p = A_c$. On the other hand $x \star (y \star z) = x \star \bigcup_{p \in e \cdot c} A_p = x \star A_c = A_c$ and therefore $x \star (y \star z) = (x \star y) \star z$. Also, if $x, z \in H$ and $y \in A_b$, where $b \neq e$ or If $y, z \in H$ and $x \in A_a$, where $a \neq e$ the proofs is similar.

(3) If $x \in A_a$, $y \in A_b$ and $z \in H$, where $a \neq e \neq b$, then $x \star y = \bigcup_{p \in a \cdot b} A_p$ and

$$(x \star y) \star z = \bigcup_{p \in a \cdot b} A_p \star z = \bigcup_{p \in a \cdot b} A_p.$$

On the other hand, we have

$$x \star (y \star z) = x \star \bigcup_{p \in b \cdot e} A_p = x \star A_b = \bigcup_{p \in a \cdot b} A_p.$$

Consequently, we get $x \star (y \star z) = (x \star y) \star z$.

(4) If $x \in A_a$, $y \in A_b$ and $z \in A_c$, where identity element $e \notin \{a, b, c\}$, then

$$(x \star y) \star z = \bigcup_{q \in a \cdot b} A_q \star z = \bigcup_{q \in a \cdot b} \bigcup_{p \in q \cdot c} A_p = \bigcup_{p \in (a \cdot b) \cdot c} A_p.$$

Also,

$$x \star (y \star z) = x \star \bigcup_{q \in b \cdot c} A_q = \bigcup_{p \in a \cdot q} \bigcup_{q \in b \cdot c} A_p = \bigcup_{p \in a \cdot (b \cdot c)} A_p.$$

Since $(a \cdot b) \cdot c = (a \cdot b) \cdot c$, it follows that $x \star (y \star z) = (x \star y) \star z$.

Therefore, the hyperstructure $(K, \star)$ is a semihypergroup. □

By Theorems 2.2 and 2.3, we immediately obtain the next result.

Corollary 2.4. *The hyperstructure $(K, \star)$ is a hypergroup.*

Definition 2.5. The hypergroup $(K, \star)$ is called an $(H, Poly(P))$-hypergroup.

Example 2.6. Let $P = \{e, a, b\}$ and $(P, \cdot)$ denote the polygroup with the corresponding Cayley table outlined in Table 3 and $(H = \{0, 1\}, \circ)$ with $0 \circ 0 = 0, 0 \circ 1 =$

Table 3: Polygroup of order 3.

$\cdot$	e	a	b
e	$\{e\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{e, a\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{e, a, b\}$

$1 \circ 0 = 1$ and $1 \circ 1 = H$. Set $K = \{0, 1, 2, 3, 4, 5\}$ with $A_e = H = \{0, 1\}, A_a = \{2\}$ and $A_b = \{3, 4, 5\}$. Then $(K, \star)$ is an $(H, Poly(P))$-hypergroup of order 5. In Table 4, we can see the Cayley table of the $(H, Poly(P))$-hypergroup $(K, \star)$.

Since every group is a polygroup, then

Corollary 2.7. *If $(P, \cdot)$ is a group, then $(K, \star) = (K, *)$, where $*$ is defined in Definition 1.1.*

Proof. For every $x, y \in K$, we have

$$\begin{aligned} x \star y &= \begin{cases} x \circ y, & \text{if } (x, y) \in H^2, \\ \bigcup_{p \in a \cdot b} A_p, & \text{if } (x, y) \in A_a \times A_b \neq H^2 \end{cases} \\ &= \begin{cases} x \circ y, & \text{if } (x, y) \in H^2, \\ A_{a \cdot b}, & \text{if } (x, y) \in A_a \times A_b \neq H^2 \end{cases} \\ &= x * y. \end{aligned}$$

This implies the result. □

Table 4: $(H, Poly(P))$-hypergroup of order 5.

$\star$	0	1	2	3	4	5
0	{0}	{1}	{2}	{3,4,5}	{2,4,5}	{3,4,5}
1	{1}	{0,1}	{2}	{3,4,5}	{2,4,5}	{3,4,5}
2	{2 }	{2}	{0,1,2 }	{3,4,5}	{2,4,5}	{3,4,5}
3	{3,4,5}	{3,4,5}	{3,4,5 }	{0,1,2,3,4,5 }	{0,1,2,3,4,5}	{0,1,2,3,4,5}
4	{3,4,5}	{3,4,5}	{3,4,5 }	{0,1,2,3,4,5 }	{0,1,2,3,4,5}	{0,1,2,3,4,5}
5	{3,4,5}	{3,4,5}	{3,4,5 }	{0,1,2,3,4,5 }	{0,1,2,3,4,5}	{0,1,2,3,4,5}

