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Reflection of Rays on Connected Flat Reflectors and Parametric Curved Mirrors

*Hossein Khorshidi** *and Afsaneh Hasani-Saleh*

Abstract

In this paper, reflections of two distinct rays and also of families of orthotomic rays, either on connected flat reflectors (as a nonsmooth surface) or on a parametric curved mirror, are investigated in 2D Cartesian plane. In this way, both situations either when the source point is at a finite distance or at infinity are considered. Although we used the usual methods in differential geometry but interestingly, in our calculations, the differential equations have not been used. In fact, at first, for two distinct rays, the intersection point of the reflected rays (which under some conditions is the interference point of simultaneous pulses) is geometrically described. Moreover, for two joint flat reflectors, conditions by which the intersection point (or image) will be in front of the reflector, are computed, such that they give us an interval for the place of incident point on the second reflector. In the continuation, considering orthotomic families of rays, the locus of interference points of reflected rays on two joint flat walls is obtained. Then, by the obtained results of two distinct rays (as a movement from discrete to continuous family), it is shown that the caustics of a family of reflected rays on a parametric curved mirror can be obtained. Finally, finding the caustic for a curved reflector which has self-intersection, and a theoretical idea to find the shape of an unknown mirror, for a given source and image curve, are described.

Keywords: Reflection of rays, Nonsmooth reflector, Parametric curved mirror, Caustic of curve.

2020 Mathematics Subject Classification: 53A04, 78A05, 78A40.

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1. Introduction

Investigations about directed rays in a 2D Cartesian plane and their reflections and refractions in different media from a geometric viewpoint are an important part of geometric optics. According to the famous law due to Snell (which was first accurately described by the Persian scientist Ibn Sahl at the Baghdad court in 984), the angle of incidence is equal to the angle of reflection and it is well known as Fermat's principal that the incident ray, normal and the reflected ray lie in the same plane. Some of the classical results about a family of rays are explained for example in [1–4], in which the Malus-Dupin theorem states that in isotopic homogenous media, the orthotomic family of rays emitted by a luminous point source remains orthotomic after reflection on a smooth surface, or refraction through a smooth surface. Bhattacharjee in [5] introduced the generalized vector laws of reflection and refraction. By this new formulation, he described in [6] the cases of reflection of a plane wavefront of light by plane as well as spherical reflecting surfaces. It should be mentioned that in order to make some proofs simpler, we use the notions of incident and reflected angles as in [7], which will be defined below.

Berry in [8, 9] has investigated the Inflection reflection in some mirrors which are special surfaces of the form $z = f(x, y)$ with the paraxial approximations. Although, in the following, we consider the parametric curves in a 2D plane, the results can be used for cylindrical reflectors, whose cross-sections are parametric curves, and the method of computation may be generalized to parametric surfaces in 3D space. But we will disregard cases of 3D space in this paper.

Here the reflection of rays on connected flat reflectors (as a nonsmooth surface) and on a parametric curved mirror are considered. We assume that the mirror is smooth and perfectly reflecting so that refraction, absorption, or dispersion of beam itself does not occur. In the following sections, we will study situations:

(1) when the rays are sent from a source point;

(2) when the incident rays are parallel before striking the reflector (or in other words, the source is at infinity).

In each situation we consider cases:

a. when two distinct incident rays strike connected flat reflectors;

b. when two distinct incident rays strike a parametric curved mirror;

c. when an orthotomic family of rays strikes a parametric curved mirror (a family of rays possessing an orthogonal wavefront is called orthotomic). In this study, the wavefronts may be spherical or flat according to situations (1) or (2), respectively.

For each case, after the first reflection of rays in a reflector, an intersection point of the reflected rays, which under some conditions will be the interference point of simultaneous pulses, is geometrically described. In this paper, we use the terminology "interference point" as "the interference point of simultaneous pulses".

The results for cases (1a) and (2a) that give us information about a reflections of two distinct rays on two flat connected reflectors can be used to describe some results about a reflection of an orthotomic family of rays on such reflectors. Furthermore, after the results for cases (1b) and (2b) in which we study the reflections of two distinct rays on a curved mirror, a question arises that "Is it possible to find the results for the orthotomic family of rays as the limits of results for the discrete family?" The answer is "Yes" as it is described in the following.

This paper is organized as follows. In Section 2, cases (1a) and (2a) will be discussed and then the locus of interference points for some orthotomic families of incident rays on two joint walls will be found in this section. In Section 3, using some results of Section 2, cases (1b) and (2b) will be considered and similar facts to previous cases for two distinct incident rays, which strike a parametric curved mirror, are described. In Section 4, by using and generalizing an idea of Mital in [10], it will be shown that by limits of the results in previous sections (and without using differential equations), it is possible to obtain the caustics of a family of reflected rays on a parametric curved mirror. It is useful to mention that the study of caustics of smooth curves, using differential equations, as described in [11] was formed in the 17th century, and up to now many caustics of smooth curves are found as the envelopes of rays. Moreover, it is well known and proved for example in [12], that using the eikonal equation, the caustic surface consists of the locus of the principal centers of curvature of the wavefront. By our formulation for cases (1c) and (2c) in this section, some examples of caustics are illustrated, which have the same shapes as shown in [13–18], or in www.mathcurve.com/courbes2d.gb/caustic, or in mathworld.wolfrom.com/topics/causticCurves.html. In Section 5, considering a parametric curved mirror that has a self-intersection for which a part of the mirror shadows on another part, finding the caustic is explained. Finally, in Section 6, a theoretical idea to find the shape of an unknown mirror, for a given source and image curve is described.

2. Connected flat reflectors

Suppose that the reflector consists of two connected flat walls, which have an angle α at the joint point (i.e. α is the angle between the normals to these walls). Here we will consider the reflection of two distinct rays on it and we will obtain some facts about their crossing. We assume that two mirrors are flat and perfectly reflecting so that refraction, absorption, or dispersion of beam itself does not occur. In the following, we will describe situations:

(1) when the rays are sent from a source;

(2) when the incident rays are parallel before striking the reflector.

In each case, we will describe the intersection point and some interference conditions when the rays are in the directions of propagation of two distinct pulses of waves.

2.1 Case (1a): Interference of two distinct rays which are sent from a source point S

At first, we consider the case in which two distinct rays SP and SP_0 are sent from a source point $S = (x_S, y_S)$ to a reflector that consists of two conjoined flat reflectors PA and AP_0 as in Figure 1 make an angle α at the joint A (i.e. α is the angle between the normals to these walls). Let ψ be the angle between these two rays and SP strike the first reflector with angle θ (note that here we use the notions of incident and reflected angles as in [7] and therefore θ is the angle from the incident ray SP to the flat reflector). Then the ray SP_0 strikes the second reflector with angle $\measuredangle SP_0A = \alpha + \theta - \psi$ (note that in triangle $^\triangle SPB$ we have $\varphi = \theta - \psi$). In Figure 1 points S' and S'_0 are symmetrical points to S with respect to normals N and N_0, respectively. It is well known that the reflected rays from points P and P_0 pass through S' and S'_0, respectively. The reflected ray from P_0 makes an angle with the line PA which equals to $2\alpha+\theta-\psi$. This ray geometrically crosses the reflected ray from P at point C in front of the reflector if and only if $2\alpha + \theta - \psi > \theta$, i.e.

$$2\alpha > \psi. \tag{1}$$

Let x-axis be in the direction of vector $\overrightarrow{PA}$ and make a right-hand frame with the y-axis in the plane. Let $P = (x_P, y_P)$, $a = |\overrightarrow{PA}|$ and $b_0 = |\overrightarrow{AP_0}|$. Then,

$$P_0 = (x_P + a + b_0 \cos\alpha, y_P + b_0 \sin\alpha).$$

Hence, the equations of lines $\overleftrightarrow{PC}$ and $\overleftrightarrow{P_0C}$ may be written and the coordinates of the intersection point C can be obtained as follows

$$\begin{cases} x_C = x_P + \cos\theta \left(\frac{a\sin(2\alpha+\theta-\psi)+b_0\sin(\alpha+\theta-\psi)}{\sin(2\alpha-\psi)}\right), \\ y_C = y_P + \sin\theta \left(\frac{a\sin(2\alpha+\theta-\psi)+b_0\sin(\alpha+\theta-\psi)}{\sin(2\alpha-\psi)}\right). \end{cases} \tag{2}$$

Now if two distinct pulses simultaneously are sent from the source S and these rays illustrate the directions of propagation of these two wavefronts, then point C will be the interference point of the waves if

$$|\overrightarrow{SP}| + |\overrightarrow{PC}| = |\overrightarrow{SP_0}| + |\overrightarrow{P_0C}|. \tag{3}$$

The above equation makes an interference condition in this case and we will try to describe it by 5 parameters a, b_0, θ, α, and ψ. By projecting vectors $\overrightarrow{SP}$, $\overrightarrow{SP_0}$, $\overrightarrow{PC}$ and $\overrightarrow{P_0C}$ on the lines $\overleftrightarrow{PA}$ and N, the following relations will be found,

$$|\overrightarrow{SP_0}|\cos(\theta-\psi) = |\overrightarrow{SP}|\cos\theta + a + b_0\cos\alpha, \tag{4}$$

$$|\overrightarrow{SP_0}|\sin(\theta-\psi) = |\overrightarrow{SP}|\sin\theta - b_0\sin\alpha, \tag{5}$$

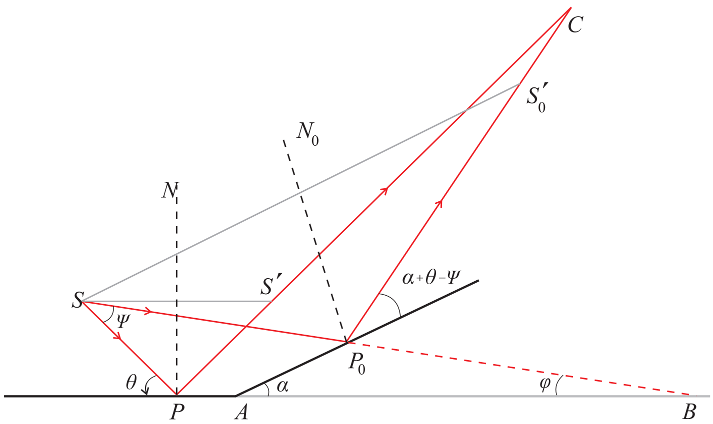

Figure 1: The reflection of two distinct rays which are sent from a source S and have a crossing point C in front of the reflector that consists of two flat reflectors. The angle α between these reflectors, which is shown at the joint point A, is the angle between two normals N and N_0. The points S' and S_0' through which pass reflected rays are symmetrical points to S with respect to normals N and N_0, respectively.

$$|\overrightarrow{PC}|\cos\theta = a + b_0\cos\alpha + |\overrightarrow{P_0C}|\cos(2\alpha+\theta-\psi), \tag{6}$$

$$|\overrightarrow{PC}|\sin\theta = b_0\sin\alpha + |\overrightarrow{P_0C}|\sin(2\alpha+\theta-\psi). \tag{7}$$

It is possible to find the values of $|\overrightarrow{SP}|$, $|\overrightarrow{SP_0}|$, $|\overrightarrow{PC}|$ and $|\overrightarrow{P_0C}|$ from the above equations and after some easy (but long) calculations by substituting these values in the interference condition (3) the following relation between a, b_0, θ, α and ψ will be obtained

$$\begin{aligned}&\tfrac{b_0}{4}[\sin 2\alpha + \sin(2\theta-2\psi) - \sin(2\alpha+2\theta-2\psi)]\\ &= a\,\sin\alpha\sin\theta\sin(\theta-\psi).\end{aligned} \tag{8}$$

By this relation, each of the values a, b_0, θ, α, and ψ may be computed with respect to 4 other values and for a given source point S the geometric behavior of the reflections of these two pulses, where C is the interference point, can be verified. For example, let the position of the reflector in the plane and values a, ψ and points S and P be given and let C be a given point on the reflected ray from P, then by Equations (2) and (8) we can find the angle α of the walls and the position of point P_0 on the second wall such that the reflected ray from P_0 passes through C as the interference point of two pulses. Note that if $\theta \neq \psi$, then the relation (8) implies

$$b_0 = \frac{a\sin\theta}{\sin(\theta+\alpha-\psi)}. \tag{9}$$

The above discussion leads us to the following fact.

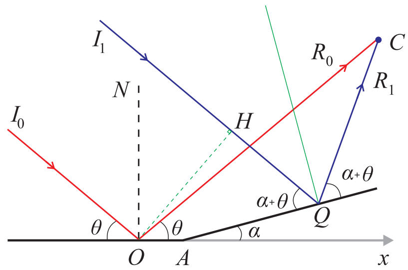

Figure 2: The reflection of two distinct parallel rays which have a crossing point C in front of the reflector consists of two flat reflectors.

