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On Power Graph of Some Finite Rings

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Abstract

Consider a ring R with order p or p^2, and let $\mathcal{P}(R)$ represent its multiplicative power graph. For two distinct rings R_1 and R_2 that possess identity element 1, we define a new structure called the unit semi-cartesian product of their multiplicative power graphs. This combined structure, denoted as $G.H$, is constructed by taking the Cartesian product of the vertex sets $V(G) \times V(H)$, where $G = \mathcal{P}(R1)$ and $H = \mathcal{P}(R2)$. The edges in $G.H$ are formed based on specific conditions: for vertices (g, h) and (g', h'), an edge exists between them if $g = g'$, g is a vertex in G, and the product hh' forms a vertex in H.

Our exploration focuses on understanding the characteristics of the multiplicative power graph resulting from the unit semi-cartesian product $\mathcal{P}(R1).\mathcal{P}(R2)$, where R_1 and R_2 represent distinct rings. Additionally, we offer insights into the properties of the multiplicative power graphs inherent in rings of order p or p^2.

Keywords: Domination number, Independence number, Split graph, Unit Semi-cartesian product.

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1. Introduction

Graph theory is employed for examining the foundational structural properties of algebraic structures, which encompass entities like rings. This approach has demonstrated particular significance in uncovering specific traits inherent to algebraic structures such as rings. Through the incorporation of various graph

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categories into the domain of rings, scholars have revealed a wide spectrum of attributes, thereby illuminating novel dimensions of these mathematical formations [1–4]. We are dealing with undirected and simple graphs exclusively in this context. Let's denote the vertex set and edge set of a graph G as $V(G)$ and $E(G)$, respectively. We first state some definitions and notations which are used throughout the paper.

For a given semigroup S, the undirected power graph $\mathcal{P}(S)$ is defined by its vertex set S. Notably, two distinct vertices x and y in this graph are connected if and only if one of the following conditions holds: either $x^n = y$ or $y^n = x$, where n is a positive integer [5]. This concept of the power graph was initially introduced in an influential work by Kelarev and Quinn [6], which was later expanded upon in their subsequent papers [7–9].

When considering a ring R, the presence of two binary operations, namely "$+$" and "$\times$", leads to the existence of two distinct power graphs associated with it. These graphs are known as the additive power graph $\mathcal{P}^+(R)$ and the multiplicative power graph $\mathcal{P}^\times(R)$. The former corresponds to the operation "$+$", while the latter corresponds to "$\times$".

Let's revisit the definitions. A graph is termed *connected* when, for any pair of distinct vertices x and y, there exists a finite sequence of distinct vertices $x = x_1, \cdots, x_n = y$ such that each consecutive pair (x_i, x_{i+1}) forms an edge. Conversely, a graph that lacks any edges is referred to as *totally disconnected*.

Given distinct vertices x and y, let $\mathrm{d}(x, y)$ represent the shortest length of a path connecting them, and if no such path exists, let $d(x, y) = \infty$. The "diameter" of a graph G is established as $diam(G) = \sup\{d(x, y)|x, y \in V(G)\}$, where $V(G)$ signifies the set of vertices within the graph.

For a given graph G, the *degree* of a vertex x within G is the count of edges in G that are connected to x, and this degree is represented as $\deg(x)$. If a graph G can be split into two vertex subsets V_1, V_2 in such a way that the complete set of vertices in G is the union of V_1 and V_2, and V_1 and V_2 are disjoint ($V_1 \cap V_2 = \emptyset$), and every edge in G connects a vertex from V_1 to a vertex in V_2, then G is termed *bipartite*. In the context of a bipartite graph, a *complete bipartite* graph encompasses all possible edges connecting vertices within V_1 and V_2. When the sizes of these vertex sets are denoted as m and n, the complete bipartite graph is symbolized as $K_{m,n}$. If m equals 1, then the outcome is a *stargraph* and a complete graph featuring n vertices is designated as K_n.

A *cycle* within graph G is a path that both begins and ends at the same vertex. The *girth* of graph G, denoted as $gr(G)$, is the length of the shortest cycle present within G, and if no cycle is found in G, then $gr(G)$ is regarded as infinite.

In graph theory, a *regular graph* is one in which each vertex has an identical number of neighbors, meaning every vertex has the same degree or valency. For a graph $G = (V(G), E(G))$ and a subset S of the vertices in G, the set of vertices in G that either belong to S or share an edge with a vertex from S is symbolized as $N_G[S]$. If $N_G[S]$ covers all vertices within $V(G)$, then S is recognized as a *dominatingset*. The *domination number* of a graph G, denoted as $\gamma(G)$, signifies the smallest number of vertices required to form a dominating set in G. An *independent set* X within the vertices of graph G is a subset where the subgraph induced by X contains no edges. The largest size achievable for an independent set within graph G is referred to as the *independence number* of G, denoted by $\alpha(G)$. A *splitgraph* refers to a graph in which the vertices can be partitioned into a clique (a complete subgraph) and an independent set.