Theorem 2.8. *The $(H, Poly(P))$-hypergroup $(K, \star)$ is a commutative hypergroup if and only if $(H, \circ)$ and $(P, \cdot)$ are commutative.*

Proof. If $(H, \circ)$ and $(P, \cdot)$ are commutative, then

$$\begin{aligned} x \star y &= \begin{cases} x \circ y, & \text{if } (x, y) \in H^2, \\ \bigcup_{p \in a \cdot b} A_p, & \text{if } (x, y) \in A_a \times A_b \neq H^2 \end{cases} \\ &= \begin{cases} y \circ x, & if\ (x, y) \in H^2, \\ \bigcup_{p \in b \cdot a} A_p, & if\ (x, y) \in A_a \times A_b \neq H^2 \end{cases} \\ &= y \star x. \end{aligned}$$

Conversely, since $(K, \star)$ is commutative, for every $x, y \in H$,

$$x \circ y = x \star y = y \star x = y \circ x.$$

Hence, $(H, \circ)$ is commutative. Moreover, if $(P, \cdot)$ is not commutative, then there are elements $a, b \in P$ such that $a \cdot b \neq b \cdot a$ and $a \cdot b - b \cdot a \neq \emptyset$. Suppose that $p \in a \cdot b - b \cdot a$. Then $A_p \subseteq x \star y$ and $A_p \nsubseteq y \star x$ and this contradicts the commutativity of $\star$. This means that $(P, \cdot)$ is commutative. □

Theorem 2.9. *The $(H, Poly(P))$-hypergroup $(K, \star)$ has an identity element if and only if $(H, \circ)$ has an identity element.*

Proof. Let $\varepsilon \in H$ be an identity element of the hypergroup $(H, \circ)$. Then, for every $x \in K$, we have one of the following two situations:

(1) If $x \in H$, then $x \in x \circ \varepsilon \cap \varepsilon \circ x = x \star \varepsilon \cap \varepsilon \star x$.

(2) If $x \in A_a$, where $a \neq e$, then

$$x \in A_a \subseteq (\bigcup_{p \in a \cdot e} A_p) \cap (\bigcup_{p \in a \cdot e} A_p) = x \star \varepsilon \cap \varepsilon \star x.$$

Therefore, we deduce that $\varepsilon \in K$ is an identity elements of $(K, \star)$.

Now, let $\varepsilon \in K$ be an identity elements of $(K, \star)$. If $\varepsilon \in H$, then the proof is obvious. If $\varepsilon \in A_b$, where $b \neq e$, then for every $x \in H$, $x \in x \star \varepsilon$ implies that $e \in e \cdot b$ and so $b \in e^{-1} \cdot e = e$. This is a contradiction and we deduce that $\varepsilon \in H$. □

Theorem 2.10. *The* $(H, Poly(P))$*-hypergroup* $(K, \star)$ *is reversible if and only if* $(H, \circ)$ *is reversible.*

Proof. It is straightforward. □

Theorem 2.11. *The* $(H, Poly(P))$*-hypergroup* $(K, \star)$ *has a scalar identity element if and only if* $(H, \circ)$ *has a scalar identity element and for every* $a \in P - \{e\}$, $\mid A_a \mid = 1$.

Proof. If $\varepsilon \in K$ is a scalar identity of $(K, \star)$, then $x = x \star \varepsilon = \varepsilon \star x$. If $x \in H$, then $x \star \varepsilon = x \in H$, hence $\varepsilon \in H$. So, $\varepsilon \in K$ is a scalar identity of $(H, \circ)$. For every $x \in A_a$, where $a \neq e$, we have $x = x \star \varepsilon = \bigcup_{p \in a \cdot e} A_p = A_a$. This yields that $A_a = \{x\}$.

Conversely, if ε is a scalar identity of $(H, \circ)$ and for every $a \in P - \{e\}$, $\mid A_a \mid = 1$, then $A_a = \{x\}$. Therefore

$$x \circ \varepsilon = x \circ \varepsilon = \begin{cases} x, & \text{if } x \in H, \\ A_a, & \text{if } x \in A_a \neq H \end{cases} = x.$$

This completes the proof. □

Now, we present a new way to construct a polygroup from other polygroups.

Theorem 2.12. *The* $(H, Poly(P))$*-hypergroup* $(K, \star)$ *is a polygroup if and only if* $(H, \circ)$ *is a polygroup and for every* $a \in P - \{e\}$, $\mid A_a \mid = 1$.