Theorem 2.1. *Suppose that two distinct rays SP and SP_0 are sent from a source point S to a reflector that consists of two connected flat reflectors PA and AP_0 that as in Figure 1 make an angle α at the connection point A. Let ψ be the angle between these two rays and SP strike the first reflector with angle θ. Let C be the intersection point of two reflected rays from P and P_0. If two distinct pulses simultaneously are sent from the source S and the rays SP and SP_0 illustrate the directions of propagation of these two wavefronts, then point C as the interference point of the waves has the following coordinates*

$$\begin{cases} x_C = x_P + a\cos\theta\left(\sin\theta\cot\left(\alpha - \frac{\psi}{2}\right) + \cos\theta\right), \\ y_C = y_P + a\sin\theta\left(\sin\theta\cot\left(\alpha - \frac{\psi}{2}\right) + \cos\theta\right). \end{cases} \tag{10}$$

Proof. It is enough to find b_0 from (8) and substitute it in (2). □

Remark 1. Note that the angle between vectors $\overrightarrow{PC}$ and $\overrightarrow{P_0C}$ is equal to $2\alpha - \psi$ and when the interference of the directions of waves happens in C, the angle between the direction of the resultant wave and the line $\overleftrightarrow{PA}$ is $\alpha + \theta - \frac{\psi}{2}$.

2.2 Case (2a): Interference of two distinct rays which are parallel before reflection

Now we consider the case in which two distinct parallel rays I_0 and I_1 strike a reflector that consists of two connected flat reflectors OA and AQ that as in Figure 2 make an angle α at the connection point A. Let the x-axis coincide with $\overrightarrow{OA}$ such that O is the origin and $a = |OA|$. If the incident angle at point O on the first wall is θ, then the incident angle at point Q on the second wall will be $\theta + \alpha$. Moreover, note that the angle between vectors $\overrightarrow{OC}$ and $\overrightarrow{QC}$ is equal to 2α and when C is the interference of the directions of waves, the angle between the direction of the resultant wave and the line $\overleftrightarrow{OA}$ is $\alpha + \theta$.

Theorem 2.2. *Suppose that $0 < \theta < \frac{\pi}{2}$ and $0 < \theta + \alpha < \pi$. Then for a given value of a, the intersection point C of the reflected rays from the points O and Q will be in front of the reflector if and only if there exists $x_1 > 0$ such that*

$$\frac{a \sin\theta + x_1 \sin(\theta - \alpha)}{\sin 2\alpha} > 0, \tag{11}$$

and

$$\frac{a \sin(\theta + 2\alpha) + x_1 \sin(\theta + \alpha)}{\sin 2\alpha} > 0, \tag{12}$$

and $x_1 = |AQ|$. In this situation if the distance $|AQ|$ is equal to

$$b_1 := \frac{a \sin\theta}{\sin(\theta + \alpha)}, \tag{13}$$

then point C will be the interference point of two simultaneous pulses in the directions of I_0 and I_1 and has the following coordinates

$$\begin{cases} x_C = (\frac{a\sin(\theta+\alpha)}{\sin\alpha}) \cos\theta, \\ y_C = (\frac{a\sin(\theta+\alpha)}{\sin\alpha}) \sin\theta. \end{cases} \tag{14}$$

Proof. At first, note that if we draw the perpendicular OH to I_1 (see Figure 2), then the behavior of two simultaneous pulses in the directions I_0 and I_1 can be verified by comparing the lengths of segments $|OC|$ and $|HQ| + |QC|$. In this way, we need to find the equations of some lines and the coordinates of some points. In fact, in this case, the interference condition is

$$|OC| = |HQ| + |QC|. \tag{15}$$

Now, let $x_1 = |AQ|$, then the equations of the lines in the right-hand coordinate system in Figure 2, where the y-axis coincides with the normal N, are as follows:

$$\overleftrightarrow{OH}: \quad y = (\cot\theta)x, \tag{16}$$

$$\overleftrightarrow{OC}: \quad y = (\tan\theta)x, \tag{17}$$

$$\overleftrightarrow{QC}: \quad y - x_1 \sin\alpha = \tan(\theta + 2\alpha)(x - x_1 \cos\alpha - a), \tag{18}$$

$$\overleftrightarrow{HQ}: \quad y - x_1 \sin\alpha = -\tan\theta.(x - x_1 \cos\alpha - a). \tag{19}$$

We can find the coordinates of points H and C, by considering the intersecting lines passing through these points. In fact, since $H \in \overleftrightarrow{OH} \cap \overleftrightarrow{HQ}$ we have

$$\begin{cases} x_H = (x_1 (\sin\alpha + \cos\alpha\tan\theta) + a\tan\theta) \sin\theta\cos\theta, \\ y_H = (x_1 (\sin\alpha + \cos\alpha\tan\theta) + a\tan\theta) \cos^2\theta. \end{cases} \tag{20}$$

Similarly, since $C \in \overleftrightarrow{OC} \cap \overleftrightarrow{QC}$ its coordinates are as follows

$$\begin{cases} x_C = (\frac{x_1\sin(\theta+\alpha)+a\sin(\theta+2\alpha)}{\sin 2\alpha}) \cos\theta, \\ y_C = (\frac{x_1\sin(\theta+\alpha)+a\sin(\theta+2\alpha)}{\sin 2\alpha}) \sin\theta. \end{cases} \tag{21}$$

In this coordinate system we have $A=(a,0)$ and therefore $Q=(a+x_1\cos\alpha, x_1\sin\alpha)$ that follows

$$\left(\overline{HQ}\right)^2=\left(x_Q-x_H\right)^2+\left(y_Q-y_H\right)^2, \tag{22}$$

in which $\overline{HQ}$ is the signed distance from H to Q, that means $\overline{HQ}=-\overline{QH}$. Hence

$$\overline{HQ}=a\cos\theta+x_1\cos(\theta+\alpha). \tag{23}$$

Similarly, for QC we have

$$\left(\overline{QC}\right)^2=\left(x_C-x_Q\right)^2+\left(y_C-y_Q\right)^2. \tag{24}$$

Therefore, the singed distance from Q to C is

$$\overline{QC}=\frac{a\sin\theta+x_1\sin\left(\theta-\alpha\right)}{\sin 2\alpha}. \tag{25}$$

By comparing this with (11), we find out that condition (11) means $\overline{QC}>0$. Note that if $\overline{QC}>0$ then the interference point C will be in front of the reflector but if $\overline{QC}=0$ or $\overline{QC}<0$ then geometrically point C will be on the reflector or behind of it, respectively, and physically we have no interference point. In Example 2.3 for some values of parameters the situations are illustrated. Now for $\overline{OC}$, we have

$$\left(\overline{OC}\right)^2=\left(x_C-x_O\right)^2+\left(y_C-y_O\right)^2, \tag{26}$$

and the signed distance from O to C is

$$\overline{OC}=\frac{a\sin(\theta+2\alpha)+x_1\sin(\theta+\alpha)}{\sin 2\alpha}. \tag{27}$$

Similarly, condition (12) means $\overline{OC}>0$, that is, the interference point C will be in front of the reflector, but if $\overline{OC}=0$ or $\overline{OC}<0$ then geometrically point C will be on the reflector or behind of it, respectively, and physically we have no interference point. Now we define the function

$$\begin{aligned} f(x_1) \quad &= \overline{HQ}+\overline{QC}-\overline{OC} \\ &= a\cos\theta+x_1\cos(\theta+\alpha) \\ &\quad +\tfrac{a\sin\theta+x_1\sin(\theta-\alpha)}{\sin 2\alpha} \\ &\quad -\tfrac{x_1\sin(\theta+\alpha)+a\sin(\theta+2\alpha)}{\sin 2\alpha}. \end{aligned} \tag{28}$$

It is clear that if the interference point exists, then for some positive x_1 we have $f(x_1)=0$, (remember that $x_1=|AQ|$). Since for some values of θ and α some of the Equations (16)-(19) may be changed or some of the signed distances $\overline{HQ}$, $\overline{OC}$ or $\overline{QC}$ may be positive or negative, we can separate the discussion into the following subcases

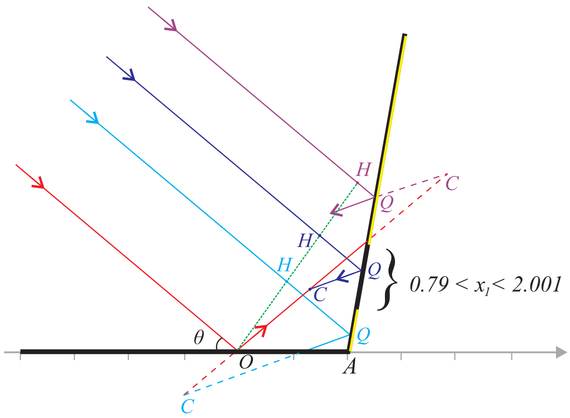

Figure 3: For a fixed ray that is reflected at point O on the first wall of the reflector, the crossing point C may be in front of or behind the reflector, depending on the place of point Q on the second wall. If the distance $x_1 = |AQ|$ is in the interval defined by Theorem 2.2, the crossing point C may have the role of the interference point and will be in front of the reflector.

1) $0 < \theta + 2\alpha < \frac{\pi}{2}$,
2) $\theta + 2\alpha = \frac{\pi}{2}$,
3) $\theta + \alpha = \frac{\pi}{2}$,
4) $\theta + \alpha > \frac{\pi}{2}$,
5) $\alpha = \frac{\pi}{2}$.

By verifying each of these situations it can be found that the answer of the linear equation $f(x_1) = 0$ is

$$x_1 = \frac{a \sin\theta}{\sin(\theta + \alpha)},$$

that is denoted in (13) by b_1. Although here we omitted the long calculations to compute and simplify this solution, the reader may verify the solution directly by substituting b_1 in (28). Now, the coordinates of the point C that is obtained in (21) for $x_1 = b_1$ will be as (14) and the proof of the theorem is completed. □

Example 2.3. In the above theorem, conditions (11) and (12) define an interval that for each x_1 in it the interference point C will be in front of the reflector. Here this interval will be computed for two cases.

a) With the above assumptions, let $\alpha = 80°$, $\theta = 40°$ and $a = 2$. The condition (11) implies that $0.79 < x_1$, otherwise the interference point is behind the second wall and the condition (12) implies that $x_1 < 2.001$, otherwise the interference point is behind the first wall (see Figure 3). By using the formula (13) we have $b_1 \approx 1.483$ and clearly $0.79 < x_1 = b_1 < 2.001$ that satisfies the conditions.

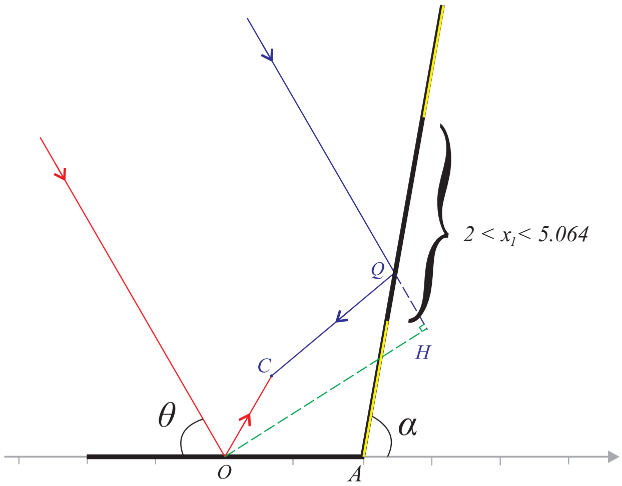

Figure 4: When the distance $x_1 = |AQ|$ is in the interval defined by Theorem 2.2, it is possible that $\overline{HQ} < 0$ and the interference point C is in front of the reflector (see Example 2.3(b) for details).

b) Suppose that $\alpha = 80°$, $\theta = 60°$ and $a = 2$. In this case, $\overline{HQ} < 0$ and the point H is behind of the second wall (see Figure 4). The conditions (11) and (12) imply that $2 < x_1 < 5.064$, otherwise the intersection point is behind the reflector. Here $b_1 \approx 2.70$.

Remark 2. By comparing formulas of the cases (1a) and (2a), it can be found that they are very similar to each other, and if in case (1a) we put $\psi = 0$, then the similar relations will be obtained for the case (2a). Moreover, it should be mentioned that in case (1a), besides condition (1), we can formulate similar conditions like (11) and (12) in Theorem 2.2. In fact, with assumptions of Theorem 2.1 for a given value of a, the intersection point C will be in front of the reflector if and only if there exists $x_0 > 0$ such that the three following conditions hold

$$\frac{a\sin\theta + x_0\sin(\theta-\alpha)}{\sin(2\alpha-\psi)} > 0, \tag{29}$$

$$\frac{a\sin(2\alpha+\theta-\psi) + x_0\sin(\alpha+\theta-\psi)}{\sin(2\alpha-\psi)} > 0, \tag{30}$$

and $x_0 = |AP_0|$. Here we omit the details of computations.

2.3 The locus of the interference points for the orthotomic family of incident rays on two joint walls

The results of the cases (1a) and (2a) lead us to find the locus of the interference points for the family of incident rays on a reflector consisting of two joint flat walls.

