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Bounds on the Minimum Edge Dominating Energy in Terms of Some Parameters of a Graph

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Abstract

The minimum edge dominating energy, denoted by $EE_F(G)$, is the sum of the absolute values of eigenvalues of the minimum edge dominating matrix of graph G. In this paper, we give some bounds and sharp bounds of $EE_F(G)$ in terms of matching number, the number of positive eigenvalues of the minimum edge dominating matrix, and the rank of G.

Keywords: Minimum edge dominating energy, Eigenvalue, Matching number, Rank.

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1. Introduction

One of the most important pieces of information about a conjugated molecule is total π-electron energy which is computed using the Hückel theory. Results on graph energy assume special significance [1]. Mathematical modeling based on graph theory is used for studying and computing total π-electron energy. According to this modeling, a molecular graph is a graph such that the vertices correspond to atoms and the edges to the bonds. Due to the importance of graph energy, many studies of it as well as new concepts of energy have been defined according to the structure of a molecular graph.

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In this paper, all the graphs are simple and undirected. Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. An edge set M of G is called a matching if any two edges in M have no common vertices. If each vertex of G is incident with exactly one edge of M, then M is a perfect matching of G. The matching number of a graph G, denoted by $\mu(G)$, is the number of edges in a maximum matching. The line graph L_G of G is the graph that each vertex of it represents an edge of G and two vertices of L_G are adjacent if and only if their corresponding edges are incident in G [2].
The adjacent matrix of G, $A(G) = (a_{ij})$ is an $n \times n$ matrix, where $a_{ij} = 1$ if $v_iv_j \in E$ and $a_{ij} = 0$ otherwise. The positive (negative) inertia of graph G, denoted by $v^+(v^-)$, is the number of the positive (negative) eigenvalues of $A(G)$. The rank of G is denoted by $r = rank(A(G))$ and is defined as the number of nonzero eigenvalues of G [2].
Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A(G)$. The graph energy $E(G)$ of G, is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$ [1]. Many kinds of energies of a graph are introduced and studied [3–8]. The edge energy of a graph G, denoted by $EE(G)$, is defined as the sum of the absolute values of eigenvalues of $A(L_G)$ [9].
A subset D of V is the dominating set of graph G if every vertex of $V \setminus D$ is adjacent to some vertices in D. Any dominating set with minimum cardinality is called a minimum dominating set [10]. Let G be a simple graph with edge set $\{e_1, e_2, \ldots, e_m\}$. A subset F of E is the minimum edge dominating set of G or the minimum dominating set of L_G. The minimum edge dominating matrix of G is the $m \times m$ matrix defined by $A_F(G) := (a_{ij})$ in which

$$a_{ij} = \begin{cases} 1, & \text{if } e_i \, and \, e_j \, are \, adjacent, \\ 1, & \text{if } i = j \, and \, e_i \in F, \\ 0, & otherwise. \end{cases}$$

The minimum edge dominating energy of G is introduced and studied in [11] as following

$$EE_F(G) = \sum_{i=1}^{m} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of $A_F(G)$.
In [11], some lower and upper bounds of this energy are obtained. Movahedi in [12] established relations between the minimum edge dominating energy, and the graph energy, the edge energy and the signless Laplacian energy of a graph.
In [13], some bounds are obtained for the minimum edge dominating energy of subgraphs of a graph. Movahedi and Akhbari in [14] investigated the new bounds of the minimum edge dominating energy of graphs. In [15], some results of eigenvalues and energy from minimum edge dominating matrix in caterpillars are investigated.
In this paper, we are interested in investigating of bounds of the minimum edge dominating energy of a graph by rank, positive inertia, and some other parameters. In some cases these bounds are sharp.

2. Preliminaries

In this section, we recall some known results that will be used in the next section.

Lemma 2.1. ([12]). *Let G be a connected graph. If F is the minimum edge dominating set of graph G, then*

$$EE_F(G) \geq 2E(G) - 4v^+,$$

where v^+ is the positive inertia of $A(G)$.