Proof. If $(K, \star, ^{-1}, \varepsilon)$ is a polygroup, then by Theorem 2.11, we obtain $\varepsilon \in H$. Moreover, for every $a \in P - \{e\}$, $\mid A_a \mid = 1$. For arbitrary elements $x, y, z \in H$, $x \in y \star z$ gives the result $y \in x \star z^{-1}$ and $z \in y^{-1} \star x$. Since the restriction of the hyperoperation $\star$ on H is equal to hyperoperation $\circ$, it follows that $x \in y \circ z$, and so $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$. (Notice that by Theorem 2.10, for all $x \in H$, $x^{-1} \in H$.) Hence, $(H, \circ)$ is a polygroup.

Conversely, let $(H, \circ)$ be a polygroup and for every $a \in P - \{e\}$, $\mid A_a \mid = 1$. Let the unitary operation $^{-I}$ on K be as follows:

$$x^{-I} = \begin{cases} x^{-1}, & \text{if } x \in H, \\ A_{a^{-1}}, & \text{if } x \in A_a \neq H. \end{cases}$$

Moreover, $x \in A_a$ if and only if $x^{-I} \in A_{a^{-1}}$. Let $x \in y \star z$, where $x, y, z \in K$. We have one of the following cases:

(1) If $y, z \in H$, then $x \in H$ and $(H, \circ)$ is a polygroup. So, we obtain $x \in y \star z = y \circ z$, which implies that $y \in x \circ z^{-1} = x \star z^{-I}$ and $z \in y^{-1} \circ x = y^{-I} \star x$.

(2) If $y \in A_b$ and $z \in A_c$ and $A_b \times A_c \neq H^2$, then $x \in y \star z = \bigcup_{a \in b \cdot c} A_a$. By reversibility of polygroup $(P, \cdot)$, $a \in b \cdot c$ yields that $b \in a \cdot c^{-1}$ and $c \in b^{-1} \cdot a$. Therefore $y \in x \star z^{-I}$ and $z \in y^{-I} \star x$.

This shows that $(K, \star, ^{-I}, \varepsilon)$ is a polygroup. □

Definition 2.13. The polygroup $(K, \star)$ is called an $(H, Poly(P))$-polygroup.

Corollary 2.14. *Let $(H, \circ, ^{-1}, \varepsilon)$ and $(P, \cdot, ^{-1}, e)$ be two polygroups, $A_e = H$ and for every $a \in P - \{e\}$, $A_a = \{a\}$. Then $(H \cup P, \star, ^{-I}, \varepsilon)$ is a polygroup.*

Theorem 2.15. *The $(H, Poly(P))$-polygroup $(K, \star)$ is a canonical hypergroup if and only if $(H, \circ)$ and $(P, \cdot)$ are canonical hypergroups and for every $a \in P - \{e\}$, $\mid A_a \mid= 1$.*

Proof. The result follows from Theorems 2.8 and 2.12. □

Definition 2.16. ([22]). A transposition hypergroup is a hypergroup $(H, \circ)$ such that for arbitrary elements $a, b, c, d \in H$,

$$b\diagdown_{\circ}a \approx c\diagup^{\circ}d \;\Rightarrow\; a \circ d \approx b \circ c,$$

where $a\diagup^{\circ}b = \{x|a \in x \circ b\}$ and $b\diagdown_{\circ}a = \{x|a \in b \circ x\}$. Every commutative transposition hypergroup is called a join space.

Theorem 2.17. *$(H, Poly(P))$-hypergroup $(K, \star)$ is a transposition hypergroup if and only if $(H, \circ)$ is a transposition hypergroup.*

Proof. Let $(H, \circ)$ be a transposition hypergroup and $y\diagdown_{\star}x \approx z\diagup^{\star}u$, where $x \in A_a, y \in A_b, z \in A_c, u \in A_d$. If $a = b = e$ then $c = d = e$ or $z\diagup^{\star}u \approx H$. If $c = d = e$, then $x, y, z, u \in H$ and $(H, \circ)$ is a transposition hypergroup. Hence, we have $x \circ z \approx z \circ u$ and therefore $x \star z \approx y \star u$. If $z\diagup^{\star}u \approx H$, then there is $h \in H$, such that $z \in h \star u$. Then $A_c = A_d$ and so $c = d$. Therefore, we conclude that $A_c \subseteq (y\diagdown_{\star}x) \cap (z\diagup^{\star}u)$.