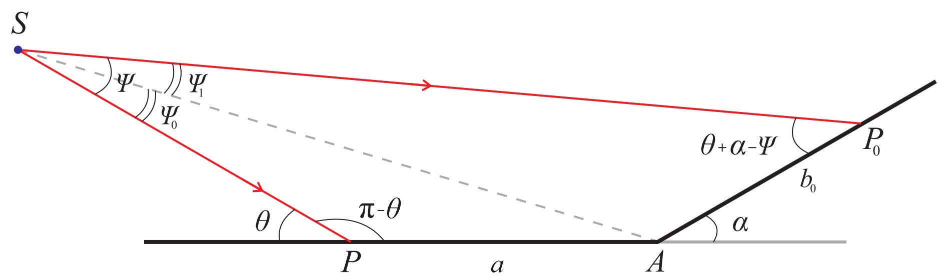

Figure 5: Incident rays SP and SP_0, which strike the reflector at points P and P_0 such that $|PA| = a$ and $|AP_0| = b_0$ as in Theorem 2.1, are symmetric with respect to $\overleftrightarrow{SA}$.

A family of rays possessing an orthogonal wavefront is called orthotomic (see for example [12]). The wavefronts that we consider in this paper, are spherical or flat, i.e. the rays are sent from a source point or they are parallel, respectively.

At first, we consider a source point $S = (x_S, y_S)$ which sends a family of orthotomic rays to the reflector. By the notations in case (1a) for Theorem 2.1 we have the following fact about the rays which make the interference point.

Theorem 2.4. *Incident rays which are sent from a source point S to the reflector (consisting of two joint flat walls) and make an interference point in front of the reflector, are symmetric with respect to the line $\overleftrightarrow{SA}$ connecting the source S to joint point A of the walls.*

Proof. By the notations of Theorem 2.1, we show that in Figure 5 angles ψ_0 and ψ_1 are equal. The law of sines in $\triangle SPA$ and $\triangle SP_0A$ respectively implies that

$$\frac{a}{\sin\psi_0} = \frac{|SA|}{\sin(\pi-\theta)}, \tag{31}$$

$$\frac{b_0}{\sin\psi_1} = \frac{|SA|}{\sin(\theta+\alpha-\psi)}, \tag{32}$$

in which we can substitute b_0 from (9) and obtain

$$\frac{a}{\sin\psi_1} = \frac{|SA|}{\sin\theta}. \tag{33}$$

Comparing this relation and (31) implies $\sin\psi_0 = \sin\psi_1$. Since both ψ_0 and ψ_1 are acute, we have $\psi_0 = \psi_1 = \frac{\psi}{2}$. By this theorem, we can better describe the behavior of a family of rays on joint flat reflectors. □

Theorem 2.5. *Suppose that an orthotomic family of rays which are sent from a source point S and bounded in angle ψ, strikes the reflector that consists of two joint flat walls (having angle α between their normals) such that one of the sides*

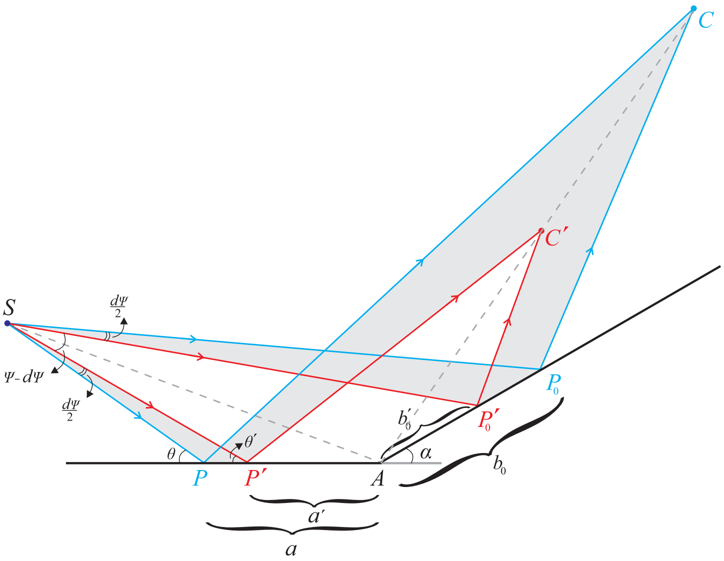

Figure 6: The interference points C and C' of the rays which are sent from a source S to the joint flat walls, are on a line passing through the joint point of the walls.

of ψ makes an angle θ with one of the walls. Then after reflection of the rays, all of the interference points are on the line

$$y - y_A = \tan(\theta + \alpha - \frac{\psi}{2})(x - x_A), \tag{34}$$

which passes throuh the joint point A of the walls.

Proof. By the result of Theorem 2.4, for simplicity, we can suppose that SA is the bisector of ψ (see Figure 6). Let SP and SP_0 be the incident rays that are edges of the angle ψ and after reflection gives the interference point C. Now consider two other incident rays SP' and SP'_0 which make a smaller angle $\psi' = \psi - d\psi$ and after reflection give another interference point C'. Since SA is the bisector of ψ' too, clearly $\measuredangle PSP' = \frac{d\psi}{2} = \measuredangle P_0SP'_0$. Then the angle between the incident ray SP' and the wall PA is $\theta' = \theta - \frac{d\psi}{2}$. The law of sines in $\triangle PSA$ and $\triangle P'SA$ respectively implies that

$$\frac{a}{\sin\frac{\psi}{2}} = \frac{|SA|}{\sin(\pi - \theta)}, \quad \frac{a'}{\sin(\frac{\psi}{2} - \frac{d\psi}{2})} = \frac{|SA|}{\sin(\theta - \frac{d\psi}{2})}.$$

Therefore,

$$a' = \frac{a \sin\theta \sin(\frac{\psi}{2} - \frac{d\psi}{2})}{\sin\frac{\psi}{2}\sin(\theta - \frac{d\psi}{2})}. \tag{35}$$

The coordinates of C' are similar to (10) and since $y_{P'} = y_P$ and $x_{P'} = x_P + a - a'$, we have

$$\begin{aligned} x_{C'} &= x_P + a - a' + a' \cos(\theta - \tfrac{d\psi}{2}) \times \left[\sin(\theta - \tfrac{d\psi}{2}) \cot(\alpha - \tfrac{\psi}{2} + \tfrac{d\psi}{2}) + \cos(\theta - \tfrac{d\psi}{2})\right], \\ y_{C'} &= y_p + a' \sin(\theta - \tfrac{d\psi}{2}) \times \left[\sin(\theta - \tfrac{d\psi}{2}) \cot(\alpha - \tfrac{\psi}{2} + \tfrac{d\psi}{2}) + \cos(\theta - \tfrac{d\psi}{2})\right]. \end{aligned}$$

The slope of the line passing through C and C' is $\frac{y_{C'} - y_C}{x_{C'} - x_C}$, in which the numerator and the denominator respectively can be calculated as follows,

$$\begin{aligned} y_{C'} - y_C &= a \sin\theta \sin(\theta + \alpha - \tfrac{\psi}{2}) \times \left[\frac{\sin(\frac{\psi}{2} - \frac{d\psi}{2})}{\sin\frac{\psi}{2} \sin(\alpha - \frac{\psi}{2} + \frac{d\psi}{2})} - \frac{1}{\sin(\alpha - \frac{\psi}{2})}\right], \\ x_{C'} - x_C &= a \sin\theta \cos(\theta + \alpha - \tfrac{\psi}{2}) \times \left[\frac{\sin(\frac{\psi}{2} - \frac{d\psi}{2})}{\sin\frac{\psi}{2} \sin(\alpha - \frac{\psi}{2} + \frac{d\psi}{2})} - \frac{1}{\sin(\alpha - \frac{\psi}{2})}\right]. \end{aligned}$$

Then, we have

$$\frac{y_{C'} - y_C}{x_{C'} - x_C} = \tan(\theta + \alpha - \frac{\psi}{2}). \tag{36}$$

Since this line passes through C, its equation reads as,

$$y - y_C = \tan(\theta + \alpha - \frac{\psi}{2})(x - x_C). \tag{37}$$

By substituting the coordinates of C from (10) in the above relation and after simplification, it follows that

$$y - y_P = \tan(\theta + \alpha - \frac{\psi}{2})(x - x_P - a), \tag{38}$$

in which $x_p - a = x_A$ and $y_P = y_A$, i.e. the line passes through A. This copmletes the proof. □

Now we consider the locus of the interference points for the orthotomic family of incident parallel rays on two joint walls. As a corollary of Theorem 2.2 and by the notations in case (2a) we have the following fact.

Theorem 2.6. *Suppose that an orthotomic family of parallel rays strikes the reflector which consists of two joint flat walls, i.e. for each ray the values θ and α are fixed. Then all of the interference points are on the line*

$$y = \tan(\theta + \alpha)(x - a), \tag{39}$$

which passes through the joint point of the walls.

Proof. By using the notations of Theorem 2.2, suppose that the reflected ray from $O_1 \in \overleftrightarrow{OA}$ for which $|O_1 A| = a_1$ gives the interference point C_1, then the coordinates of C_1 with respect to the origin O will be

$$\begin{cases} x_{C_1} = a - a_1 + \left(\frac{a_1 \sin(\theta + \alpha)}{\sin\alpha}\right) \cos\theta, \\ y_{C_1} = \left(\frac{a_1 \sin(\theta + \alpha)}{\sin\alpha}\right) \sin\theta. \end{cases} \tag{40}$$

Considering (14) and (40), it is clear that the coordinates of the interference points C and C_1 have linear relations with respect to a_1 and the line that contains both of them can be obtained as follows

$$y - y_C = \frac{y_C - y_{C_1}}{x_C - x_{C_1}}(x - x_C), \tag{41}$$

in which $\frac{y_C - y_{C_1}}{x_C - x_{C_1}} = \tan(\theta + \alpha)$, that is independent from choices of C and C_1. Therefore, the equation of the line reads as,

$$y - \left(\tfrac{a \sin(\theta+\alpha)}{\sin \alpha}\right) \sin \theta = \tan(\theta + \alpha) \left(x - \left(\tfrac{a \sin(\theta+\alpha)}{\sin \alpha}\right) \cos \theta\right). \tag{42}$$

Simplification of this relation shows that it is exactly similar to (39). Note that this line passes through the point A. The proof is completed. □

It may be interesting to verify a similar fact to the result of Theorem 2.4 for parallel rays.

Theorem 2.7. *Incident parallel rays which strike the reflector (consisting of two joint flat walls) and make an interference point in front of the reflector, are symmetric with respect to the line that is parallel to the incident rays and passes through the joint point A of the walls.*

Proof. Recall the notations of the case (2a) and suppose that the incident parallel rays strike the reflector at points O and Q such that $|OA| = a$ and $|AQ| = b_1$. Let l be the line passing through A which is parallel to the incident rays. Let OK and QK' be perpendicular to l (see Figure 7). In triangles $\triangle OAK$ and $\triangle QAK'$ respectively we have

$$|OK| = a \sin \theta, \quad |QK'| = b_1 \sin(\theta + \alpha).$$

By substituting b_1 from (13), it shows that $OK = QK'$ and therefore the incident parallel rays which make an interference point in front of the reflector, are symmetric with respect to l. □

3. The parametric curved reflector and the reflection of two distinct rays

Now suppose that we have a curved mirror defined by the parametric equation $\gamma(t) = (x(t), y(t))$ in $2D$ Cartesian plane. As in the previous section, we will consider two situations **(1)** and **(2)** for two distinct incident rays and we will obtain some facts about their crossing. In the following, we consider some notations and facts about the family of reflected rays which are needed to study the next cases.

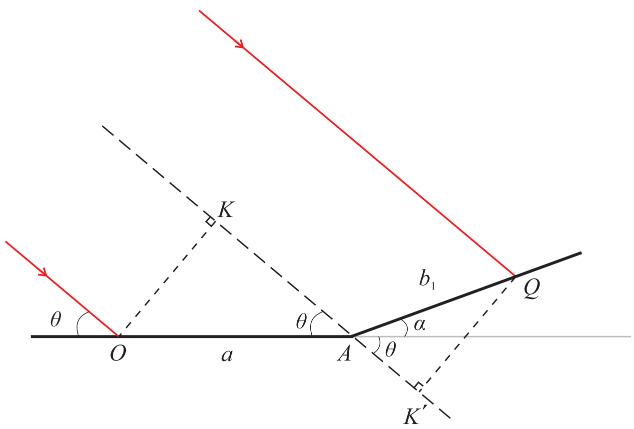

Figure 7: Incident parallel rays that strike the reflector at points O and Q such that $|OA| = a$ and $|AQ| = b_1$ as in Theorem 2.2, are symmetric with respect to the line passing through A and parallel to the incident rays.

3.1 Equation of a family of reflected rays which are sent from a source

Let $S = (x_S, y_S)$ be a point object in the plane of the mirror that as a source sends a family of rays to the mirror. We assume that the mirror is smooth and perfectly reflecting so that refraction, absorption, or dispersion of beam itself does not occur. Here we want to find the equation of a family of reflected rays that are sent from S. It should be mentioned that in this section our method generalizes the results described by Mittal in [10].