Assuming p is a prime, Fine [10] categorized all rings of order p^2 as follows:

$$
\begin{array}{rcl}
A & = & \langle a : p^2 a = 0, a^2 = a\rangle \cong \mathbb{Z}_{p^2},\\
B & = & \langle a : p^2 a = 0, a^2 = pa\rangle,\\
C & = & \langle a : p^2 = 0, a^2 = 0\rangle,\\
D & = & \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = ba = 0\rangle = \mathbb{Z}_p \oplus \mathbb{Z}_p,\\
E & = & \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle,\\
F & = & \langle a, b : pa = pb = 0, a^2 = a, b^2 = b, ab = b, ba = a\rangle,\\
G & = & \langle a, b : pa = pb = 0, a^2 = 0, b^2 = 0, ab = ba = a\rangle,\\
H & = & \langle a, b : pa = pb = 0, a^2 = 0, b^2 = b, ab = ba = 0\rangle,\\
I & = & \langle a, b : pa = pb = 0, a^2 = b, ab = 0\rangle,\\
J & = & \langle a, b : pa = pb = 0, a^2 = b^2 = 0\rangle,\\
K & = & \left\{ \begin{array}{ll} \langle a, b : 2a = 2b = 0, a^2 = a, b^2 = a + b, ab = b, ba = b\rangle, & p = 2,\\ \langle a, b : pa = pb = 0, a^2 = a, b^2 = ja, ab = ba = b\rangle, & p \neq 2, \end{array}\right.
\end{array}
$$

where j is not a square in $\mathbb{Z}_p$. According to the reference [5], the additive power graph of a ring is indicative of the ring's additive structure. As a result, our attention is directed towards the multiplicative power graph denoted as $\mathcal{P}(R) = \mathcal{P}^{\times}(R)$, as it holds significance in understanding the multiplicative aspects of the ring. The multiplicative power graphs of these 11 non-isomorphic rings have already been studied. Simply we can see $\mathcal{P}(A) \cong \mathcal{P}(G)$, $\mathcal{P}(B) \cong \mathcal{P}(I)$, $\mathcal{P}(C) \cong \mathcal{P}(J)$ and $\mathcal{P}(E) \cong \mathcal{P}(F)$. Accordingly, it is sufficient to consider the rings A, B, C, D, E, H and K in order to investigate the multiplicative power graphs of rings of order p^2. Consider a prime number p, and let R be a ring with an order of p. Regarding its additive group, R is isomorphic to the ring of integers modulo p, represented as $\mathbb{Z}_p$. Consequently, there are two distinct rings with an order of p: the ring $\mathbb{Z}_p$ and the zero ring associated with the additive group, denoted as N_p.

In this document, we denote the "cardinality" of a set A as $|A|$, and we use $\mathcal{P}^*(R)$ to indicate the power graph of R excluding the vertex 0. Euler's phi function, denoted by $\varphi(n)$, plays a role in number theory by counting the positive integers up to a given integer n that are coprime with n.

In Section 2 of the paper, we introduce the concept of the unit semi-cartesian product of power graphs for specific rings of order p and p^2. This unit semi-cartesian product involves two distinct simple graphs, referred to as G and H. The notation $G.H$ represents this product, defined as an undirected graph with a vertex set $V(G.H) = V(G) \times V(H)$ and an edge set $E(G.H) = \{(g, h)(g', h') \mid g =$

$g', g1 \in E(G)$ and $hh' \in E(H)\}$. In other sections, we study some graph properties of the power graph $\mathcal{P}(R)$ of a ring R and also $\mathcal{P}(R_1).\mathcal{P}(R_2)$ where R, R_1, and R_2 are rings of order p or p^2. Note that in this paper we consider only rings with an identity element showed by 1.

Our other notations are standard and can be found in the major survey articles on power graphs [11–16] and the books [17–20].

2. Unit semi-cartesian product

In this section, we introduce the unit semi-cartesian product of two simple graphs G and H, denoted by $G.H$, as the undirected graph that has the set $V(G.H) = V(G) \times V(H)$ as vertices set and the edges set $E(G.H) = \{(g,h)(g',h') \mid g = g', g1 \in E(G)$ and $hh' \in E(H)\}$. Now we mention this product for our graph. In fact, we investigate the graphs $\mathcal{P}(R_1).\mathcal{P}(R_2)$, where R_1 and R_2 are two rings of order p^i or q^j for $i, j = 1$ or 2 and prime numbers p and q, respectively.