Lemma 2.2. ([12]). *Let G be a bipartite graph of order n with $m \geq 1$ edges and rank r. Then*

$$EE_F(G) \geq 2\big(E(G) - r\big).$$

Lemma 2.3. ([16]). *For any graph G, $E(G) \geq 2\mu(G)$ in which $\mu(G)$ is the matching number of G.*

Lemma 2.4. ([17]). *Let G be a graph of order n. Then $E(G) \geq rank(G)$ and equality holds if and only if $G \simeq \frac{r}{2}K_2 \cup (n-r)K_1$ for some even positive integer r.*

Lemma 2.5. ([17]). *Let G be a bipartite graph with at least 4 vertices. If G is not of full rank, then $E(G) \geq r + 1$.*

Lemma 2.6. ([18]). *A connected graph G is of rank 2 if and only if it is a complete bipartite graph.*

Lemma 2.7. ([16]). *Let G be a connected graph of rank r. If G has at least one odd cycle, then $E(G) \geq \sqrt{r^2 + r - 1}$. Further if G is not of full rank, then $E(G) \geq r + \frac{1}{2}$.*

Lemma 2.8. ([12]). *Let G be a simple graph and F the minimum edge dominating set of G with cardinality k. Then*

$$EE_F(G) \leq EE(G) + k.$$

Lemma 2.9. ([19]). *For $n \times n$ matrices A and B,*

$$|rank(A) - rank(B)| \leq rank(A + B) \leq rank(A) + rank(B).$$

Lemma 2.10. ([20]). *Let G be a connected graph with no perfect matching. Then $E(G) \geq 2\mu(G) + 1$ except for $G = K_{k,k+1}$.*

Lemma 2.11. ([21]). *Let $\lambda_n, \ldots, \lambda_{s+1} \leq \bar{\lambda} \leq \lambda_s, \ldots, \lambda_1$ be the eigenvalues of a matrix M. Assume that arithmetic mean $\bar{\lambda} \geq 0$. Then*

$$E = \sum_{i=1}^{n} |\lambda_i| \geq n\bar{\lambda} \geq 0, \tag{1}$$

and

$$\sum_{i=1}^{s} \lambda_i - (n-s)\bar{\lambda} \leq E(M) \leq \Big(1 + \frac{n-s}{n}\Big) \sum_{i=1}^{s} \lambda_i. \tag{2}$$

Lemma 2.12. ([22]). *Let G be a graph of size m. Then*

$$2\sqrt{m} \leq E(G) \leq 2m,$$

with equality on the left if and only if G is a complete bipartite graph and equality on the right if and only if G is matching of m edges.

Lemma 2.13. ([23]). *Let A and B be $n \times n$ Hermitian matrices and $C = A + B$. Then*

$$\lambda_i(C) \leq \lambda_j(A) + \lambda_{i-j+1}(B), \quad 1 \leq j \leq i \leq n,$$

$$\lambda_i(C) \geq \lambda_j(A) + \lambda_{i-j+n}(B), \quad 1 \leq i \leq j \leq n,$$

where $\lambda_i(M)$ is the ith largest eigenvalue of matrix M $(M = A, B, C)$.

Lemma 2.14. ([17]). *If T is a tree with no perfect matching, then the product of its nonzero eigenvalues is at least 2.*

3. Main results

In this section, we present some bounds of the minimum edge dominating energy in terms of the rank of a graph and positive inertia.

Theorem 3.1. *Let G be a connected graph. If $\mu(G)$ and $v^+(G)$ are the matching number of G and the positive inertia of $A(G)$, respectively, then*

$$EE_F(G) \geq 4\big(\mu(G) - v^+(G)\big).$$

Proof. According to Lemma 2.1, we have $EE_F \geq 2E(G) - 4v^+$. Using Lemma 2.3, $E(G) \geq 2\mu(G)$. Therefore

$$EE_F(G) \geq 2\big(2\mu(G)\big) - 4v^+ = 4\big(\mu(G) - v^+(G)\big).$$

□

Theorem 3.2. *For any tree T with $n \geq 4$ vertices and $r = rank(T) < n$, then*

$$EE_F(G) > 2.$$

Proof. Using Lemma 2.2, we have $EE_F(G) \geq 2\big(E(G) - r\big)$. Since every tree is a bipartite graph, we consider the following cases.