At first, We need some notations which help us to make formulas short and clear. Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and w be 3 vectors in the plane, we set

$$\begin{aligned} &|u, v| = u_1 v_2 - u_2 v_1 = \det(u, v), \\ &\langle u, v\rangle = u_1 v_1 + u_2 v_2, \\ &U_{(u,v,w)} = \begin{pmatrix} |u, v| \\ \langle u, w\rangle \end{pmatrix}. \end{aligned} \tag{43}$$

Note that in the construction of the components of $U_{(u,v,w)}$ we have a determinant in the first and an inner product in the second component. In the following, this order will be regarded in the description of important points or vectors. Moreover, the determinant $|u, v|$ is the signed area of parallelogram constructed by u and v, thus the absolute value of this determinant is equal to the length of cross product $u \times v$ when the vectors u and v are considered in 3D space. Indeed, $|u, v| = \pm|u \times v|$. Although our discussion is in the plane, this note is useful if we consider the vectors in 3D space.

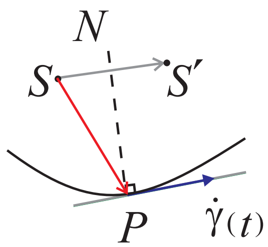

Figure 8: The positions of vectors $\overrightarrow{SS'}$ and $\overrightarrow{SP}$ with respect to the normal N and the tangent vector $\dot{\gamma}(t)$.

Now let $P = (x(t), y(t))$ be a point on the curve $\gamma(t)$ and the tangent vector $\dot{\gamma}$ be the differentiation of γ with respect to t, whose length is $|\dot{\gamma}(t)| = \sqrt{\langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle} = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}$. Let S' be the symmetrical point to S with respect to the normal N at point P on γ (see Figure 8). In order to find the equation of the family of reflected rays from mirror γ it is enough to find the equation of line $\overleftrightarrow{PS'}$.

It is clear that the vector $\overrightarrow{SS'}$ is parallel to the tangent line at P and we can consider $\frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}$ as the unit vector on this tangent line. Hence

$$S' = S + \overrightarrow{SS'}, \qquad \overrightarrow{SS'} = 2\langle\overrightarrow{SP}, \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}\rangle \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}, \tag{44}$$

in which $\langle\ ,\ \rangle$ is the inner product of vectors (as in (43)) and the value $\langle\overrightarrow{SP}, \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}\rangle$ is the component of vector $\overrightarrow{SP}$ on the tangent line and its absolute value gives the half of length of $\overrightarrow{SS'}$. Since $P = \gamma(t)$, the coordinates of $\overrightarrow{SP}$ can be found as

$$\overrightarrow{SP} = \gamma(t) - S = \begin{pmatrix} x(t) - x_S \\ y(t) - y_S \end{pmatrix}, \tag{45}$$

by which

$$S' = \frac{\langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle\, S + 2\langle\gamma(t) - S, \dot{\gamma}(t)\rangle\, \dot{\gamma}(t)}{|\dot{\gamma}(t)|^2}. \tag{46}$$

Therefore, by some easy calculations the coordinates of the point S' can be obtained as follows

$$\begin{cases} x_{S'} = \frac{\dot{x}(t)\langle\dot{\gamma}(t), 2\gamma(t)-S\rangle - \dot{y}(t)\det(\dot{\gamma}(t),S)}{|\dot{\gamma}(t)|^2}, \\ y_{S'} = \frac{\dot{y}(t)\langle\dot{\gamma}(t), 2\gamma(t)-S\rangle + \dot{x}(t)\det(\dot{\gamma}(t),S)}{|\dot{\gamma}(t)|^2}, \end{cases} \tag{47}$$

which by (43) can be written in the following shorter form

$$\begin{cases} x_{S'} = \frac{1}{|\dot{\gamma}(t)|^2} \left|\dot{\gamma}(t), U_{(\dot{\gamma},S,2\gamma-S)}\right|, \\ y_{S'} = \frac{1}{|\dot{\gamma}(t)|^2} \langle\dot{\gamma}(t), U_{(\dot{\gamma},S,2\gamma-S)}\rangle. \end{cases} \tag{48}$$

Now having two points P and S' of the line $\overleftrightarrow{PS'}$ we can obtain the equation of this line

$$\overleftrightarrow{PS'}: \quad y = mx + h, \tag{49}$$

in which for $\Gamma = \begin{pmatrix} |(\gamma(t) - S), \dot{\gamma}(t)| \\ \langle(\gamma(t) - S), \dot{\gamma}(t)\rangle \end{pmatrix}$ and $\Gamma_1 = \begin{pmatrix} |\dot{\gamma}(t), \gamma(t)| \\ \langle\dot{\gamma}(t), \gamma(t)\rangle \end{pmatrix}$ we have

$$\begin{aligned} m = m(t) &= \tfrac{y_{S'}(t)-y(t)}{x_{S'}(t)-x(t)} \\ &= \tfrac{\dot{y}(t)\langle(\gamma(t)-S),\dot{\gamma}(t)\rangle+\dot{x}(t)\det((\gamma(t)-S),\dot{\gamma}(t))}{\dot{x}(t)\langle(\gamma(t)-S),\dot{\gamma}(t)\rangle-\dot{y}(t)\det((\gamma(t)-S),\dot{\gamma}(t))} \\ &= \tfrac{\langle\dot{\gamma}(t),\Gamma\rangle}{|\dot{\gamma}(t),\Gamma|}, \end{aligned} \tag{50}$$

and

$$\begin{aligned} h = h(t) &= y(t) - mx(t) \\ &= \tfrac{\langle(\gamma(t)-S),\dot{\gamma}(t)\rangle\det(\dot{\gamma}(t),\gamma(t))-\langle\dot{\gamma}(t),\gamma(t)\rangle\det((\gamma(t)-S),\dot{\gamma}(t))}{\dot{x}(t)\langle\dot{\gamma}(t),(\gamma(t)-S)\rangle-\dot{y}(t)\det((\gamma(t)-S),\dot{\gamma}(t))} \\ &= \tfrac{|\Gamma_1,\Gamma|}{|\dot{\gamma}(t),\Gamma|}. \end{aligned} \tag{51}$$

Thus, it is proved that:

Theorem 3.1. *By the above formulation when the point P and so the normal N change on the curve $\gamma(t)$, we have the family of reflected rays from mirror $\gamma(t)$ with the equation*

$$y = m(t)x + h(t), \quad m(t) = \frac{\langle\dot{\gamma}(t), \Gamma\rangle}{|\dot{\gamma}(t), \Gamma|}, \quad h(t) = \frac{|\Gamma_1, \Gamma|}{|\dot{\gamma}(t), \Gamma|}.$$

3.2 Case (1b): Reflection of two distinct rays which are sent from a source to γ

Now, in the case that two distinct rays are sent from the source S, suppose that these two rays strike the curve in points $P = \gamma(t)$ and $P_0 = \gamma(t_0)$, such that the intersection point C is in front of the reflector (see Figure 9). The point C is on the lines $\overleftrightarrow{PS'}$ and $\overleftrightarrow{P_0S'_0}$, whose equations can be described by (49), (50) and (51) for the parameters t and t_0. Hence, the coordinates of point C satisfy the following equations

$$\begin{cases} y_C = m(t)x_C + h(t), \\ y_C = m(t_0)x_C + h(t_0), \end{cases} \tag{52}$$

Therefore,

$$x_C = \frac{h(t_0) - h(t)}{m(t) - m(t_0)}. \tag{53}$$

Furthermore, by (53), (50) and (51), it is possible to describe x_C and y_C by the points S, $\gamma(t)$, $\gamma(t_0)$ and the vectors $\dot{\gamma}(t)$ and $\dot{\gamma}(t_0)$ (see Theorem 3.2 in below).

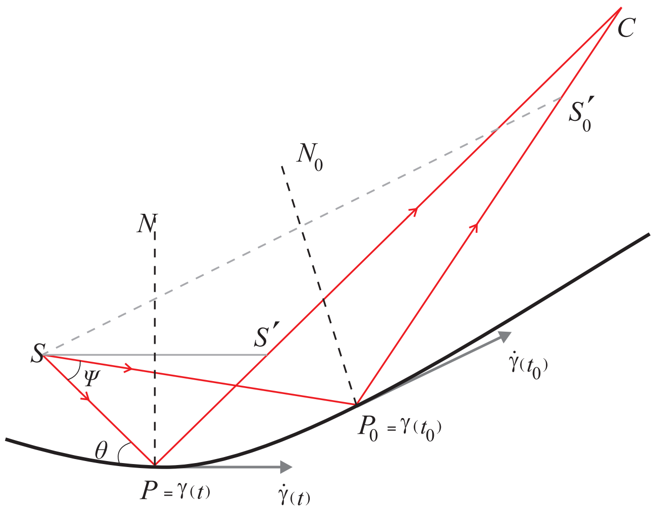

Figure 9: The reflection of two distinct rays which are sent from a source S and have a crossing point C in front of the parametric curved reflector γ.

Now if the rays illustrate the directions of propagation of two pulses, then point C will be the interference point of the waves if

$$|\overrightarrow{SP}| + |\overrightarrow{PC}| = |\overrightarrow{SP_0}| + |\overrightarrow{P_0C}|, \tag{54}$$

that is an interference condition as (3). Note that

$$|\gamma(t) - S| = |\overrightarrow{SP}| = |\overrightarrow{PS'}|,$$

and

$$|\gamma(t_0) - S| = |\overrightarrow{SP_0}| = |\overrightarrow{P_0S_0'}|.$$

Moreover, with respect to the x-coordinate

$$|\overrightarrow{PC}| = |\overrightarrow{PS'}|\frac{x_C - x(t)}{x_{S'} - x(t)},$$

and similarly

$$|\overrightarrow{P_0C}| = |\overrightarrow{P_0S_0'}|\frac{x_C - x(t_0)}{x_{S_0'} - x(t_0)}.$$

Therefore, the condition (54) can be written as follows

$$|\gamma(t) - S|\left[1 + \frac{x_C - x(t)}{x_{S'} - x(t)}\right] = |\gamma(t_0) - S|\left[1 + \frac{x_C - x(t_0)}{x_{S_0'} - x(t_0)}\right]. \tag{55}$$

Now, if we consider some notations for simplicity, as in (50), i.e. $\Gamma = \Gamma(t) = \begin{pmatrix} |(\gamma(t) - S), \dot{\gamma}(t)| \\ \langle(\gamma(t) - S), \dot{\gamma}(t)\rangle \end{pmatrix}$, $\Gamma^0 = \Gamma(t_0)$, $\gamma_0 = \gamma(t_0)$ and $\dot{\gamma}_0 = \dot{\gamma}(t_0)$, then clearly

$$\frac{1}{x_{S'} - x(t)} = \frac{|\dot{\gamma}|^2}{|\dot{\gamma}, \Gamma|},$$

in which the denominator of the right-hand fraction is the same as the denominator of m in (50). Obviously, there is a similar relation between the fractions

$$\frac{1}{x_{S'_0} - x(t_0)} = \frac{|\dot{\gamma}_0|^2}{|\dot{\gamma}_0, \Gamma^0|},$$

and $m_0 = m(t_0)$. Moreover, it can be seen that

$$x_C - x(t) = \frac{y(t_0) - y(t) - m_0\,(x(t_0) - x(t))}{m - m_0},$$

and

$$x_C - x(t_0) = \frac{y(t_0) - y(t) - m\,(x(t_0) - x(t))}{m - m_0}. \tag{56}$$

Therefore, Equation (55), after some calculations by using enumerators and denominators of m and m_0 will be as follows

$$\begin{aligned} &|\gamma - S| \left[|\dot{\gamma}|^2 \begin{vmatrix} x(t) - x(t_0) & |\dot{\gamma}_0, \Gamma^0| \\ y(t) - y(t_0) & \langle\dot{\gamma}_0, \Gamma^0\rangle \end{vmatrix} - \begin{vmatrix} |\dot{\gamma}, \Gamma| & |\dot{\gamma}_0, \Gamma^0| \\ \langle\dot{\gamma}, \Gamma\rangle & \langle\dot{\gamma}_0, \Gamma^0\rangle \end{vmatrix} \right] \\ &= \\ &|\gamma_0 - S| \left[|\dot{\gamma}_0|^2 \begin{vmatrix} x(t) - x(t_0) & |\dot{\gamma}, \Gamma| \\ y(t) - y(t_0) & \langle\dot{\gamma}, \Gamma\rangle \end{vmatrix} - \begin{vmatrix} |\dot{\gamma}, \Gamma| & |\dot{\gamma}_0, \Gamma^0| \\ \langle\dot{\gamma}, \Gamma\rangle & \langle\dot{\gamma}_0, \Gamma^0\rangle \end{vmatrix} \right], \end{aligned} \tag{57}$$

that can be written in a shorter relation as the following

$$\frac{|\gamma - S|}{|\gamma_0 - S|} = \frac{|\dot{\gamma}_0|^2\,|(\gamma - \gamma_0), \Gamma_2| - \left|\Gamma_2, \Gamma_2^0\right|}{|\dot{\gamma}|^2\,|(\gamma - \gamma_0), \Gamma_2^0| - |\Gamma_2, \Gamma_2^0|}, \tag{58}$$

in which $\Gamma_2 = \Gamma_2(t) = \begin{pmatrix} |\dot{\gamma}, \Gamma| \\ \langle\dot{\gamma}, \Gamma\rangle \end{pmatrix}$ and $\Gamma_2^0 = \Gamma_2(t_0)$.