Theorem 2.1. *Suppose R_1 and R_2 are two rings with orders p and q, respectively. Then, in this context one of the following statements hold:*
(1) $\mathcal{P}(R_1).\mathcal{P}(R_2)$ is empty.
(2) $\mathcal{P}(R_1).\mathcal{P}(R_2)$ has $2q$ isolated vertices and $p-2$ connected components $K_{1,q-1}$.
(3) The graph $\mathcal{P}(R_1).\mathcal{P}(R_2)$ has $p+2q-2$ isolated vertices and $q-2$ components isomorphic to $\mathcal{P}^*(\mathbb{Z}_q)$.

Proof. First, We will examine three distinct scenarios:
Case 1: If $R_1 \cong N_q$ and $R_2 \cong \mathbb{Z}_q$,or $R_2 \cong N_q$. Then this graph is empty since R_1 is a ring without identity.
Case 2: If $R_1 \cong \mathbb{Z}_p$ and $R_2 \cong \mathbb{Z}_q$. Then for $p = 2$ or $q = 2$, this graph is empty.
Case 3: If $R_1 \cong \mathbb{Z}_p$ and $R_2 \cong N_q$. Then by definition, all vertices with the first component 0 or 1 are isolated. Also, in $\mathcal{P}(\mathbb{Z}_p)$ the number of adjacent vertices to 1 is $p-2$.
In other cases part (3) holds. □

Theorem 2.2. *Let R_1 and R_2 represent two rings with orders p and q^2 respectively. In this situation, the following analysis holds: (1) The graph $\mathcal{P}(R_1).\mathcal{P}(R_2)$ is totally disconnected.*
(2) The graph $\mathcal{P}(R_1).\mathcal{P}(R_2)$ In this context, the graph has a total of $2q^2$ isolated vertices, and $p-2$ components isomorphic to $\mathcal{P}(R_2)$.

Proof. Clearly, if $R \cong N_p$ or $\mathbb{Z}_2$, then part (1) holds. In other cases by definition of unit semi-cartesian product, only all vertices $(0,a)$ and $(1,b)$, with $a, b \in V(\mathcal{P}(R_2))$ are isolated. Also $\mathcal{P}(\mathbb{Z}_p).\mathcal{P}(R_2)$ has $deg(1) = p-2$ components isomorphic to $\mathcal{P}(R_2)$, where $1 \in R_1$. □

Theorem 2.3. *Let R_1 and R_2 be two rings of order p^2 and q, respectively. Then one of the following cases happens:*

(1) $\mathcal{P}(R_1).\mathcal{P}(R_2)$ *forms a graph that is completely disconnected.*
(2) $\mathcal{P}(R_1).\mathcal{P}(R_2)$ *has* $q(p+1)$ *isolated vertices and* $deg(1)$ *copies of* $\mathcal{P}(N_q)$*, where* $1 \in R_1$*.*
(3) The graph $\mathcal{P}(R_1).\mathcal{P}(R_2)$ *has* $(p+1)q + deg(1)$ *isolated vertices and* $deg(1)$ *copies of* $\mathcal{P}^*(\mathbb{Z}_q)$*, where* $1 \in R_1$*.*

Proof. Assume that $R_1 \cong B, C, E$ or H. Then the graph $\mathcal{P}(R_1).\mathcal{P}(N_q)$ forms a graph that is completely disconnected, because these rings are not with identity. If $R_2 \cong \mathbb{Z}_2$, then this graph does not have any edges since the power graph $\mathcal{P}(\mathbb{Z}_2)$ is a totally disconnected graph. Now, let $R_1 \cong A, D$, or K and $R_2 \cong N_q$. Then all vertices (a, b), where $a = 0$ or 1, or $a1 \notin E(\mathcal{P}(R_1))$ and $b = 0$, or $b0 \in E(N_q)$ are isolated. It is easy to see that this graph has $deg(1)$ copies of $\mathcal{P}(N_q)$. Similarly, if $R_1 \cong A, D$ or K, then (3) is true. □

Corollary 2.4. *Consider two rings denoted as* R_1 *and* R_2*, where* R_1 *has an order of* p^2 *and* R_2 *has an order of* q*. If at least one of the graphs* $\mathcal{P}(R_1).\mathcal{P}(R_2)$ *and* $\mathcal{P}(R_2).\mathcal{P}(R_1)$ *is not a totally disconnected, then* $\mathcal{P}(R_1).\mathcal{P}(R_2) \ncong \mathcal{P}(R_2).\mathcal{P}(R_1)$*.*

Proof. This follows directly from Theorems 2.2 and 2.3. □

Theorem 2.5. *Consider two rings denoted as* R_1 *and* R_2*, where* R_1 *has an order of* p^2 *and* R_2 *has an order of* q^2*. Then* $\mathcal{P}(R_1).\mathcal{P}(R_2)$ *has* $deg(1)$ *component(s) isomorphic to* $\mathcal{P}(R_2)$ *where* $1 \in R_1$*, and the number of its isolated vertices is equal to one of the following:*
(1) $(p^2 - p + 1)q^2$*.*
(2) $2pq^2$*.*
(3) $2q^2$*.*