If $c = d = e$ and $y\diagdown_{\star}x \approx H$, by the similar way we obtain $A_b \subseteq (y\diagdown_{\star}x) \cap (z\diagup^{\star}u)$.

Else, $y\diagdown_{\star}x \approx z\diagup^{\star}u$ implies that $t \in \{v \in K|x \in y \star v\} \cap \{w \in K|z \in w \star u\}$. If $t \in A_f$ then $a \in b \cdot f$ and $c \in f \cdot d$. This yields that $f \in c \cdot d^{-1} \cap b^{-1} \cdot a$. Since $(P, \cdot)$ is a polygroup, it follows that $(P, \cdot)$ is a transposition hypergroup and $c\diagup^{\cdot}d = c \cdot d^{-1}$ and $b\diagdown_{\cdot}a = b^{-1} \cdot a$. Now, $c\diagup^{\cdot}d \approx b\diagdown_{\cdot}a$ implies $a \cdot d \approx b \cdot c$.

But $a \cdot d \cap b \cdot c$ if and only if $c\diagup^{\cdot}d \approx b\diagdown_{\cdot}a$. This completes the proof of this part.

Conversely, assume that $(K, \star)$ is a transposition hypergroup and $y\diagdown_{\circ}x \approx z\diagup^{\circ}u$. Then, we obtain $y\diagdown_{\star}x \approx z\diagup^{\star}u$. So, $x \star z \approx y \star u$, which implies that $x \circ z \approx y \circ u$. □

Corollary 2.18. *$(H, Poly(P))$-hypergroup $(K, \star)$ is a join space if and only if $(H, \circ)$ is a join space and $(P, \cdot)$ is a canonical hypergroup.*

Proof. The result follows from Theorems 2.8 and 2.17. □

Two relations β and γ in hyperstructures, especially in hypergroups, play the role of congrunce relations in group and semigroup theory. Many mathematicians explored and introduced this two relation, for instant see Koskas [23, 24], Corsini [2], Freni [25–27], Vougiouklis [28] and others.

Consider ρ be a strongly regular relation on a (semi)hypergroup $(H, \circ)$. Then the quotient structure $(H/\rho, \circ_\rho)$ is a (semi)group where $H/\rho = \{\rho(u) | u \in H\}$ and $\circ_\rho$ is an operation defined on H/ρ by

$$\rho(u) \circ_\rho \rho(v) = \rho(w), \quad \text{for all } w \in u \circ v.$$

Now, we recall definitions of strongly regular relations γ and β on a (semi)hypergroup [23, 25]. Koskas [23] introduced the relation β on a hypergroup by the following formula:

$$u\beta v \Leftrightarrow \exists l \in \mathbb{N}, w_1, \dots, w_n \in H \textit{ such that } u, v \in \prod_{k=1}^{l} w_k.$$

Moreover, in [25], Freni defined the relation γ as follows:

$$u\gamma v \Leftrightarrow \exists l \in \mathbb{N}, \exists w_1, \dots, w_n \in H \textit{ and } \exists \tau \in \mathbb{S}_n : u \in \prod_{k=1}^{l} w_k \textit{ and } v \in \prod_{k=1}^{l} w_{\tau(k)},$$

where $\mathbb{S}_n$ is the permutaion group on $\{1, 2, \dots, n\}$. It has been proven that β and γ are the smallest equivalence relations on H such that H/β and H/γ^* are a group and an Abelian group. The group H/β and the Abelian group H/γ^* are named the *fundamental group* and *commutative fundamental group*, respectively.

Theorem 2.19. *Let $(K, \star)$ be an $(H, Poly(P))$-hypergroup and $K = \bigcup_{p\in P} A_p$. For all $n \geq 1$ and $x_1, x_2, \dots, x_n \in K$ we have one of the following two cases:*

(1) There is a $Q \subseteq P$ such that $\prod_{k=1}^{n} x_k = \bigcup_{p\in Q} A_p$;

(2) There is $B \in P^*(H)$ such that $\prod_{k=1}^{n} x_k = B$;

Proof. Suppose that $x_k \in A_{a_k}$, where $1 \leq k \leq n$. If for every $1 \leq k \leq n$, $a_k = e$ then $B = \prod_{k=1}^{n} x_k \subseteq H$ and case (2) holds. Else $\prod_{k=1}^{n} x_k = \bigcup_{p\in \prod_{k=1}^{n} a_k} A_p$ and $Q = \prod_{k=1}^{n} a_k \subseteq P$. Therefore, we conclude that the case (1) holds. □

Theorem 2.20. *Let $(K, \star)$ be an $(H, Poly(P))$-hypergroup and $K = \bigcup_{p\in P} A_p$. For all $(x, y) \in A_a \times A_b$, where $a, b \in P$, we have*

$$x\beta_K y \Longleftrightarrow a\beta_P b.$$

and so $K/\beta_K \cong P/\beta_P$.