Case 1: Assume that $r \geq 3$. According to the proof of Theorem 3 in [17], for $r \geq 3$, $E(G) > 1 + r$. Thus,

$$EE_F(G) \geq 2\big(E(G) - r\big) > 2(1 + r - r) = 2.$$

Therefore, the result holds.
Case 2: Let $r = 2$. Using Lemma 2.6, G is a complete bipartite graph. Since every complete bipartite graph which is the tree is a star thus, $G \simeq K_{1,n-1}$. So, G has a spectrum $\pm\sqrt{n-1}$ and 0 with the multiplicity $n-2$. Therefore, $E(G) = 2\sqrt{n-1}$.
On the other hand, according to the proof in Theorem 3 in [17], if $r = 2$ then $EE(G) \geq 1 + r$. Therefore,

$$EE_F(G) \geq 2\big(E(G) - r\big) \geq 2(1 + r - r) = 2.$$

If the equality holds in the above relations, then $2 = 2(E(G) - r)$. So, $E(G) = 1 + r = 3$. Since $E(G) = 2\sqrt{n-1}$ thus, $\sqrt{n-1} = \frac{3}{2}$ and consequently, $n = \frac{13}{4}$. Therefore, $EE_F(G) > 2$. □

The following theorem is an immediate consequence of Lemma 2.1 and Lemma 2.7.

Theorem 3.3. *Let G be a connected graph of rank r. If G has at least one odd cycle, then $EE_F(G) \geq 2\big(\sqrt{r^2+r-1} - 2v^+\big)$. Further, if G is not of full rank, then $E(G) \geq (2r+1) - 4v^+$ where v^+ is the number of positive eigenvalues of $A(G)$.*

Theorem 3.4. *Let G be a graph of order n with rank r. Then $EE_F(G) \geq 2\big(r - 2v^+\big)$ where v^+ is the number of positive eigenvalues of $A(G)$. Equality holds if and only if $G \simeq \frac{r}{2}K_2 \cup (n-r)K_1$ for some even positive integer r.*

Proof. Using Lemma 2.1 and Lemma 2.4, we get

$$EE_F(G) \geq 2\big(E(G) - 2v^+\big) \geq 2\big(r - 2v^+\big).$$

If equality holds in the above inequality, then $EE_F(G) = 2\big(E(G) - 2v^+\big) = 2\big(r - 2v^+\big)$. Thus, $E(G) = r$. According to the equality in Lemma 2.4, the result follows. □

Theorem 3.5. *Let G be a bipartite connected graph of rank r with no perfect matching. If $\mu(G)$ is the matching number of G, then*

$$EE_F(G) \geq 2\big(2\mu(G) - r + 1\big),$$

except for $G = K_{k,k+1}$.

Proof. Suppose that $rank(G) = r$ and $\mu(G)$ is the matching number of G. According to Lemma 2.2 and Lemma 2.10, we have

$$EE_F(G) \geq 2E(G) - 2r \geq 2(2\mu(G) + 1) - 2r = 2\big(2\mu(G) - r + 1\big).$$

□

Theorem 3.6. *Let G be a connected graph of order n and size m with $rank(G) = 2$. Then*

$$EE_F(G) \geq 4(\sqrt{m} - 1).$$

Proof. Let F be the minimum edge dominating set of G. According to Lemma 2.6 since $rank(G) = 2$, then G is a complete bipartite graph. By applying Lemma 2.1 and Lemma 2.12, we get

$$EE_F(G) \geq 2E(G) - 4v^+ = 2(2\sqrt{m}) - 4v^+ = 4(\sqrt{m} - v^+).$$

Since complete bipartite graph $K_{p,q}$ has spectrum $\pm\sqrt{pq}$ and 0 with the multiplicity $p + q - 2$, then $v^+(G) = 1$. So, the result is complete. □

The following result follows from Theorem 3.6.

Corollary 3.7. *If G is a complete bipartite graph $K_{n,n}$ and $n \geq 1$, then*

$$EE_F(G) \geq 4(n - 1).$$

Theorem 3.8. *Let G be a connected graph with m edges. If $\bar{\lambda}$ is the mean of eigenvalues $A_F(G)$ such that $\lambda_m, \dots, \lambda_{v^++1} \leq \bar{\lambda} \leq \lambda_{v^+}, \dots, \lambda_1$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ is a non-increasing sequence of eigenvalues of $A_F(G)$ and v^+ is the number of positive eigenvalues, then*

$$k\Big(\frac{2v^+}{m} - 1\Big) \leq EE_F(G) \leq k\Big(\frac{2m}{v^+} - 1\Big),$$

where F is the minimum edge dominating set with $|F| = k$. Equality holds if and only if $A_F(G)$ is the positive semidefinite matrix.