Proof. The initial portion of the theorem is a straightforward consequence of the definition. Now, let $R_1 \cong A, D$ or K. By the structure of $\mathcal{P}(R_1)$, the second part is clear. So we illustrate the structure of the multiplicative power graph of these rings. $\mathcal{P}(A)$ has 2 connected components. The first component is the star graph $K_{1,p-1}$ with p vertices since $(np)^2 = n^2p^2 = 0$ for all $1 \le n \le p-1$. The second component is the graph of the group of multiplicative units of $\mathbb{Z}_{p^2}$, denoted U_{p^2}. It has $p^2 - p$ vertices. Each generator is connected to all other vertices in this component, so the valency of each vertex in the second component is ≥ 2 when $p \ge 3$. Ring D has 4 connected components corresponding to the idempotents $a, b, a+b$ and 0. The component with 0 is an isolated vertex. The component connected to a contains vertices of the form $ja + 0b$ with $1 \le j \le p-1$. This graph represents the power graph of the group of units U_p. The component connected to b is the vertices of the form $0a + kb$ for $1 \le k \le p-1$ and the component connected to $a+b$ is the graph of the group of units of the ring. The multiplicative power graph of K has two components, the isolated vertex 0 and the graph of the group of units of K connected to the multiplicative identity, a.

□

3. Domination and independence numbers

Now, we determine the domination and independence numbers of the graphs $\mathcal{P}(R)$ and $\mathcal{P}(R_1).\mathcal{P}(R_2)$, where R, R_1 and R_2 are rings of order p^i or q^j for $i, j = 1$ or 2 and prime numbers p and q. First, we characterize the domination number of these graphs.

Theorem 3.1. *Let R be a ring of order p or p^2. Then $\gamma(\mathcal{P}(R)) \in \{1, 2, 4, 3+|L|\}$, where*

$$L = \{sa + tb \mid 1 \leq s, t \leq p-1, s+t-1 \equiv^p 0\},$$

such that a and b are two generators of ring E.

Proof. Let $R \cong \mathbb{Z}_p$. Then the graph $\mathcal{P}(R)$ has 0 as an isolated vertex and all other vertices form a connected component in which all vertices are adjacent to vertex 1, so the subset $S = \{0, 1\}$ of $V(\mathcal{P}(R))$ implies that $N_{\mathcal{P}}[S] = V(\mathcal{P}(R))$, then $\gamma(\mathcal{P}(R)) = 2$. Let $R \cong N_p$. Then $\mathcal{P}(R) \cong K_{1,p-1}$, so the smallest dominating set is $\{0\}$ and obviously $\gamma(\mathcal{P}(R)) = 1$.
Now, assume that R is a ring of order p^2. By the structure of $\mathcal{P}(R)$, the smallest dominating set S such that $N_{\mathcal{P}}(S) = V(\mathcal{P})$ can be obtained as follows:

1. Let $R \cong A$. Then $S = \{0, va\}$, where a serves as a generator of A, while v acts as a generator of cyclic group $U(\mathbb{Z}_{p^2})$. Hence, $\gamma(\mathcal{P}(R)) = 2$.
2. Let $R \cong B$ or C. Then $S = \{0\}$, thus $\gamma(\mathcal{P}(R)) = 1$.
3. Let $R \cong D$. Then the set of idempotents 0, $a, b, a+b$ stands for S. Hence, $\gamma(\mathcal{P}(R)) = 4$.
4. Let $R \cong E$. Then $S = \{0, a, b, sa + tb \mid 1 \leq s, t \leq p-1, s+t-1 \equiv^p 0\}$ in which a and b are two generators of ring E.
5. Let $R \cong H$. Then $S = \{0, b\}$, where b is an idempotent generator of ring H.
6. Let $R \cong K$. Then $S = \{0, s\}$, where $0 \neq s \in V(\mathcal{P}(K))$.

□

Now, by considering R as a ring of order p or p^2, we determine the independence number of $\mathcal{P}(R)$

Theorem 3.2. *Let R be a ring of order p. Then $\alpha(\mathcal{P}(R)) \geq 2$. Particularly, if $R \cong N_p$, then $\alpha(\mathcal{P}(R)) = p-1$.*

Proof. It is straightforward by considering the structure of $\mathcal{P}(R)$. □

Theorem 3.3. *Let R be a ring of order p^2. Then the following statements hold:*
(1) If $R \cong A$ and $|R| = 4$, or $R \cong K$, then $\alpha(\mathcal{P}(R)) = 2$.
(2) If $R \cong A$ and $|R| \neq 4$, then $\alpha(\mathcal{P}(R)) \leq p^2 - \varphi(\varphi(p^2)) - 2$.
(3) If $R \ncong A$ or K, then $\alpha(\mathcal{P}(R)) \in \{p^2-p, p^2-1, 2p-1, p+3, 4, p^2-|L|\}$, where $L = \{kpa \mid 0 \leq k \leq p-1,\ k \text{ is square modulo } p\}$.