Proof. If $a = b = e$, then there are two elements $p, q \in P$ such that $e \in p \cdot q$ and so $H \subseteq u \star v$, where $u \in A_p$ and $v \in A_q$. It implies that $x\beta_K y$. If $x\beta_K y$ and $x, y \in A_e$ then $a = e$ and $b = e$ and $e\beta_P e$.

Now, let $(a, b) \neq (e, e)$, $x \in A_a$ and $y \in A_b$. $x\beta_K y$ if and only if there are $x_1, x_2, \ldots, x_n \in K$ such that $\{x, y\} \subseteq x_1 \star x_2 \star \ldots \star x_n = \bigcup_{p \in a_1 \cdot a_2 \cdot \ldots \cdot a_n} A_p$, where $x_i \in A_{a_i}$. Since $x \in A_a$ and $y \in A_b$ then A_a and A_B are subset of $\bigcup_{p \in a_1 \cdot a_2 \cdot \ldots \cdot a_n} A_p$. This imply that $\{a, b\} \subseteq a_1 \cdot a_2 \cdot \ldots \cdot a_n$ and so $a\beta_P b$.

$a\beta_P b$ implies that $a, b \in a_1 \cdot a_2 \cdot \ldots \cdot a_n$, for some $a_1, a_2, \ldots, a_n \in P$. Then for every $x_i \in A_{a_i}$ we have $A_a, A_b \subseteq x_1 \star x_2 \star \ldots \star x_n$ and so $x\beta_K y$.

Define $\psi : K/\beta_K \to P/\beta_P$ by $\psi(\beta_K(x)) = \beta_P(a)$, where $x \in A_a$ for some $a \in P$. With a bit calculation, we can see that ψ is an isomorphism and the proof is completed. □

Theorem 2.21. *Let $(K, \star)$ be an $(H, Poly(P))$-hypergroup and $K = \bigcup_{p \in P} A_p$. For all $(x, y) \in A_a \times A_b$, where $a, b \in P$, we have*

$$x\gamma_K y \Longleftrightarrow a\gamma_P b.$$

and so $K/\gamma_K \cong P/\gamma_P$.

Proof. See proof of Theorem 2.19. □

Corollary 2.22. *Let $(K, \star)$ be an $(H, Poly(P))$-hypergroup and $(P, \circ)$ be a canonical hypergroup. Then $K/\gamma_K \cong P/\beta_P$.*

Corollary 2.23. *Let $(K, \star)$ be an $(H, Poly(P))$-hypergroup. Then*

(1) *If $(P, \circ)$ is a group, then $K/\beta_K \cong P$.*

(2) *If $(P, \circ)$ is a group, then $K/\gamma_K \cong P/P'$, where P' is the derived subgroup of P.*

(3) *If $(P, \circ)$ is an abelian group, then $K/\gamma_K \cong P$.*

3. Comparison with other subclasses of hypergroups

In this section, we compare $(H, Poly(P))$-hypergroups with (H, G)-hypergroups, K_H-hypergroup and Extension of polygroups by polygroups . First, we introduce some concepts in the theory of hyperstructures.

Definition 3.1. Given two hypergroups $(A, \circ)$ and$(B, \bullet)$, we say that the hypergroup $(B, \bullet)$ is an enlargement of $(A, \circ)$ if

(1) $A \subseteq B$;

(2) For every element x, y belong A, we have the set $x \bullet y$ includes $x \circ y$.

Definition 3.2. ([29]). Let $(H, \circ)$ be An H_v-group. We called $(H, \circ)$ is an H_b-group if we can find a group $(H, \diamond)$ such that for all elements $x, y \in H$, $x \diamond y \in x \circ y$. If H_b-group $(H, \circ)$ is a hypergroup(polygroup) then $(H, \circ)$ is called $\diamond$-hypergroup($\diamond$-polygroup).