By the above discussion we have the following description of C.

Theorem 3.2. *Suppose that two distinct rays are sent from the source point S and the reflector is a parametric planar curve $\gamma(t) = (x(t), y(t))$. Let these two rays strike the curve in points $P = \gamma(t)$ and $P_0 = \gamma(t_0)$, such that the reflected rays have the intersection point C in front of the reflector (see Figure 9). If C is the interference of two simultaneous pulses from S then C has the following coordinates*

$$\begin{cases} x_C = x(t_0) + \frac{|\Gamma_2, \gamma - \gamma_0|}{|\Gamma_2, \Gamma_2^0|}|\dot{\gamma}_0, \Gamma^0|, \\ y_C = y(t_0) + \frac{|\Gamma_2, \gamma - \gamma_0|}{|\Gamma_2, \Gamma_2^0|}\langle\dot{\gamma}_0, \Gamma^0\rangle. \end{cases} \tag{59}$$

Remark 3. The relation (58) shows that if the equation of the curve $\gamma(t)$ and the positions of S and P are given, then we may try to find the parameter t_0 such that reflected rays from $P = \gamma(t)$ and $P_0 = \gamma(t_0)$ have an intersection which is an interference point for two simultaneous pulses. For instance, if γ is a polynomial or a B-spline curve, then finding the value of t_0 by numerical methods can be considered.

3.3 Case (2b): Reflection of two distinct rays which are parallel before reflecting on γ

As the last case of reflection of two distinct incident rays, suppose that two parallel rays I_0 and I_1 strike the curved mirror that is the parametric planar curve $\gamma(t) = (x(t), y(t))$. Let $O = \gamma(t_0)$ and $Q = \gamma(t)$ be two points on the mirror that reflect the incident rays I_0 and I_1, respectively (see Figure 10). Now if we draw a horizontal line at O that is parallel to the x-axis, then we need some notations as follows

θ: the angle from I_0 to the tangent line at O, that is also the angle between the reflected ray R_0 and the tangent vector $\dot{\gamma}(t_0)$;

$\bar{\theta}$: the angle from I_0 to the horizontal line at O (that is parallel to the x-axis);

τ: the angle between the vector $\dot{\gamma}(t)$ and the x-axis that for $t = t_0$ will be denoted by τ_0;

A: or $A(t)$ is the intersection point of the tangent lines at O and Q;

α: or $\alpha(t)$ is the angle between the tangent vectors $\dot{\gamma}(t_0)$ and $\dot{\gamma}(t)$.

Remark 4. As in Figure 10, it is clear that $\theta = \bar{\theta} + \tau_0$.

Let $a(t) = \overline{OA}$. In the following lemma we show how the value of $a(t)$ can be computed by γ.

Lemma 3.3. *For the curve $\gamma(t)$ and the points $O = \gamma(t_0)$ and $Q = \gamma(t)$ we have:*

$$a(t) = \frac{|\dot{\gamma}(t_0)|\,|\dot{\gamma}(t), (\gamma(t) - \gamma(t_0))|}{|\dot{\gamma}(t), \dot{\gamma}(t_0)|}. \tag{60}$$

Proof. First, note that the lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{AQ}$ are in the directions of $\dot{\gamma}(t_0)$ and $\dot{\gamma}(t)$, respectively, and their equations can be described as follows

$$\begin{aligned} \overleftrightarrow{OA}: &\qquad y - y(t_0) = \tfrac{\dot{y}(t_0)}{\dot{x}(t_0)}(x - x(t_0)), \\ \overleftrightarrow{AQ}: &\qquad y - y(t) = \tfrac{\dot{y}(t)}{\dot{x}(t)}(x - x(t)). \end{aligned} \tag{61}$$

Since the point A is the intersection of these two lines, its coordinates satisfy both of the above equations, hence

$$\begin{aligned} y_A &= \tfrac{\dot{y}(t_0)}{\dot{x}(t_0)}x_A + \tfrac{|\dot{\gamma}(t_0), \gamma(t_0)|}{\dot{x}(t_0)}, \\ y_A &= \tfrac{\dot{y}(t)}{\dot{x}(t)}x_A + \tfrac{|\dot{\gamma}(t), \gamma(t)|}{\dot{x}(t)}, \end{aligned} \tag{62}$$

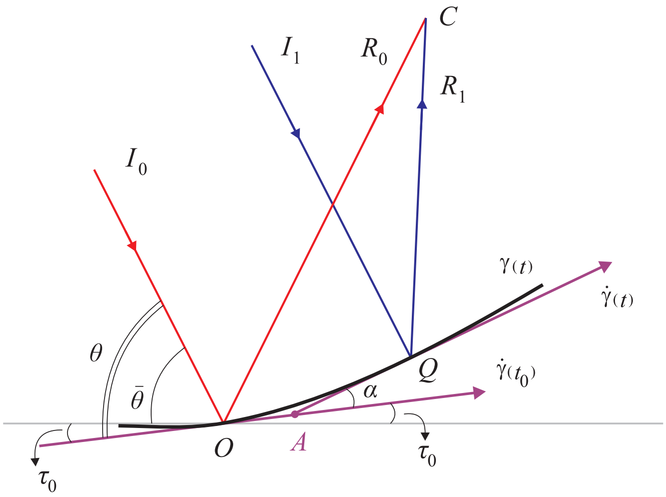

Figure 10: Angles θ, $\bar{\theta}$, τ_0 and α are shown for the reflection of two distinct rays which are parallel before incidence to the reflector and have a crossing point C in front of the parametric curved reflector γ.

in which we used the notations in (43). Therefore

$$\begin{cases} x_A = \frac{(\dot{x}(t_0)|\dot{\gamma}(t),\gamma(t)| - \dot{x}(t)|\dot{\gamma}(t_0),\gamma(t_0)|)}{|\dot{\gamma}(t),\dot{\gamma}(t_0)|}, \\ y_A = \frac{(\dot{y}(t_0)|\dot{\gamma}(t),\gamma(t)| - \dot{y}(t)|\dot{\gamma}(t_0),\gamma(t_0)|)}{|\dot{\gamma}(t),\dot{\gamma}(t_0)|}. \end{cases} \tag{63}$$

Now, we calculate the distance $\overline{OA}$ as follows

$$\begin{aligned} a^2(t) \quad &= (x_A - x(t_0))^2 + (y_A - y(t_0))^2 \\ &= \dot{x}^2(t_0)\frac{(\dot{x}(t)(y(t)-y(t_0)) - \dot{y}(t)(x(t)-x(t_0)))^2}{(\dot{x}(t)\dot{y}(t_0)-\dot{y}(t)\dot{x}(t_0))^2} \\ &+\dot{y}^2(t_0)\frac{(\dot{x}(t)(y(t)-y(t_0)) - \dot{y}(t)(x(t)-x(t_0)))^2}{(\dot{x}(t)\dot{y}(t_0)-\dot{y}(t)\dot{x}(t_0))^2} \\ &= \frac{\left(\dot{x}^2(t_0)+\dot{y}^2(t_0)\right)[\dot{x}(t)(y(t)-y(t_0)) - \dot{y}(t)(x(t)-x(t_0))]^2}{(\dot{x}(t)\dot{y}(t_0)-\dot{y}(t)\dot{x}(t_0))^2}. \end{aligned} \tag{64}$$

Thus by our notation, we have

$$a(t) = \overline{OA} = \frac{|\dot{\gamma}(t_0)|\,|\dot{\gamma}(t), (\gamma(t) - \gamma(t_0))|}{|\dot{\gamma}(t), \dot{\gamma}(t_0)|}. \tag{65}$$

□

Theorem 3.4. *By the above assumptions about the curve γ and parallel incident rays I_0 and I_1, if the reflected ray from $O = \gamma(t_0)$ intersects the reflected ray from $Q = \gamma(t)$ at point C such that C is the interference point of two pulses in the* *directions of I_0 and I_1, then*

$$\frac{2\langle T_{\bar{\theta}}, \dot{\gamma}(t_0)\rangle}{|\dot{\gamma}(t), \dot{\gamma}(t_0)|} = \frac{\langle T_{\bar{\theta}}, (\gamma(t) - \gamma(t_0))\rangle}{|\dot{\gamma}(t), (\gamma(t) - \gamma(t_0))|}, \tag{66}$$

in which $T_{\bar{\theta}} = \begin{pmatrix} \sin\bar{\theta} \\ \cos\bar{\theta} \end{pmatrix}$ *is the unit vector in the plane of the curve γ which is perpendicular to the direction of the incidents rays.*

Proof. First, note that we can consider the reflections at the points O and Q on the curve γ as the reflections from the tangent lines at O and Q. Therefore, the results in the case (2a) can be used here. Let $b(t) = \overline{AQ}$. If C is the interference point of the reflected rays in front of the mirror, then by Theorem 2.2 we have

$$b(t) = \frac{a(t)\sin\theta}{\sin(\theta + \alpha)}. \tag{67}$$

Moreover, as the sum of vectors in Figure 10, we have

$$\overrightarrow{OA} + \overrightarrow{AQ} = \overrightarrow{OQ},$$

that is

$$a(t)\frac{\dot{\gamma}(t_0)}{|\dot{\gamma}(t_0)|} + b(t)\frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|} = \gamma(t) - \gamma(t_0). \tag{68}$$

□

Remark 5. Note that since the vectors $\dot{\gamma}(t_0)$ and $\dot{\gamma}(t)$ are coplanar if $\overrightarrow{k}$ is perpendicular to the plane containing $\dot{\gamma}(t_0)$ and $\dot{\gamma}(t)$, then for their vector product we have

$$\begin{aligned} &\dot{\gamma}(t_0) \times \dot{\gamma}(t) = |\dot{\gamma}(t_0), \dot{\gamma}(t)|\, \overrightarrow{k}, \\ &|\dot{\gamma}(t_0), \dot{\gamma}(t)| = |\dot{\gamma}(t_0)|\, |\dot{\gamma}(t)| \sin\alpha. \end{aligned} \tag{69}$$

It should be mentioned that if we consider α as a signed angle from $\dot{\gamma}(t_0)$ to $\dot{\gamma}(t)$, then $\pm|\dot{\gamma}(t_0) \times \dot{\gamma}(t)| = |\dot{\gamma}(t_0), \dot{\gamma}(t)|$.

Now, we can modify the relation (67). In fact

$$b(t) \; = \frac{a(t)\sin\theta}{\sin(\theta+\alpha)} = \frac{a(t)\sin(\bar{\theta}+\tau_0)}{\sin(\bar{\theta}+\tau_0+\alpha)} = \frac{a(t)\sin(\bar{\theta}+\tau_0)}{\sin(\bar{\theta}+\tau_0)\cos\alpha + \cos(\bar{\theta}+\tau_0)\sin\alpha}. \tag{70}$$

Note that

$$\sin\alpha = \frac{|\dot{\gamma}(t_0), \dot{\gamma}(t)|}{|\dot{\gamma}(t_0)||\dot{\gamma}(t)|}, \quad \cos\alpha = \frac{\langle\dot{\gamma}(t_0), \dot{\gamma}(t)\rangle}{|\dot{\gamma}(t_0)||\dot{\gamma}(t)|}, \quad \frac{\dot{\gamma}(t_0)}{|\dot{\gamma}(t_0)|} = \begin{pmatrix} \cos\tau_0 \\ \sin\tau_0 \end{pmatrix}. \tag{71}$$

Moreover,

$$\begin{aligned} \sin\theta &= \sin\left(\bar{\theta} + \tau_0\right) = \sin\bar{\theta}\cos\tau_0 + \cos\bar{\theta}\sin\tau_0, \\ &= \left\langle \frac{\dot{\gamma}(t_0)}{|\dot{\gamma}(t_0)|}, T_{\bar{\theta}} \right\rangle, \\ \cos\theta &= \cos\left(\bar{\theta} + \tau_0\right) = \cos\bar{\theta}\cos\tau_0 - \sin\bar{\theta}\sin\tau_0, \\ &= \left| \frac{\dot{\gamma}(t_0)}{|\dot{\gamma}(t_0)|}, T_{\bar{\theta}} \right|. \end{aligned} \tag{72}$$

By substituting (71) and (72) in the denominator of the fraction in (70) we have

$$\sin(\bar{\theta}+\tau_0+\alpha)=\frac{\langle\dot{\gamma}(t_0),T_{\bar{\theta}}\rangle\langle\dot{\gamma}(t_0),\dot{\gamma}(t)\rangle+|\dot{\gamma}(t_0),T_{\bar{\theta}}||\dot{\gamma}(t_0),\dot{\gamma}(t)|}{|\dot{\gamma}(t_0)|^2|\dot{\gamma}(t)|}. \tag{73}$$