Proof. Let the minimum edge dominating matrix $A_F(G)$ has v^+ positive eigenvalues with the condition $\lambda_m, \dots, \lambda_{v^++1} \leq \bar{\lambda} \leq \lambda_{v^+}, \dots, \lambda_1$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ is its non-increasing sequence of eigenvalues.
According to inequality (2) in Lemma 2.11, we have

$$\sum_{i=1}^{v^+} \lambda_i - (m - v^+)\bar{\lambda} \leq EE_F(G) \leq \Big(1 + \frac{m - v^+}{m}\Big) \sum_{i=1}^{v^+} \lambda_i. \tag{3}$$

Using Theorem 1 in [11], $\sum_{i=1}^{m} \lambda_i = |F| = k$. Therefore, we have

$$\begin{aligned} EE_F(G) &= \sum_{i=1}^{m} |\lambda_i| = \sum_{i=1}^{v^+} \lambda_i - \sum_{i=1}^{v^-} \lambda_i \\ &= 2\sum_{i=1}^{v^+} \lambda_i - \sum_{i=1}^{m} \lambda_i = 2\sum_{i=1}^{v^+} \lambda_i - k. \end{aligned}$$

Thus, $\sum_{i=1}^{v^+} \lambda_i = \frac{EE_F(G)+k}{2}$. Therefore, we get for inequality of (3)

$$\frac{EE_F(G)+k}{2} - m\bar{\lambda} + v^+\bar{\lambda} \leq EE_F(G).$$

Since $\bar{\lambda} = \frac{\sum_{i=1}^{m} \lambda_i}{m} = \frac{k}{m}$ thus,

$$\frac{EE_F(G)}{2} - \frac{k}{2} + v^+\bar{\lambda} \leq EE_F(G).$$

Therefore,

$$\begin{aligned} EE_F(G) &\geq 2v^+\bar{\lambda} - k = 2v^+\big(\frac{k}{m}\big) - k \\ &= k\big(\frac{2v^+}{m} - 1\big). \end{aligned}$$

On the other hand, from the right hand of (3), we have

$$\begin{aligned} EE_F(G) &\leq \Big(1 + \frac{m - v^+}{m}\Big) \sum_{i=1}^{v^+} \lambda_i = \big(2 - \frac{v^+}{m}\big) \sum_{i=1}^{v^+} \lambda_i \\ &= 2\sum_{i=1}^{v^+} \lambda_i - \frac{v^+}{m} \sum_{i=1}^{v^+} \lambda_i \\ &= EE_F(G) + k - \frac{v^+}{m}\Big(\frac{EE_F(G)+k}{2}\Big) \\ &= EE_F(G)\Big(1 - \frac{v^+}{2m}\Big) + k\Big(1 - \frac{v^+}{2m}\Big). \end{aligned}$$

So,

$$EE_F(G)\big(\frac{v^+}{2m}\big) \leq k\big(1 - \frac{v^+}{2m}\big).$$

Therefore, we have

$$EE_F(G) \leq k\Big(\frac{2m}{v^+} - 1\Big).$$

Equality holds if and only if

$$k\Big(\frac{2v^+}{m} - 1\Big) = k\Big(\frac{2m}{v^+} - 1\Big).$$

Thus, $(v^+)^2 = m^2$. Since $v^+ \geq 0$ and $m \geq 0$, then $v^+ = m$. Therefore, all of the eigenvalues of the matrix $A_F(G)$ are positive. Consequently, $A_F(G)$ is the positive semidefinite matrix. $\square$

Corollary 3.9. *Let G be a graph with m edges. If $\bar{\lambda}$ is the mean of eigenvalues $A_F(G)$ such that $\lambda_m, \ldots, \lambda_{v^+ +1} \leq 0 \leq \bar{\lambda} \leq \lambda_{v^+}, \ldots, \lambda_1$ where λ_i's are the eigenvalues of $A_F(G)$ for $i = 1, \ldots, m$ and v^+ is the positive inertia, then*

$$EE_F(G) \geq k,$$

where F is a minimum edge dominating set of G with $|F| = k$.

In the following theorem, a sharp lower bound of the minimum edge dominating energy of G can be characterized by the rank of line graph G.

Theorem 3.10. *Let G be a graph of order n and size m. If F is the minimum edge dominating set with cardinality k, then*

$$EE_F(G) \geq rank(L_G) - k.$$

Equality holds if and only if $G \simeq kK_2 \cup (n-2k)K_1$.