Theorem 3.3. *Let $(P, \cdot)$ be a $\diamond$-polygroup ($(P, \diamond)$ is a group). Then the hypergroup $(H, Poly(P))$-group $(K, \star)$ is an enlargement of the hypergroup (H, P)-group $(K, *)$.*

Proof. This holds since $\diamond \subseteq \cdot$ so $* \subseteq \star$. □

We recall the definitions of K_H-hypergroups, complete hypergroups and extension of polygroups by polygroups:

Definition 3.4. ([14]). **K_H-hypergroups and complete hypergroups**

Let $(H, \circ)$ be a hypergroup and $A_a \subseteq K$, where $a \in H$, be non-empty subsets and $\{A_a \mid a \in h\}$ be a partition of set K. We obtain a hypergroup $(K, \otimes)$ when $\otimes$ is the following hyperoperation:

$$x \otimes y = \bigcup_{c \in a \circ b} A_c, \text{ for all } x \in A_a, y \in A_b.$$

The hypergroup $(K, \otimes)$ is called a K_H-hypergroup, made by the hypergroup H.

If $(K, \otimes)$ is a K_H-hypergroup and $(H, \circ)$ is a group then we say that $(K, \otimes)$ is a complete hypergroup.

For the first time, Comer [21] made a larger polygroup from two smaller polygroups.

Definition 3.5. ([21]). **Extensions of polygroups by polygroups**

Let $\mathfrak{A} = (A, \circ, e, ^{-1})$ be a polygroup with identity e and $\mathfrak{B} = (B, \cdot, e, ^{-1})$ be an other polygroup with identity e such that $A \cap B = \{e\}$. Then we construct a polygroup $\mathfrak{A}[\mathfrak{B}] = (M, \odot, e, ^{-I})$, where $M = A \cup B$, $x^{-I} := x^{-1}$, $x \odot e = e \odot x = x$, for all $x \in M$. For all $x, y \in M - \{e\}$ the hyperoperation $\cdot$ is defined as follows:

$$x \odot y = \begin{cases} x \circ y, & \text{if } x, y \in A, \\ x, & \text{if } x \in B \text{ and } y \in A, \\ y, & \text{if } x \in A \text{ and } y \in B, \\ x \cdot y, & \text{if } x, y \in B \text{ and } x \neq y^{-1}, \\ x \cdot y \cup A, & \text{if } x, y \in B \text{ and } x \neq y^{-1}. \end{cases}$$

Then the hyperstructure $\mathfrak{A}[\mathfrak{B}] = (M, \odot, e, ^{-I})$ made by the above method has a polygroup structure and it is called polygroup extension $\mathfrak{A}$ by $\mathfrak{B}$.

Example 3.6. Let $(G = \{e, a, b\}, \cdot) \cong (\mathbb{Z}_3, +)$ and $(H = \{0, 1\}, \circ)$ with $0 \circ 0 = 0, 0 \circ 1 = 1 \circ 0 = 1$ and $1 \circ 1 = H$. Set $K = \{0, 1, 2, 3, 4, 5\}$ with $A_e = H = \{0, 1\}$, $A_a = \{2\}$ and $A_b = \{3, 4, 5\}$ then we obtain the complete hypergroup $(K, \otimes)$ denote the hypergroup with the corresponding Cayley table outlined in Table 5. It can be seen with a little care that $(K, \odot)$ is an enlargement of (H, G)-hypergroup $(K, *)$ in Example 1.2.

Table 5: Complete hypergroup.

$\otimes$	0	1	2	3	4	5
0	{0,1}	{0,1}	{2}	{3,4,5}	{3,4,5}	{3,4,5 }
1	{0,1}	{0,1}	{2}	{3,4,5}	{3,4,5}	{3,4,5 }
2	{2}	{2}	{3,4,5}	{0,1}	{0,1}	{0,1}
3	{3,4,5}	{3,4,5 }	{0,1}	{2}	{2}	{2}
4	{3,4,5}	{3,4,5 }	{0,1}	{2}	{2}	{2}
5	{3,4,5}	{3,4,5 }	{0,1}	{2}	{2}	{2}

Table 6: Polygroup of order 2.

$\circ$	0	1
0	{0}	{1}
1	{1}	{0,1}

Example 3.7. Let $H = \{0, 1\}$ and $(H, \circ)$ denote the hypergroup(polygroup) with the corresponding Cayley table outlined in Table 6. Also, let $P = \{e, a, b\}$ and $(P, \cdot)$ denote the polygroup with the corresponding Cayley table outlined in Table 7. Let $K = \{0, 1, 2, 3, 4, 5\}$ and set $A_e = H = \{0, 1\}, A_a = \{2\}$ and $A_b = \{3, 4, 5\}$ then we obtain the $(H, Poly(P))$-hypergroup $(K, \star)$ denote the hypergroup with the corresponding Cayley table outlined in Table 8.