Proof. By the definition of $\alpha(\mathcal{P})$, it is sufficient to determine the biggest independent set X of the graph $\mathcal{P}(R)$.

(1) $R \cong A$ and $|R| = 4$. Then it is obvious that $X = \{0, va\}$ or $X = \{npa, va \mid 1 \leq n \leq p-1\}$, where a and v are generators of R and $U(\mathbb{Z}_4)$, respectively. Now, suppose that $R \cong K$. Thus $\mathcal{P}(R)$ has two components, one of them is the isolated vertex 0 and another is a complete graph. Hence $\alpha(\mathcal{P}(R)) = 2$.
(2) By structure of $\mathcal{P}(R)$, clearly X contains $p-1$ elements npa as well as all non-identity non-generator elements of $U(\mathbb{Z}_{p^2})$ that they are not adjacent to each other, so $\alpha(p) \leq p^2 - \varphi(\varphi(p^2)) - 2$.
(3) We consider the following cases:

1. If $R \cong B$ or H. Then one can check that $\alpha(\mathcal{P}(R))$ is $p^2 - p + |kpa|$, where $0 \leq k \leq p-1$ and k is not modulo square p.
2. If $R \cong C$. Then we have $X = \{v \mid\ 0 \neq v \in V(\mathcal{P}(\mathcal{R}))\}$ since the graph $\mathcal{P}(R)$ is a star.
3. If $R \cong D$ and $|R| = 4$. Then $\alpha(\mathcal{P}(R)) = 4$ since the graph $\mathcal{P}(R)$ is totally disconnected. Let $|R| \neq 4$, then $\mathcal{P}(R)$ has four components corresponding to the idempotent elements $a, b, a+b$ and 0 as an isolated vertex. The component corresponding to a and b are isomorphic to K_{p-2}. In the component corresponding to $a+b$ one can see that the maximum number of the elements which can choose is p, that is the vertex $a + (p-1)b$ as well as all vertices of the form $p'a + kb$, where p' is the first prime number before p and $1 \leq k \leq p-1$.
4. If $R \cong E$. Then $\mathcal{P}(R)$ has $p+1$ connected components, p components K_{p-1} and a component $K_{1,p-1}$, so X has exactly $2p-1$ vertices.

□

Theorem 3.4. *Consider two rings denoted as R_1 and R_2, where R_1 has an order of p and R_2 has an order of q. Then*

$$\gamma(\mathcal{P}(R_1).\mathcal{P}(R_2)) \in \{pq, p+2q-2, 2p, 2q, 4, p+3q-4\},$$

and

$$\alpha(\mathcal{P}(R_1).\mathcal{P}(R_2)) = \begin{cases} pq, & if\ \ R_1 \cong N_q, \\ pq - p + 2, & if\ \ R_1 \cong \mathbb{Z}_p \ and\ R_2 \cong N_q, \end{cases}$$

otherwise $\alpha(\mathcal{P}(R_1).\mathcal{P}(R_2)) \geq p + 3q - 4$.