Since vectors $\dot{\gamma}(t_0)$, $\dot{\gamma}(t)$, and $T_{\bar{\theta}}$ are coplanar, the vector product of each pair of these vectors is a vector perpendicular to this plane. Therefore by our notations

$$|\dot{\gamma}(t_0),T_{\bar{\theta}}|\,|\dot{\gamma}(t_0),\dot{\gamma}(t)|=\langle\dot{\gamma}(t_0)\times T_{\bar{\theta}},\ \dot{\gamma}(t_0)\times\dot{\gamma}(t)\rangle\,, \tag{74}$$

because both of the vectors in the above inner product are on the same line. By Lagrange's identity,

$$\begin{aligned}\langle\dot{\gamma}(t_0)\times T_{\bar{\theta}},\dot{\gamma}(t_0)\times\dot{\gamma}(t)\rangle &= \begin{vmatrix}\langle\dot{\gamma}(t_0),\dot{\gamma}(t_0)\rangle & \langle\dot{\gamma}(t),\dot{\gamma}(t_0)\rangle\\ \langle\dot{\gamma}(t_0),T_{\bar{\theta}}\rangle & \langle\dot{\gamma}(t),T_{\bar{\theta}}\rangle\end{vmatrix}\\ &= |\dot{\gamma}(t_0)|^2\langle\dot{\gamma}(t),T_{\bar{\theta}}\rangle-\langle\dot{\gamma}(t),\dot{\gamma}(t_0)\rangle\langle\dot{\gamma}(t_0),T_{\bar{\theta}}\rangle\,.\end{aligned} \tag{75}$$

By substituting (75) in (73) we have

$$\sin(\bar{\theta}+\tau_0+\alpha)=\frac{1}{|\dot{\gamma}(t)|}\langle\dot{\gamma}(t),T_{\bar{\theta}}\rangle\,. \tag{76}$$

Thus, the relation (70) will be as follows

$$b(t)=\frac{a(t)\frac{1}{|\dot{\gamma}(t_0)|}\langle\dot{\gamma}(t_0),T_{\bar{\theta}}\rangle}{\frac{1}{|\dot{\gamma}(t)|}\langle\dot{\gamma}(t),T_{\bar{\theta}}\rangle}. \tag{77}$$

Now, by substituting this relation in (68) we have

$$a(t)\left(\langle\dot{\gamma}(t),T_{\bar{\theta}}\rangle\dot{\gamma}(t_0)+\langle\dot{\gamma}(t_0),T_{\bar{\theta}}\rangle\dot{\gamma}(t)\right)=|\dot{\gamma}(t_0)|\langle\dot{\gamma}(t),T_{\bar{\theta}}\rangle(\gamma(t)-\gamma(t_0))\,. \tag{78}$$

If we use the formula for $a(t)$ from Lemma 3.3 in this equation and make the inner product of both sides of the above relation with $T_{\bar{\theta}}$, then it follows that

$$\frac{|\dot{\gamma}_0||\dot{\gamma},\gamma-\gamma_0|}{|\dot{\gamma},\dot{\gamma}_0|}\left(\langle\dot{\gamma},T_{\bar{\theta}}\rangle\langle\dot{\gamma}_0,T_{\bar{\theta}}\rangle+\langle\dot{\gamma}_0,T_{\bar{\theta}}\rangle\langle\dot{\gamma},T_{\bar{\theta}}\rangle\right)=|\dot{\gamma}_0|\langle\dot{\gamma},T_{\bar{\theta}}\rangle\langle\gamma-\gamma_0,T_{\bar{\theta}}\rangle\,, \tag{79}$$

in which for simplicity we denoted $\gamma(t_0)$, $\dot{\gamma}(t_0)$, $\gamma(t)$, and $\dot{\gamma}(t)$ by γ_0, $\dot{\gamma}_0$, γ, and $\dot{\gamma}$, respectively. Now after simplifying this relation, Equation (66) will be obtained. The proof is completed.

Remark 6. The above theorem and Equation (66) show that for a given value of t_0, we may obtain value or values of t such that the reflected rays from the points $\gamma(t_0)$ and $\gamma(t)$ have an intersection in front of the mirror. Moreover, in Equation (66), if we change the role of t by t_0 and vice versa, then the equation will not be changed. Therefore without loss of generality, we can suppose that $t>t_0$. This fact is useful for making the numerical methods for finding the intersection point C.

Corollary 3.5. *Let* $J_{\bar{\theta}} = \begin{pmatrix} \cos\bar{\theta} \\ -\sin\bar{\theta} \end{pmatrix}$ *be the unit vector in the direction of incident rays* I_0 *and* I_1*. Then, Equation (66) can be written as follows*

$$\frac{2\,|J_{\bar{\theta}}, \dot{\gamma}_0|}{|\dot{\gamma}, \dot{\gamma}_0|} = \frac{|J_{\bar{\theta}}, \gamma - \gamma_0|}{|\dot{\gamma}, \gamma - \gamma_0|}. \tag{80}$$

Proof. It is enough to note that

$$\begin{aligned} \langle T_{\bar{\theta}}, \dot{\gamma}_0\rangle &= |J_{\bar{\theta}}, \dot{\gamma}_0|\,, \\ \langle J_{\bar{\theta}}, \dot{\gamma}_0\rangle &= -\,|T_{\bar{\theta}}, \dot{\gamma}_0|\,. \end{aligned} \tag{81}$$

□

Theorem 3.6. *With the assumptions of Theorem 3.4, the coordinates of the intersection point* C *are as follows*

$$\begin{cases} x_C = x(t_0) + \frac{|\dot{\gamma}, \gamma-\gamma_0||\dot{\gamma}, J_{\bar{\theta}}|}{|\dot{\gamma}_0, \dot{\gamma}|^2}\left|\dot{\gamma}_0, U_{(J_{\bar{\theta}}, \dot{\gamma}_0, \dot{\gamma}_0)}\right|, \\ y_C = y(t_0) + \frac{|\dot{\gamma}, \gamma-\gamma_0||\dot{\gamma}, J_{\bar{\theta}}|}{|\dot{\gamma}_0, \dot{\gamma}|^2}\left\langle\dot{\gamma}_0, U_{(J_{\bar{\theta}}, \dot{\gamma}_0, \dot{\gamma}_0)}\right\rangle. \end{cases} \tag{82}$$

Proof. The coordinates of the point Q are

$$Q = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(t_0) + a(t)\cos\tau_0 + b(t)\cos(\alpha+\tau_0) \\ y(t_0) + a(t)\sin\tau_0 + b(t)\sin(\alpha+\tau_0) \end{pmatrix}. \tag{83}$$

On the other hand, the point C is the intersection of two lines $\overleftrightarrow{OC}$ and $\overleftrightarrow{QC}$, whose equations are as follows

$$\begin{aligned} y - y(t_0) &= \tan(\theta+\tau_0)\,(x - x(t_0))\,, \\ y - y(t) &= \tan(\theta+2\alpha+\tau_0)\,(x - x(t))\,, \end{aligned} \tag{84}$$

which implies

$$\begin{aligned} x_C &= \tfrac{y(t_0)-y(t)+x(t)\tan(\theta+2\alpha+\tau_0)-x(t_0)\tan(\theta+\tau_0)}{\tan(\theta+2\alpha+\tau_0)-\tan(\theta+\tau_0)}, \\ y_C &= y(t_0) - x(t_0)\tan(\theta+\tau_0) + \tan(\theta+\tau_0)x_C. \end{aligned} \tag{85}$$

By using $b(t) = \frac{a(t)\sin\theta}{\sin(\theta+\alpha)}$ and (83), the above coordinates will be simplified to the following

$$\begin{aligned} x_C &= x(t_0) + \tfrac{a(t)\cos(\theta+\tau_0)\sin(\theta+\alpha)}{\sin\alpha}, \\ y_C &= y(t_0) + \tfrac{a(t)\sin(\theta+\tau_0)\sin(\theta+\alpha)}{\sin\alpha}. \end{aligned} \tag{86}$$

Now, with respect to (81), the relation (76) will have the following form

$$\sin(\theta+\alpha) = \sin(\bar{\theta}+\tau_0+\alpha) = \frac{\langle T_{\bar{\theta}}, \dot{\gamma}(t)\rangle}{|\dot{\gamma}(t)|} = \frac{|J_{\bar{\theta}}, \dot{\gamma}(t)|}{|\dot{\gamma}(t)|}. \tag{87}$$

By (65), (71), (87), and the relations

$$\begin{aligned}
\cos(\theta+\tau_0) &= \cos\theta\cos\tau_0 - \sin\theta\sin\tau_0 \\
&= \tfrac{|\dot\gamma(t_0),T_{\bar\theta}|}{|\dot\gamma(t_0)|}\tfrac{\dot x(t_0)}{|\dot\gamma(t_0)|} - \tfrac{\langle\dot\gamma(t_0),T_{\bar\theta}\rangle}{|\dot\gamma(t_0)|}\tfrac{\dot y(t_0)}{|\dot\gamma(t_0)|} \\
&= \tfrac{\langle J_{\bar\theta},\dot\gamma(t_0)\rangle}{|\dot\gamma(t_0)|}\tfrac{\dot x(t_0)}{|\dot\gamma(t_0)|} - \tfrac{|J_{\bar\theta},\dot\gamma(t_0)|}{|\dot\gamma(t_0)|}\tfrac{\dot y(t_0)}{|\dot\gamma(t_0)|} \\
&= \tfrac{1}{|\dot\gamma_0|^2}\left|\dot\gamma_0, \begin{pmatrix} |J_{\bar\theta},\dot\gamma_0| \\ \langle J_{\bar\theta},\dot\gamma_0\rangle \end{pmatrix}\right| \\
&= \tfrac{1}{|\dot\gamma_0|^2}\left|\dot\gamma_0, U_{(J_{\bar\theta},\dot\gamma_0,\dot\gamma_0)}\right|, \\
\sin(\theta+\tau_0) &= \sin\theta\cos\tau_0 + \cos\theta\sin\tau_0 \\
&= \tfrac{|J_{\bar\theta},\dot\gamma_0|}{|\dot\gamma_0|}\tfrac{\dot x(t_0)}{|\dot\gamma_0|} + \tfrac{\langle J_{\bar\theta},\dot\gamma_0\rangle}{|\dot\gamma_0|}\tfrac{\dot y(t_0)}{|\dot\gamma_0|} \\
&= \tfrac{1}{|\dot\gamma_0|^2}\left\langle\dot\gamma_0, \begin{pmatrix} |J_{\bar\theta},\dot\gamma_0| \\ \langle J_{\bar\theta},\dot\gamma_0\rangle \end{pmatrix}\right\rangle \\
&= \tfrac{1}{|\dot\gamma_0|^2}\left\langle\dot\gamma_0, U_{(J_{\bar\theta},\dot\gamma_0,\dot\gamma_0)}\right\rangle,
\end{aligned} \tag{88}$$

we can obtain the coordinates of intersection point as it is described in (82). $\square$

4. Reflection of an orthotomic family of rays on a parametric curved mirror

In this section for a parametric curved reflector we consider the reflection of a family of rays in two situations **(1)** and **(2)**, i.e. when the source is at a point S or at infinity. In the following, we suppose that the family of rays are orthotomic.

4.1 The intersection of a family of lines

As it is described in [10] we can obtain the intersection of a parametric family of lines in the following. Consider a family of lines

$$y = m(t)x + h(t), \tag{89}$$

with parameter $t \in \mathbb{R}$. Now, let the point $(x_I(t), y_I(t))$ be the intersection of two infinitesimally apart lines $y = mx + h$ and $y = (m + dm)x + (h + dh)$, therefore

$$y_I(t) = mx_I(t) + h, \tag{90}$$

$$y_I(t) = (m + dm)\,x_I(t) + (h + dh), \tag{91}$$

in which

$$dm = \tfrac{dm}{dt}dt, \qquad dh = \tfrac{dh}{dt}dt. \tag{92}$$

Substituting (90) in (91) we have

$$x_I(t) = -\tfrac{\dot h(t)}{\dot m(t)}, \tag{93}$$

and by using Equation (51) it will be

$$x_I(t) = x(t) - \frac{\dot{y}(t) - m(t)\dot{x}(t)}{\dot{m}(t)}. \tag{94}$$

Moreover,

$$y_I(t) = h(t) + m x_I(t). \tag{95}$$

Remark 7. Although here we considered the approach of [10], clearly from (53) when $t_0 \to t$ the relation (93) can be obtained and $x_C \to x_I$. Thus we may have another approach for the above results by limits.