Example 3.8. Let $(H, \circ)$ denote the hypergroup with the corresponding Cayley table outlined 6 and $(P, \cdot)$ denote the polygroup with the corresponding Cayley table outlined in Table 7. Let $K = \{0, 1, 2, 3, 4, 5\}$ and set $A_e = H = \{0, 1\}, A_a = \{2\}$ and $A_b = \{3, 4, 5\}$ then we obtain the K_P-hypergroup $(K, \otimes)$ by denote the hypergroup with the corresponding Cayley table outlined in Table 9.

Example 3.9. Let $H = \{e, 1\}$ and $(H, \circ)$ denote the hypergroup(polygroup) with the corresponding Cayley table outlined in Table 10. and let $(P, \cdot)$ denote the polygroup with the corresponding Cayley table outlined in Table 7. Then $M = \{e, 1, a, b\}$ and extension of polygroup H by the polygroup P is denote the polygroup with the corresponding Cayley table outlined in Table 11.

Theorem 3.10. *Let $(H, \circ)$ and $(P, \cdot)$ be a hypergroup and a polygroup, respectively. Set $K = \bigcup_{a \in P} A_a$ with $A_e = H$, where e is an identity element of $(P, \cdot)$. Then the K_P-hypergroup $(K, \otimes)$ is an enlargement of the $(H, Poly(P))$-hypergroup $(K, \star)$.*

Proof. For every $x, y \in K$ we have one of two following cases:

(1) If $x, y \in H$ then $x \circ y \in H$ and so $x \star y \subseteq x \otimes y$.

Table 7: Polygroup of order 3.

$\cdot$	e	a	b
e	{e}	{a}	{b}
a	{a}	{e,a}	{b}
b	{b}	{b}	{e,a,b}

Table 8: $(H, Poly(P))$-hypergroup.

$\star$	0	1	2	3	4	5
0	{0}	{1}	{2}	{3,4,5}	{2,4,5}	{3,4,5}
1	{1}	{0,1}	{2}	{3,4,5}	{2,4,5}	{3,4,5}
2	{2}	{2}	{0,1,2}	{3,4,5}	{2,4,5}	{3,4,5}
3	{3,4,5}	{3,4,5}	{3,4,5 }	{0,1,2,3,4,5 }	{0,1,2,3,4,5}	{0,1,2,3,4,5}
4	{3,4,5}	{3,4,5}	{3,4,5 }	{0,1,2,3,4,5 }	{0,1,2,3,4,5}	{0,1,2,3,4,5}
5	{3,4,5}	{3,4,5}	{3,4,5 }	{0,1,2,3,4,5 }	{0,1,2,3,4,5}	{0,1,2,3,4,5}

(2) If $x, y \in A_a \otimes A_b \neq H^2$, then $x \star y = \bigcup_{p \in a \cdot b} A_p = x \otimes y$.

This yields that $(K, \otimes)$ is an enlargement of $(K, \star)$. □

Corollary 3.11. *Let $(H, \circ)$ be a hypergroup, $(G, \cdot)$ be a group and $K = \bigcup_{a \in P} A_a$, where $A_e = H$. Then the complete hypergroup $(K, \otimes)$ is an enlargement of the (H, G)-hypergroup $(K, *)$.*

Theorem 3.12. *Let $(H, \circ)$ be a hypergroup. If the polygroup $(P, \cdot)$ is an enlargement of the polygroup $(P', \cdot')$ then the $(H, Poly(P))$-hypergroup $(K, \star)$ is an enlargement of $(H, Poly(P'))$-hypergroup $(K, \star')$.*

Proof. For every $x, y \in K$ we have one of two following cases:

(1) If $x, y \in H$ then $x \star y = x \circ y = x \star' y$.

(2) If $(x, y) \in A_a \times A_b \neq H^2$, then $x \star' y = \bigcup_{p \in a \cdot' b} A_p \subseteq \bigcup_{p \in a \cdot b} A_p = x \star y$.

Therefore, we conclude that $(K, \star)$ is an enlargement of $(K, \star')$. □

Theorem 3.13. *Every $(H, Poly(P))$-hypergroup $(K, \star)$ is an enlargement of the hypergroup H.*

Proof. It follows from $H \subseteq H$ and $\circ \subseteq \star$. □

Theorem 3.14. *The $(A, Poly(B))$-hypergroup $(K, \star)$ is an enlargement of the polygroup $A[B] = (M, \odot, e, ^{-1})$ where for all $a \in B - \{e\}$, $a \in A_a$.*

Table 9: K_P-hypergroup.