Proof. It is sufficient to mention the graphs $\mathcal{P}(N_p).\mathcal{P}(N_q)$, $\mathcal{P}(\mathbb{Z}_p).\mathcal{P}(N_q)$, $\mathcal{P}(\mathbb{Z}_p).\mathcal{P}(\mathbb{Z}_q)$ and $\mathcal{P}(N_p).\mathcal{P}(\mathbb{Z}_q)$. By Theorem 2.1, the graphs $\mathcal{P}(N_p).\mathcal{P}(N_q)$, $\mathcal{P}(N_p).\mathcal{P}(\mathbb{Z}_q)$, $\mathcal{P}(\mathbb{Z}_2).\mathcal{P}(\mathbb{Z}_q)$, $\mathcal{P}(\mathbb{Z}_p).\mathcal{P}(\mathbb{Z}_2)$ and $\mathcal{P}(\mathbb{Z}_2).\mathcal{P}(\mathbb{Z}_2)$ are totally disconnected graphs, so $\gamma(\mathcal{P}(N_p).\mathcal{P}(N_q) = \gamma(\mathcal{P}(N_p).\mathcal{P}(\mathbb{Z}_q)) = pq$. Also, the graph $\mathcal{P}(\mathbb{Z}_p).\mathcal{P}(N_q)$ has $2q$ isolated vertices and $p-2$ components $K_{1,q-1}$, so we have $\gamma(\mathcal{P}(\mathbb{Z}_p).\mathcal{P}(N_q)) = p+2q-2$. On the other hand, if $p, q \neq 2$, then $\mathcal{P}(\mathbb{Z}_p).\mathcal{P}(\mathbb{Z}_q)$ has $p+2q-2$ isolated vertices and $q-2$ components isomorphic to $\mathcal{P}^*(\mathbb{Z}_q)$. By definition of dominating set S, in every component the vertex $1 \in V(\mathcal{P}^*(\mathbb{Z}_q))$ lies down in S, moreover all the isolated vertices are in S. Thus $\gamma(\mathbb{Z}_p).\mathcal{P}(\mathbb{Z}_q)) = p+3q-4$. Now, we consider some cases to find the independent number. Let $R_1 \cong N_q$. Then $\mathcal{P}(R_1).\mathcal{P}(R_2)$ is totally disconnected so $\alpha(\mathcal{P}(R_1).\mathcal{P}(R_2)) = pq$. Let $R_1 \cong \mathbb{Z}_p$ and $R_2 \cong N_q$. Then again by Theorems 2.1 and 3.2, $\alpha(\mathcal{P}(R_1).\mathcal{P}(R_2)) = (p-2)(q-1)+2q$. Moreover, if $R_1 \cong \mathbb{Z}_p$ and $R_2 \cong \mathbb{Z}_q$, then the graph $\mathcal{P}(R_1).\mathcal{P}(R_2)$ is empty or it has $p+2q-2$ isolated vertices and $q-2$ components isomorphic to $\mathcal{P}^*(\mathbb{Z}_q)$. Thus $\alpha(\mathcal{P}(R_1).\mathcal{P}(R_2)) = pq$ or $\alpha(\mathcal{P}(R_1).\mathcal{P}(R_2)) \geq p+3q-4$. □

Theorem 3.5. *Let R_1 and R_2 be two rings of order p and q^2, respectively. Then*

$$\gamma(\mathcal{P}(R_1).\mathcal{P}(R_2)) \in \{pq^2, (p-2)(\gamma(\mathcal{P}(R_2)) + 2q^2\},$$

and one of the following relations hold:
(1) $\alpha(\mathcal{P}(R_1).\mathcal{P}(R_2)) = pq^2$.
(2) If $R \cong A$ *and* $|R| = 4$, *or* $R \cong K$, *then* $\alpha(\mathcal{P}(R_1).\mathcal{P}(R_2)) = 2(p-2) + 2q^2$.
(3) If $R \ncong A$ *or* K, *then*

$$\alpha(\mathcal{P}(R_1).\mathcal{P}(R_2)) \in \{(p-2)(p^2-p)+2q^2, (p-2)(p^2-1)+2q^2, (p-2)(2p-1)+2q^2,$$

$$(p-2)(p+3) + 2q^2, 4(p-2) + 2q^2, (p-2)(p^2 - |L|) + 2q^2\},$$

where $L = \{kpa \mid 0 \leq k \leq p-1,\ k \text{ is square modulo } p\}$.

Proof. This is obtained directly from Theorems 2.2 and 3.1. □

Theorem 3.6. *Consider two rings denoted as R_1 and R_2, where R_1 has an order of p^2 and R_2 has an order of q. Then*

$$\gamma(\mathcal{P}(R_1).\mathcal{P}(R_2)) \in \{p^2q, (p+1)q + deg(1), (p+1)q + 2deg(1) \mid 1 \in R_1\},$$

and one of the following relations hold:
(1) $\alpha(\mathcal{P}(R_1).\mathcal{P}(R_2)) = p^2q$.
(2) $\alpha(\mathcal{P}(R_1).\mathcal{P}(R_2)) = (p+1)q + (q-1)deg(1)$, *where* $1 \in R_1$.
(3) $\alpha(\mathcal{P}(R_1).\mathcal{P}(R_2)) \geq (p+1)q + 2deg(1)$, *where* $1 \in R_1$.

Proof. Suppose that $R_1 \cong B, C, E$ or H and $R_2 \cong N_q$, or $R_2 \cong \mathbb{Z}_2$. Then $\mathcal{P}(R_1).\mathcal{P}(R_2)$ is a totally disconnected graph, so

$$\gamma(\mathcal{P}(R_1).\mathcal{P}(R_2)) = \alpha(\mathcal{P}(R_1).\mathcal{P}(R_2)) = p^2q.$$