Proof. Assume that $\lambda_1, \ldots, \lambda_r$ are all the nonzero eigenvalues of $A_F(G)$. So, one can consider $\lambda^{m-r}\big(\lambda^r + a_1\lambda^{r-1} + \cdots + a_r\big)$ as a characteristic polynomial of $A_F(G)$ where a_r is a nonzero integer. Using the arithmetic-geometric inequality, we get

$$\frac{|\lambda_1| + \cdots + |\lambda_r|}{r} \geq \sqrt[r]{|\lambda_1| \ldots |\lambda_r|} = \sqrt[r]{|a_r|} \geq 1. \tag{4}$$

Thus, $EE_F(G) \geq rank(A_F(G))$. Assume that $A_F(G) = A(L_G) + B$ where $B = \begin{bmatrix} \mathbf{1}_k & \mathbf{0}_{k\times m-k} \\ \mathbf{0}_{m-k\times k} & \mathbf{0}_{m-k\times m-k} \end{bmatrix}$, where $\mathbf{1}_i$ and $\mathbf{0}_i$ denoting the identity and zero matrices of order i, respectively. According to Lemma 2.9, we have

$$\begin{aligned} EE_F(G) &\geq rank(A_F(G)) = rank\big(A(L_G) + B\big) \\ &\geq \big|rank(A(L_G)) - rank(B)\big| \geq \big|rank(A(L_G))\big| - \big|rank(B)\big|. \end{aligned}$$

Since $rank(A(L_G)) \geq 0$ and $rank(B) = k \geq 0$, then $EE_F(G) \geq rank(L_G) - k$. We consider the equality case. So,

$$EE_F(G) = rank(A_F(G)) = rank(A_F(G)) - k.$$

Thus, equality holds in (4). This means that $|\lambda_1| = \cdots = |\lambda_r| = 1$ and therefore $k = |F| = \sum_{i=1}^{m} \lambda_i = r$. Hence, $rank(A_F(G)) = k$.
On the other hand, $\sum_{i=1}^{m} \lambda_i^2 = k + \sum_{i=1}^{n} d_i^2 - 2m$ in which d_i is the degree of vertex v_i in G. Since $k = k + \sum_{i=1}^{n} d_i^2 - 2m$ then, $\sum_{i=1}^{n} d_i^2 = 2m = \sum_{i=1}^{n} d_i$. Therefore, the degree of vertices in graph G is 0 or 1. Thus, $G \simeq kK_2 \cup (n-2k)K_1$. $\square$

In the following theorem, we obtain a lower bound of the minimum edge dominating energy of a connected graph G in terms of the rank of the line graph G.

Theorem 3.11. *Let G be a connected graph. If the line graph of G, L_G, is the connected bipartite graph with m' edge and $rank(L_G) = r'$, then*

$$EE_F(G) \geq \sqrt{r'^2 + 2r' - 4}.$$

Proof. Let F be the minimum edge dominating set of G with cardinality k. Let v^+ be the positive inertia of L_G. Using Lemma 2.13, for $A_F(G) = A(L_G) + B$ where $B = \begin{bmatrix} \mathbf{1}_k & \mathbf{0}_{k\times m-k} \\ \mathbf{0}_{m-k\times k} & \mathbf{0}_{m-k\times m-k} \end{bmatrix}$, such that the matrix B is defined in the proof of Theorem 3.10, we get

$$\lambda_i(A_F(G)) \geq \lambda_i(L_G) + \lambda_m(B),$$

where $1 \leq i \leq m$ and $\lambda_m(B)$ is the smallest eigenvalues of B. Note that the spectrum of B consists of 1 with the multiplicity k and 0 with the multiplicity $m - k$. Thus, $\lambda_m(B) = 0$. Hence, we can consider

$$2\sum_{i=1}^{v^+} \lambda_i(A_F(G)) \geq 2\sum_{i=1}^{v^+} \lambda_i(L_G).$$

Using the above inequality, we get

$$\begin{aligned} EE_F(G) &= \max_{1\leq k\leq m} \sum_{i=1}^{k} \lambda_i(A_F(G)) \geq 2\sum_{i=1}^{v^+} \lambda_i(A_F(G)) \\ &\geq 2\sum_{i=1}^{v^+} \lambda_i(L_G). \end{aligned}$$