$\otimes$	0	1	2	3	4	5
0	{0,1}	{0,1}	{2}	{3,4,5}	{2,4,5}	{3,4,5}
1	{0,1}	{0,1}	{2}	{3,4,5}	{2,4,5}	{3,4,5}
2	{2}	{2}	{0,1,2}	{3,4,5}	{2,4,5}	{3,4,5}
3	{3,4,5}	{3,4,5}	{3,4,5 }	{0,1,2,3,4,5 }	{0,1,2,3,4,5}	{0,1,2,3,4,5}
4	{3,4,5}	{3,4,5}	{3,4,5 }	{0,1,2,3,4,5 }	{0,1,2,3,4,5}	{0,1,2,3,4,5}
5	{3,4,5}	{3,4,5}	{3,4,5 }	{0,1,2,3,4,5 }	{0,1,2,3,4,5}	{0,1,2,3,4,5}

Table 10: Polygroup of order 2.

$\circ$	e	1
e	{e}	{1}
1	{1}	{e,1}

Proof. We have $A \cup B \subseteq K$. Moreover, for all $a, b \in B$, we have $x \star y = \bigcup_{p \in a \cdot b} A_p$ and so $a \cdot b \subseteq a \star b$. Now, we have the following cases:

(1) If $x, y \in A$, then $x \odot y = x \circ y = x \star y$.

(2) If $x \in A = A_e$ and $y \in B$, then $y = x \odot y \in A_y \subseteq \bigcup_{p \in e \cdot y} A_p = x \star y$.

(3) If $x \in B$ and $y \in A$, then $x = x \odot y \in A_x \subseteq \bigcup_{p \in x \cdot e} A_p = x \star y$.

(4) If $x, y \in B$ and $x \neq y^{-1}$, then $x \odot y = x \cdot y \subseteq A_{x \cdot y} \subseteq x \star y$.

(5) If $x, y \in A$ and $x = y^{-1}$, then $x \odot y = A \cup \{e\} = A = A_e \subseteq\subseteq \bigcup_{p \in x \cdot e} A_p = x \star y$, because $e \in x \cdot y$.

Therefore for all $x, y \in A \cup B$, $x \odot b \subseteq x \star y$ and so the hypergroup$(K, \star)$ is an enlargement of the polygroup $A[B]$. □

Theorem 3.15. *Let* $(B, \cdot, e,^{-1})$ *be a group and* $A_x = \{x\}$, *for all* $x \in B - \{e\}$. *Then* $(K, \star) = (M, \odot)$.

Proof. We have $M = A \cup B = K$. Moreover, we can consider the following cases:

(1) If $x, y \in A$, then $x \odot y = x \circ y = x \star y$.

(2) If $x \in A = A_e$ and $y \in B$, then $x \odot y = y = A_y = A_{e \cdot y} = x \star y$.

(3) If $x \in B$ and $y \in A = A_e$, then $x \odot y = x = A_x = A_{x \cdot e} = x \star y$.

(4) If $x, y \in B$ and $x \neq y^{-1}$, then $x \odot y = x \cdot y = A_{x \cdot y} = x \star y$.

Table 11: Extension of polygroup by polygroup.

$\odot$	e	1	a	b
e	{e}	{1}	{a}	{b}
1	{1}	{e,1}	{a}	{b}
a	{a}	{a}	{e,1,a}	{b}
b	{b}	{b}	{b}	{e,1,a,b}

(5) If $x, y \in A$ and $x = y^{-1}$, then $x \odot y = A \cup \{e\} = A = A_e = A_{x \cdot y} = x \star y$, because $e = x \cdot y$.

Therefore $(K, \star) = (M, \odot)$.. □

4. Conclusions

In this paper, we proposed and characterized new classes of hypergroups named the $(H, Poly(P))$-hypergroups, thus, it is an generalization of (H, G)-hypergroups explored by De Salvo[16]. We compared $(H, Poly(P))$-hypergroups with K_H-hypergroups, complete hypergroups and extension of polygroups by polygroups.

Our future objective is to expand this study to encompass other hyperstructures. Also, it can be interesting to define and introduce the reducibility for $(H, Poly(P))$-hypergroups like in [17, 30].

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Saeed Mirvakili
Department of Mathematics,
Payame Noor University,
Tehran, Iran
e-mail: saeed_ mirvakili@pnu.ac.ir

Somayeh Hosseini
Department of Mathematics,
Payame Noor University,
Tehran, Iran
e-mail: hosseiniii.somayeh@yahoo.com

Bijan Davvaz
Department of Mathematical Sciences,
Yazd University,
Yazd, Iran
e-mail: davvaz@yazd.ac.ir