Let $R_1 \cong A, D$ or K and $R_2 \cong N_q$. Then, by Theorems 2.3 and 3.1, we get $\gamma(\mathcal{P}(R_1).\mathcal{P}(R_2)) = (p+1)q + deg(1)$ in which $1 \in R_1$. Hence part (2) holds. Now, let $R_2 \cong \mathbb{Z}_q$, then again by Theorem 2.3 and the structure of $\mathcal{P}^*(R_2)$, we have $\gamma(\mathcal{P}(R_1).\mathcal{P}(R_2)) = p^2q$ or $(p+1)q + 2deg(1)$, where $1 \in R_1$. Thus part (3) is true. □

Theorem 3.7. *Let R_1 and R_2 be two rings of order p^2 and q^2, respectively. If $1 \in R_1$ and $\mathcal{I}$ is the set of isolated vertices of $\mathcal{P}(R_1).\mathcal{P}(R_2)$, then*
(1) $\gamma(\mathcal{P}(R_1).\mathcal{P}(R_2)) = |\mathcal{I}| + deg(1)\gamma(\mathcal{P}(R_2))$,
(2) $\alpha(\mathcal{P}(R_1).\mathcal{P}(R_2)) = |\mathcal{I}| + deg(1)(\alpha(\mathcal{P}(R_2))$.

Proof. This is obtained directly from Theorems 2.5, 3.1 and 3.3. □

4. Regular graph and split graph

Within this section, we examine the graphs $\mathcal{P}(R)$ and $\mathcal{P}(R_1).\mathcal{P}(R_2)$, where R, R_1, and R_2 represent rings with orders p^i or q^j for $i, j = 1$ or 2 and prime numbers p and q and determine which one is regular or split graph.

Theorem 4.1. *Let R be a ring of order p. Then $\mathcal{P}(R)$ is not a regular graph.*

Proof. By the definition of regular graph, one can check it for just two possible cases that we have. □

Theorem 4.2. *Let R be a ring of order p^2. Then $\mathcal{P}(R)$ is a regular graph if and only if $p = 2$ and $R \cong A$, D or H.*

Proof. We consider some cases:

1. Let $R \cong A$. Then if $p = 2$ it is obvious that $\mathcal{P}(R)$ is a regular graph. Now suppose that this graph is regular and $p \neq 2$, therefore $1 = deg(npa) \neq deg(0) = p - 1$, where $1 \leq n \leq p - 1$ and a is a generator of A, which is a contradiction.

2. Let $R \cong B$. Then we claim that $\mathcal{P}(R)$ is not a regular graph. By contrary, suppose that $\mathcal{P}(R)$ is regular. Thus $deg(0) = deg(kpa)$, where $1 \leq k \leq p-1$, k is not a square modulo p and a is a generator of B, so $p^2 - 1 = 1$ which is a contradiction.

3. Let $R \cong C$. Then as the previous case, suppose that $\mathcal{P}(R)$ is a regular graph so we must have $deg(v) = deg(0)$ where $0 \neq v \in V(\mathcal{P})$, therefore $1 = p^2 - 1$, which is a contradiction.

4. Let $R \cong D$. Then if $|R| = 4$, the graph $\mathcal{P}(R)$ is regular, since the graph $\mathcal{P}(R)$ is totally disconnected. Thus $deg(v) = 0$ for every $v \in V(\mathcal{P})$. If $|R| \neq 4$, then obviously $deg(0) \neq deg(v)$ in which $0 \neq v \in V(\mathcal{P})$.

5. Let $R \cong E$. Then in $\mathcal{P}(R)$ we have $1 = deg(ia + jb)$, where a and b are two generators of R, $1 \leq i, j \leq p-1$, $i + j = p$ and $deg(0) = p - 1$. Suppose that $\mathcal{P}(R)$ is a regular graph so we obtain $p = 2$, on the other hand if $|R| = 4$ then $0 = deg(a)$, where a is a generator of R so $deg(0) \neq deg(a)$, which is a contradiction.

6. Let $R \cong H$. Then if $p = 2$, then $\mathcal{P}(R)$ is two copies of K_2. Thus it is regular. If $p \neq 2$, then $\mathcal{P}(R)$ contains a star as a connected component. Hence, $\mathcal{P}(R)$ is not regular.

7. Let $R \cong K$. Then 0 in $\mathcal{P}(R)$ represents an isolated vertex, on the other hand, $\mathcal{P}(R)$ is not a completely disconnected graph so $\mathcal{P}(R)$ is not regular.

□

Theorem 4.3. *Let R_1 and R_2 be two rings of order p or q^2. Then the graph $\mathcal{P}(R_1).\mathcal{P}(R_2)$ is regular if and only if it is totally disconnected.*

Proof. By Theorems 2.1 to 2.3 and Theorem 2.5, this graph is totally disconnected or contains some isolated vertices in addition some other components so, it is regular if and only if it is totally disconnected. □

In the next theorems, we determine $\mathcal{P}(R)$, where R is a ring of order p or p^2 is split.