4.2 Case(1c): The intersection of reflected rays on γ which are sent from a source point S

We know that for finding the coordinates of the intersection of lines that are reflected rays from the parametric curve $\gamma(t)$, it is necessary to obtain $\dot{m}(t)$ by Equation (50), that is after some calculations as follows

$$\dot{m}(t) = \frac{|\dot{\gamma}(t)|^2 \big(2|(\gamma(t)-S)|^2 \, |\dot{\gamma}(t),\ddot{\gamma}(t)| - |\dot{\gamma}(t)|^2 \, |(\gamma(t)-S),\dot{\gamma}(t)|\big)}{(\dot{x}(t)\langle\dot{\gamma}(t),(\gamma(t)-S)\rangle - \dot{y}(t) \, |(\gamma(t)-S),\dot{\gamma}(t)|)^2}, \tag{96}$$

or when

$\Gamma_3 = \begin{pmatrix} |\dot{\gamma}(t),\ddot{\gamma}(t)| \\ \langle\dot{\gamma}(t),\dot{\gamma}(t)\rangle \end{pmatrix}$

and

$\Gamma_4 = \begin{pmatrix} |\gamma(t)-S,\dot{\gamma}(t)| \\ 2\langle\gamma(t)-S,\gamma(t)-S\rangle \end{pmatrix}$

it will have the smaller form

$$\dot{m}(t) = |\dot{\gamma}(t)|^2 \frac{|\Gamma_3,\Gamma_4|}{|\dot{\gamma}(t),\Gamma|^2}.$$

Now by substituting Equation (96) in Equation (94) we can obtain the x-coordinate of the intersection point of reflected rays, namely

$$x_I(t) = x(t) - \frac{|\dot{\gamma}(t),(\gamma(t)-S)|(\dot{x}(t)\langle(\gamma(t)-S),\dot{\gamma}(t)\rangle - \dot{y}(t) \, |(\gamma(t)-S),\dot{\gamma}(t)|)}{2|(\gamma(t)-S)|^2 \, |\dot{\gamma}(t),\ddot{\gamma}(t)| - |\dot{\gamma}(t)|^2 \, |(\gamma(t)-S),\dot{\gamma}(t)|}, \tag{97}$$

that has a shorter form

$$x_I(t) = x(t) - |\dot{\gamma}(t),\gamma(t)-S| \frac{|\dot{\gamma}(t),\Gamma|}{|\Gamma_3,\Gamma_4|}.$$

By replacing (97) in (95) we find the y-coordinate of the intersection point of reflected rays that is

$$y_I(t) = y(t) - \frac{|\dot{\gamma}(t),(\gamma(t)-S)|(\dot{y}(t)\langle(\gamma(t)-S),\dot{\gamma}(t)\rangle + \dot{x}(t) \, |(\gamma(t)-S),\dot{\gamma}(t)|)}{2|(\gamma(t)-S)|^2 \, |\dot{\gamma}(t),\ddot{\gamma}(t)| - |\dot{\gamma}(t)|^2 \, |(\gamma(t)-S),\dot{\gamma}(t)|}, \tag{98}$$

whose smaller form will be

$$y_I(t) = y(t) - |\dot{\gamma}(t), \gamma(t) - S| \frac{\langle \dot{\gamma}(t), \Gamma \rangle}{|\Gamma_3, \Gamma_4|}.$$

It may be useful to mention that

$$\Gamma \ = U_{(\gamma - S, \dot{\gamma}, \dot{\gamma})}, \ \Gamma_1 = U_{(\dot{\gamma}, \gamma, \gamma)}, \ \Gamma_3 = U_{(\dot{\gamma}, \ddot{\gamma}, \dot{\gamma})}, \ \Gamma_4 = U_{(\gamma - S, \dot{\gamma}, 2\gamma - 2S)}.$$

Therefore, as a generalization of [10], it is shown that

Theorem 4.1. *The intersection point of reflected rays from the mirror $\gamma(t)$ by the above notations has the following coordinates*

$$\begin{aligned} x_I(t) &= x(t) - |\dot{\gamma}(t), \gamma(t) - S| \tfrac{|\dot{\gamma}(t), \Gamma|}{|\Gamma_3, \Gamma_4|}, \\ y_I(t) &= y(t) - |\dot{\gamma}(t), \gamma(t) - S| \tfrac{\langle \dot{\gamma}(t), \Gamma \rangle}{|\Gamma_3, \Gamma_4|}, \end{aligned} \tag{99}$$

that gives the parametric curve of the image of the source point S.

4.3 Some examples of caustics for the case(1c)

In this section, we express some examples from different parametric curves as the mirror or the reflector curve when the locus of the point object (or the source point) S can be changed. Note that in each of examples, the thin (red color) curve is the curve of images of S. The blue dot is the source point S. Although, by changing the point S for a fixed curved mirror we can illustrate many different shapes of the images curves, here we show only two of them in each example.

Mittal in [10] has shown that if the reflector curve is a parabolic mirror (e.g. $y^2 = 4x$) and the source point S is located in the focus of the parabola, then there isn't any intersection point for the family of reflected rays and these rays are Parallel. This agree with the property of parabola. Here using our formulation it is also can be observed since in Equations (97) and (98) the denominators of both fractions are zero and so both coordinates $x_I(t)$ and $y_I(t)$ are undefined.

Example 4.2. Let the reflector curve be $\gamma(t) = (t, t^3)$. In Figure 11 the images of source point S are shown when $S = (0, 0)$ and $S = (0.2, 0.05)$.

Example 4.3. If the reflector is a circle, then it is well known that the images of a source point that is on the circle will be a cardioid. This is shown in Figure 12 (i), and the shape of images when $S = (-0.9, 0)$ is illustrated in 12 (ii).

Example 4.4. In Figure 13 the reflector curve is the Archimede's spiral $\gamma(t) = (t \cos t, t \sin t)$, $t \in [0, \frac{3\pi}{2}]$, and the curve of images is shown (in red) when $S = (0, 0)$ and $S = (-1, 0.5)$.

Example 4.5. Let the reflector be a Cycloid with parametric equation $\gamma(t) = (t - \sin t, 1 - \cos t)$, $t \in [0, 2\pi]$, in Figure 14 the curves of the images are shown (in red color) only for two cases $S = (0, 0)$ and $S = (2, -1)$.

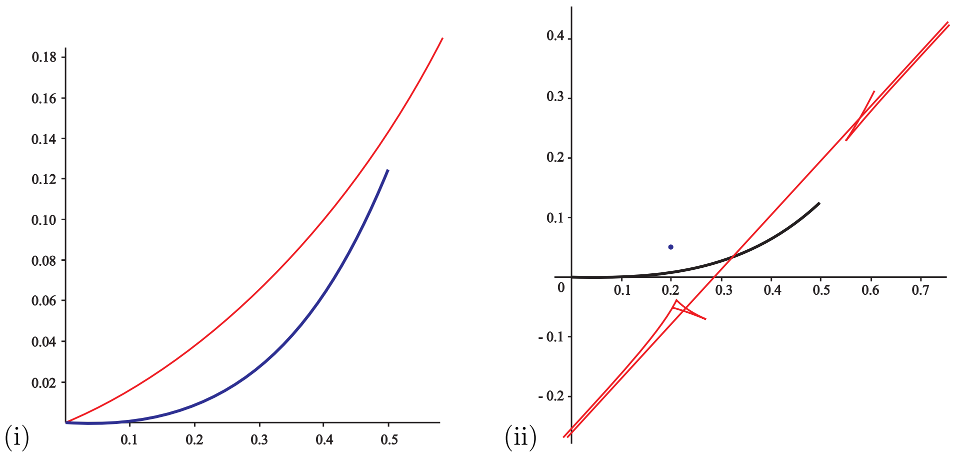

Figure 11: The reflector curve is $\gamma(t) = (t, t^3)$. The (red color) thin curves of intersections of reflected rays are drawn for the source point $S = (0, 0)$ in (i), and $S = (0.2, 0.05)$ in (ii).

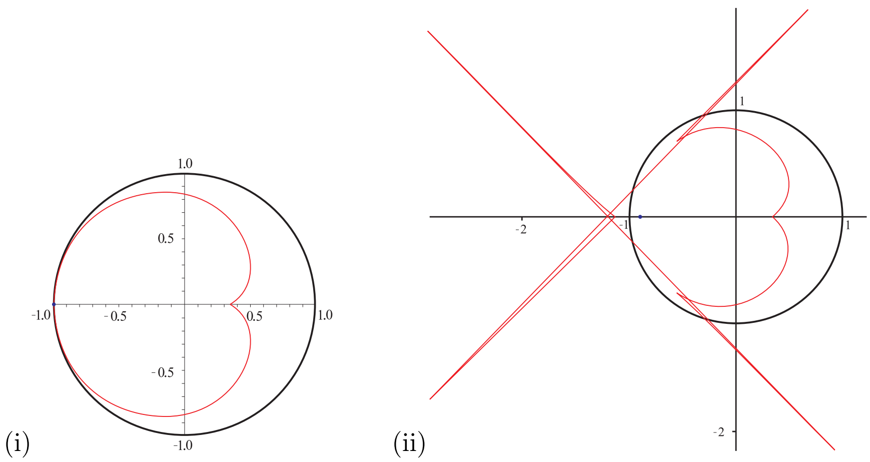

Figure 12: For the circle $\gamma(t) = (\cos t, \sin t)$ the intersections of reflected rays (red color thin curves) are drawn for $S = (-1, 0)$ in (i), and $S = (-0.9, 0)$ in (ii), that in (i) the image curve is a cardioid.

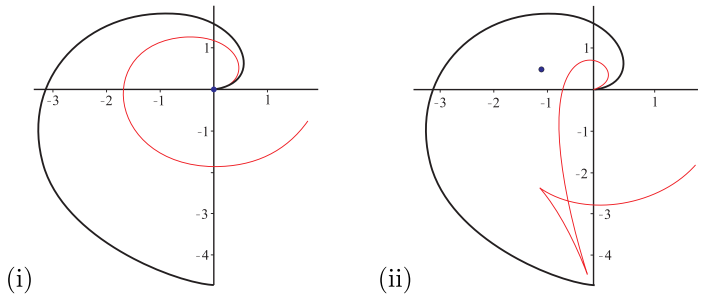

Figure 13: The mirror $\gamma(t) = (t\cos t, t\sin t)$ (black) is the Archimede's spiral. The location of the source are $S = (0, 0)$ in (i), and $S = (-1, 0.5)$ in (ii).

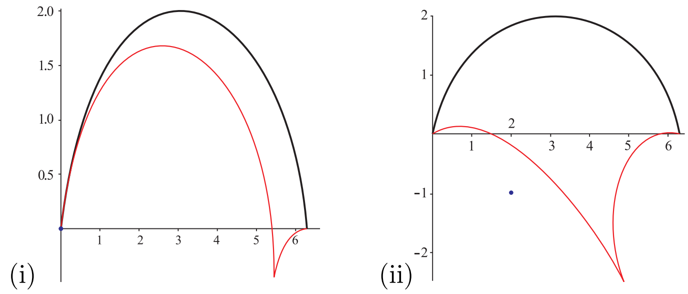

Figure 14: For the Cycloid $\gamma(t) = (t - \sin t, 1 - \cos t)$ (in black color), the images curves are drawn (in red color) for $S = (0, 0)$ in (i), and $S = (2, -1)$ in (ii).

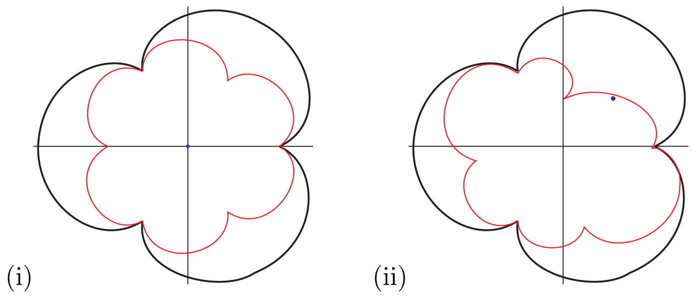

Figure 15: $\gamma(t) = (12\cos t - 3\cos(4t), 12\sin t - 3\sin(4t))$ is an Epicycloid. The source locations are $S = (0,0)$ in (i), and $S = (5,5)$ in (ii).

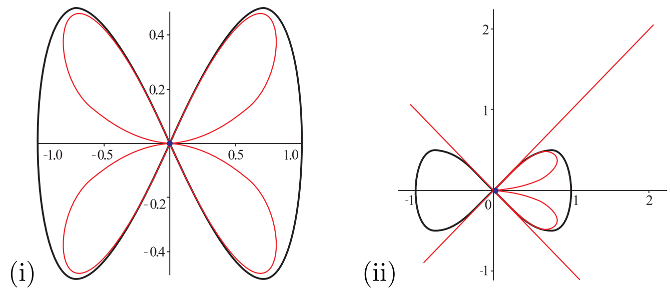

Figure 16: The mirror $\gamma(t) = (\sin t, \sin t\cos t)$ (black) is the Eight curve and the locations of the source point are $S = (0,0)$ in (i), and $S = (0.0001, 0)$ in (ii).

When the reflector is defined by a closed curve, it is clear that the light rays will be reflected infinite times. Here we will find only the first intersection of the reflected rays.