Since L_G is a bipartite graph, then $v^+ = \frac{rank(L_G)}{2} = \frac{r'}{2}$. Therefore,

$$\begin{aligned} EE_F^2(G) &\geq \Big(2\sum_{i=1}^{v^+} \lambda_i(L_G)\Big)^2 = 4\Big(\sum_{i=1}^{v^+} \lambda_i^2(L_G) + \sum_{i\neq j} \lambda_i\lambda_j\Big) \\ &= 4\big(m' + v^+(v^+ - 1)\theta\big), \end{aligned}$$

where m' is the number of edges of L_G and θ is the arithmetic mean of $\{\lambda_i\lambda_j\}_{i\neq j}$. The geometric mean of $\{\lambda_i\lambda_j\}_{i\neq j}$ is

$$\big(\prod_{i\neq j} \lambda_i\lambda_j\big)^{\frac{1}{v^+(v^+-1)}} = \big(\lambda_1^2 \dots \lambda_{v^+}^2\big)^{\frac{1}{v^+}}.$$

Since L_G is connected, $m' \geq r' - 1$ and $\lambda_1^2 \dots \lambda_{v^+}^2 \geq 1$. So, we get

$$\begin{aligned} EE_F(G) &\geq \sqrt{4m' + r'(r' - 2)\sqrt[r']{\big(\lambda_1^2 \dots \lambda_{v^+}^2\big)^2}} \\ &\geq \sqrt{(r' + 1)^2 - 5} = \sqrt{r'^2 + 2r' - 4}. \end{aligned}$$

□

Theorem 3.12. *Let G be a connected graph with n vertices and m edges such that the line graph of G of order $m \geq 4$ and size m'. If L_G is a connected bipartite graph and rank$(L_G) = r' < m$, then*

$$EE_F(G) > r'.$$

Proof. According to the proof from Theorem 3.11, we have

$$EE_F(G) \geq \sqrt{4m' + r'(r'-2)\sqrt[r']{\left(\lambda_1^2 \dots \lambda_{v^+}^2\right)^2}},$$

where $\lambda_1, \dots, \lambda_{v^+}$ are the positive eigenvalues of L_G and $v^+ = \frac{r'}{2}$. We consider the following cases.
Case 1: Suppose that L_G is a tree. Then by Theorem 8.1 in [24] and Lemma 2.14, G has no perfect matching and $\lambda_1^2 \dots \lambda_{v^+}^2 \geq 2$. Hence,

$$\begin{aligned} EE_F(G) &\geq \sqrt{4(m-1) + r'(r'-2)\sqrt[r']{4}} \\ &\geq \sqrt{4r' + r'(r'-2)\sqrt[r']{4}}. \end{aligned}$$

If $r' \geq 3$, then $\sqrt[r']{4} = (4)^{\frac{1}{r'}} > 1 + \frac{1}{r'} \geq 1 + \frac{1}{r'(r'-2)}$. Therefore,

$$\begin{aligned} EE_F(G) &\geq \sqrt{4r'(r'^2 - 2r' + 1)} = \sqrt{r'^2 + 2r' + 1} \\ &= r' + 1 > r'. \end{aligned}$$

If $r' = 2$, then

$$\begin{aligned} EE_F(G) &\geq \sqrt{4(m-1) + r'(r'-2)\sqrt[r']{4}} = \sqrt{4(m-1)} \\ &= 2\sqrt{(m-1)} = 2\sqrt{r'} \geq r'. \end{aligned}$$

Case 2: If L_G is not a tree, then $m' \geq n' \geq r' + 1$. So,

$$\begin{aligned} EE_F(G) &\geq \sqrt{4m' + r'(r'-2)\sqrt[r']{4}} \geq \sqrt{4(r'+1) + r'(r'-2)\sqrt[r']{4}} \\ &\geq \sqrt{4r' + 4 + (r'^2 - 2r' + 1)} = \sqrt{(r'^2 + 2r' + 5)} \\ &\geq \sqrt{(r'+1)^2} \geq r' + 1 > r'. \end{aligned}$$

□

4. Conclusion

One spectral quantity of studies on the structure of a graph is graph energy which can bring important structural information about the graph. Nowadays, variants

of graph energy have been proposed, based on the matrices of a graph. Graph energies have various applications in the fields of science and engineering. Applications of graph energy in the chemistry of unsaturated conjugated molecules are rather numerous. Also, this set of invariants of a molecular graph related are applications in crystallography, theory of macromolecules, as well as analysis and comparison of protein sequences [25–27].
In this paper, we focused on the minimum edge dominating energy of a graph and obtained theoretical results on this invariant. The results obtained in this study help to better understand the minimum edge dominating energy for graphs for which it is difficult to obtain this energy.

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