Theorem 4.4. *Let R be a ring of order p. Then $\mathcal{P}(R)$ is a split graph if $R \cong N_p$.*

Proof. It is straightforward. □

Now, we mention a useful theorm to prove the next theorem.

Theorem 4.5. *(See [6, Theorem 6.7]). Consider an undirected graph $G(V(G), E(G))$ with a degree sequence $d_1 \geq d_2 \geq ... \geq d_n$. Let $m = \max\{i | d_i \geq i - 1\}$. The graph G is a split graph if and only if $\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m}^{n} d_i$.*

Theorem 4.6. *Let R be a ring of order p^2. Then $\mathcal{P}(R)$ is a split graph if and only if $R \cong B, D$ or E and $|R| = 4$, $R \cong B$ and $|R| = 9$, or $R \cong C$ or K.*

Proof. Consider the following possible cases:

1. Let $R \cong A$. Then by notation of Theorem 4.4, we have $m = p^2 - 2p + 2$ and

$$\begin{aligned}
d_1 &= p^2 - p - 1 \geq ... \geq d_{\varphi(\varphi(p^2))+1} = p^2 - p - 1 \geq d_{\varphi(\varphi(p^2))+2} = m \\
&\geq ... \geq d_{m-p+1} = m \geq d_{m-p+2} = m - 1 \\
&\geq ...d_m = m - 1 \geq d_{r_1} \geq ... \geq d_{r_{p^2-p-m}} \geq d_{s_1} = p - 1 \geq \\
d_{s_2} &= 1 \geq ... \geq d_{s_p} = 1,
\end{aligned}$$

where
$$\sum_{i=1}^{p^2-p-m} d_{r_i} = (p-2)(m-p)+1,$$
therefore, $\sum_{i=1}^{m} d_i - \sum_{i=m+1}^{n} d_i \neq m(m-1)$ since otherwise we obtain $p^3 - p^2 = -4$ which is not valid. Thus by Theorem 4.4, $\mathcal{P}(R)$ is not split.

2. Let $R \cong B$ and $|R| = 4$ or 9. Then it is easy to check that $\mathcal{P}(R)$ is split. For $p \geq 5$ by notation of Theorem 4.4, we have
$$d_1 = p^2 - 1 \geq d_2 = 2p+1 \geq ... \geq d_m = 2p+1 \geq d_{m+1} = 2 \geq ... \geq$$
$$d_{m+p^2-p} = 2 \geq d_{s_1} = 1 \geq ... \geq d_n = d_{s_{|A|}} = 1,$$
where $A = \{kpa \mid 0 \leq k \leq p-1,\ \text{k is not square modulo } p\}$. It implies that $\sum_{i=1}^{m} d_i - \sum_{i=m+1}^{n} d_i \neq m(m-1)$ so, $\mathcal{P}(R)$ is not split.
3. Let $R \cong C$. Then the vertex 0 consider as a clique and all other vertices form an independent set.
4. Let $R \cong D$ and $|R| = 4$, then the graph $\mathcal{P}(R)$ is totally disconnected, so we consider one vertex as a clique and all other vertices as an independent set so, $\mathcal{P}(R)$ is split. Let $R \cong E$ and $|R| = 4$, then $\mathcal{P}(R)$ has two isolated vertices and a component K_2, so the graph $\mathcal{P}(R)$ is split. Obviously, in this case for other orders of R we can not split $V(\mathcal{P})$ into a clique and an independent set so $\mathcal{P}(R)$ is not split by definition.
5. Let $R \cong H$. Then $\mathcal{P}(R)$ is not split.
6. Let $R \cong K$. Then the isolated vertex 0 mentioned as an independent set along with other components is a clique, so $\mathcal{P}(R)$ is split.
7. Let $R \cong B$ and $|R| = 9$. Then $0(3a) \in E(\mathcal{P}(R))$ is a clique and other vertices form an independent set.

□

Theorem 4.7. *Let R_1 and R_2 be considered as two rings with the orders p or q^2. Then the graph $\mathcal{P}(R_1).\mathcal{P}(R_2)$ is split if and only if one of the following statements holds:*
(1) $\mathcal{P}(R_1).\mathcal{P}(R_2)$ is empty.
(2) Either $R_1 \cong \mathbb{Z}_3$ and $R_2 \cong N_2$, or $R_1 \cong \mathbb{Z}_2$.
(3) $R_1 \cong \mathbb{Z}_p$ and $R_2 \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$.
(4) $R_1 \cong A$, $R_2 \cong N_2$ and $|R_2| = 4$.
(5) $R_1 \cong A$, $R_2 \cong \mathbb{Z}_q$, $|R_1| = 4$ and $q = 2, 3$ or 5.

Proof. By Theorems 2.1 to 2.3 and Theorem 2.5, it is obvious that in every above parts we have a split graph, otherwise the graph is not split. □

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