Example 4.6. If the mirror is an Epicycloid defined by $\gamma(t) = (12\cos t - 3\cos(4t), 12\sin t - 3\sin(4t))$, $t \in [0, 2\pi]$, then in Figure 15, it is shown that when the place of S is changed, how the curve of images is deformed.

Example 4.7. Let the reflector be the Eight curve with parametric equation $\gamma(t) = (\sin t, \sin t\cos t)$, $t \in [0, 2\pi]$, for which the images of S are shown in Figure 16, when $S = (0,0)$ and $S = (0.0001, 0)$. It should be mentioned that in Figure 16 (ii) we have two crossed-line segments, i.e. the image is bounded.

Example 4.8. In Figure 17, the reflector is the Astroid defined by $\gamma(t) = (\cos^3(t), \sin^3(t))$, $t \in [0, 2\pi]$, for which it is illustrated change of the first intersection point of reflected rays when the source S moves from origin to point $(0.3, 0.3)$.

Note that, we consider the object point which is located in front of the reflector curve, but the image curve may be in front or behind the reflector. For instance in

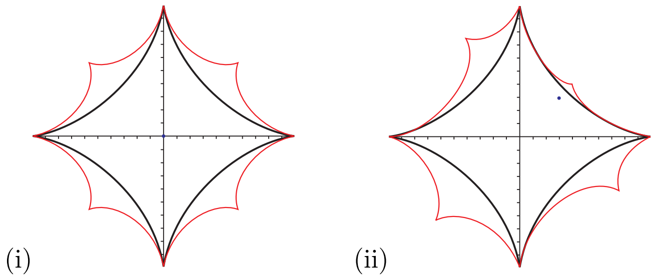

Figure 17: The mirror $\gamma(t) = (\cos^3(t), \sin^3(t))$ is an Astroid (with black color) and the locations of the source point are $S = (0, 0)$ in (i), and $S = (0.3, 0.3)$ in (ii).

the diagrams of Example 4.8 we have no image curve in front of the reflector but in Examples 4.2 and 4.7 a part of the image curve is placed behind the reflector. Therefore when we have a mechanical wavefront the diagrams (and discussion of the reflected wavefront) may have some changes.

4.4 Reflection of an orthotomic family of parallel rays on a parametric curved mirror

Suppose that an orthotomic family of parallel rays in a $2D$ Cartesian plane strikes a curved mirror defined by the parametric equation $\gamma(t) = (x(t), y(t))$. It is clear that the reflected rays of this family can be considered as a parametric family of rays and the results of Subsections 3.1 and 4.1 can be used here. Since in this case, we have not coordinates of the source point, we need to describe similar formulas to those in Theorem 3.1.

Let I be an incident ray of the orthotomic family of parallel rays that strikes the reflector γ at point $Q = (x(t), y(t))$ and R be its reflection, see Figure 18. The angles $\theta, \bar{\theta}$ and τ, which are shown in Figure 18, can be considered as in the case (2b) in Subsection 3.3, by replacing I_0 and R_0 with I and R, respectively. Note that for different incident rays in a family of parallel rays, the angle $\bar{\theta}$ is constant.

Lemma 4.9. *The slope of the reflected ray from the curved reflector $\gamma(t)$ in an arbitrary point $Q = (x(t), y(t))$ is*

$$m(t) = \frac{\left|U_{(T_{\bar{\theta}}, \dot{\gamma}(t), \dot{\gamma}(t))}, \dot{\gamma}(t)\right|}{\left\langle U_{(T_{\bar{\theta}}, \dot{\gamma}(t), \dot{\gamma}(t))}, \dot{\gamma}(t)\right\rangle}, \tag{100}$$

in which $T_{\bar{\theta}} = \begin{pmatrix} \sin\bar{\theta} \\ \cos\bar{\theta} \end{pmatrix}$ and $\bar{\theta}$ is the acute angle between incident ray I and the horizontal line at Q (that is parallel to the x-axis).

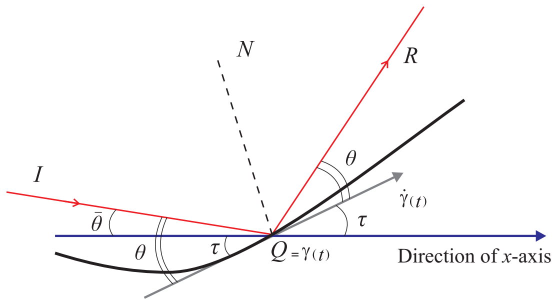

Figure 18: The angles $\bar{\theta}, \theta$ and τ.

Proof. The slope of the reflected ray from the point $Q = \gamma(t) = (x(t), y(t))$ on the curve is

$$m = \tan(\bar{\theta} + 2\tau) = \frac{\sin(\bar{\theta}+\tau+\tau)}{\cos(\bar{\theta}+\tau+\tau)} = \frac{\sin(\bar{\theta}+\tau)\cos\tau+\cos(\bar{\theta}+\tau)\sin\tau}{\cos(\bar{\theta}+\tau)\cos\tau-\sin(\bar{\theta}+\tau)\sin\tau}. \tag{101}$$

By the definition of τ, it is the angle between $\dot{\gamma}(t)$ and the x-axis, therefore $\frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|} = \begin{pmatrix} \cos\tau \\ \sin\tau \end{pmatrix}$. Hence

$$\begin{aligned} \sin(\bar{\theta}+\tau) &= \sin\bar{\theta}\cos\tau + \cos\bar{\theta}\sin\tau \\ &= \sin\bar{\theta}\frac{\dot{x}(t)}{|\dot{\gamma}(t)|} + \cos\bar{\theta}\frac{\dot{y}(t)}{|\dot{\gamma}(t)|} = \frac{1}{|\dot{\gamma}(t)|}\left\langle \dot{\gamma}(t), T_{\bar{\theta}} \right\rangle, \\ \cos(\bar{\theta}+\tau) &= \cos\bar{\theta}\cos\tau - \sin\bar{\theta}\sin\tau \\ &= \cos\bar{\theta}\frac{\dot{x}(t)}{|\dot{\gamma}(t)|} - \sin\bar{\theta}\frac{\dot{y}(t)}{|\dot{\gamma}(t)|} = \frac{1}{|\dot{\gamma}(t)|}\left| \dot{\gamma}(t), T_{\bar{\theta}} \right|. \end{aligned} \tag{102}$$

By substituting Equation (102) in Equation (101) we have

$$\begin{aligned} m(t) &= \frac{\frac{1}{|\dot{\gamma}(t)|^2}(\langle\dot{\gamma}(t),T_{\bar{\theta}}\rangle\dot{x}(t)+|\dot{\gamma}(t),T_{\bar{\theta}}|\dot{y}(t))}{\frac{1}{|\dot{\gamma}(t)|^2}(|\dot{\gamma}(t),T_{\bar{\theta}}|\dot{x}(t)-\langle\dot{\gamma}(t),T_{\bar{\theta}}\rangle\dot{y}(t))} \\ &= \frac{|T_{\bar{\theta}},\dot{\gamma}(t)|\dot{y}(t)-\langle T_{\bar{\theta}},\dot{\gamma}(t)\rangle\dot{x}(t)}{|T_{\bar{\theta}},\dot{\gamma}(t)|\dot{x}(t)+\langle T_{\bar{\theta}},\dot{\gamma}(t)\rangle\dot{y}(t)}, \end{aligned} \tag{103}$$

that by using the notations in (43), it is equal to (100). $\square$

Theorem 4.10. *Suppose that an orthotomic family of parallel rays strikes the reflector $\gamma(t)$. Then the reflected rays form a family of parametric rays with the equation $y = m(t)x + h(t)$ such that*

$$m(t) = \frac{\left|U_{(T_{\bar{\theta}},\dot{\gamma}(t),\dot{\gamma}(t))},\dot{\gamma}(t)\right|}{\left\langle U_{(T_{\bar{\theta}},\dot{\gamma}(t),\dot{\gamma}(t))},\dot{\gamma}(t)\right\rangle}, \quad \textit{and} \quad h(t) = \frac{\langle\gamma(t),\Gamma_5\rangle}{\left\langle U_{(T_{\bar{\theta}},\dot{\gamma}(t),\dot{\gamma}(t))},\dot{\gamma}(t)\right\rangle}, \tag{104}$$

in which $\Gamma_5 = \begin{pmatrix} \left|\dot{\gamma}(t), U_{(T_{\bar{\theta}},\dot{\gamma}(t),\dot{\gamma}(t))}\right| \\ \left\langle\dot{\gamma}(t), U_{(T_{\bar{\theta}},\dot{\gamma}(t),\dot{\gamma}(t))}\right\rangle \end{pmatrix}$.

Proof. By changing the point Q on the curve γ as in Lemma 4.9, we can calculate the slope $m(t)$ of the reflected rays. Now for $h(t)$ we have,

$$\begin{aligned} h(t) &= y(t) - m(t)x(t) \\ &= y(t) - \frac{\left|U_{(T_{\bar\theta},\dot\gamma(t),\dot\gamma(t))},\dot\gamma(t)\right|}{\left\langle U_{(T_{\bar\theta},\dot\gamma(t),\dot\gamma(t))},\dot\gamma(t)\right\rangle}x(t) \\ &= \frac{y(t)\left\langle U_{(T_{\bar\theta},\dot\gamma(t),\dot\gamma(t))},\dot\gamma(t)\right\rangle - x(t)\left|U_{(T_{\bar\theta},\dot\gamma(t),\dot\gamma(t))},\dot\gamma(t)\right|}{\left\langle U_{(T_{\bar\theta},\dot\gamma(t),\dot\gamma(t))},\dot\gamma(t)\right\rangle} \\ &= \frac{x(t)\left|\dot\gamma(t),U_{(T_{\bar\theta},\dot\gamma(t),\dot\gamma(t))}\right| + y(t)\left\langle \dot\gamma(t),U_{(T_{\bar\theta},\dot\gamma(t),\dot\gamma(t))}\right\rangle}{\left\langle U_{(T_{\bar\theta},\dot\gamma(t),\dot\gamma(t))},\dot\gamma(t)\right\rangle}, \end{aligned} \tag{105}$$

which by (43) describes $h(t)$ as in (104). □

4.5 Case(2c): Finding the intersection of reflected rays on γ for the parallel incident rays

As in Subsection 4.1 for finding the coordinates of the intersection of reflected rays from the parametric curve $\gamma(t)$, it is necessary to obtain $\dot m(t)$. By (101) and (103) we have

$$m(t) = \frac{\dot x^2(t)\sin\bar\theta + 2\dot x(t)\dot y(t)\cos\bar\theta - \dot y^2(t)\sin\bar\theta}{\dot x^2(t)\cos\bar\theta - 2\dot x(t)\dot y(t)\sin\bar\theta - \dot y^2(t)\cos\bar\theta} = \frac{E}{D}, \tag{106}$$

in which E and D denote the numerator and denominator of m, respectively. Therefore,

$$\dot m(t) = \frac{\dot E D - \dot D E}{D^2}, \tag{107}$$

which after some calculations can be obtained as,

$$\begin{aligned} \dot m(t) &= \frac{2\left(\dot x^2(t)+\dot y^2(t)\right)(\dot x(t)\ddot y(t) - \dot y(t)\ddot x(t))}{\left(\dot x^2(t)\cos\bar\theta - 2\dot x(t)\dot y(t)\sin\bar\theta - \dot y^2(t)\cos\bar\theta\right)^2} \\ &= \frac{2}{D^2}\left|\dot\gamma\right|^2\left|\dot\gamma,\ddot\gamma\right|, \end{aligned} \tag{108}$$

and by using the notations in (43) we have

$$\dot m(t) = \frac{2\left|\dot\gamma(t)\right|^2\left|\dot\gamma(t),\ddot\gamma(t)\right|}{\left\langle U_{(T_{\bar\theta},\dot\gamma(t),\dot\gamma(t))},\dot\gamma(t)\right\rangle^2}. \tag{109}$$

Theorem 4.11. *The intersection point of reflected rays from the reflector $\gamma(t)$ by the above notations has the following coordinates*

$$\begin{aligned} x_I(t) &= x(t) - \frac{\langle T_{\bar\theta},\dot\gamma(t)\rangle\left\langle U_{(T_{\bar\theta},\dot\gamma(t),\dot\gamma(t))},\dot\gamma(t)\right\rangle}{2\left|\dot\gamma(t),\ddot\gamma(t)\right|}, \\ y_I(t) &= y(t) - \frac{\langle T_{\bar\theta},\dot\gamma(t)\rangle\left|U_{(T_{\bar\theta},\dot\gamma(t),\dot\gamma(t))},\dot\gamma(t)\right|}{2\left|\dot\gamma(t),\ddot\gamma(t)\right|}. \end{aligned} \tag{110}$$

Proof. Substituting (106) and (108) in equation (94), we can obtain the x coordinate of the intersection point as follows

$$\begin{aligned} x_I(t) &= x(t) - \frac{\dot{y}(t) - m(t)\dot{x}(t)}{\dot{m}(t)} \\ &= x(t) - \frac{\dot{y}D - \dot{x}E}{D \times \frac{2}{D^2}|\dot{\gamma}|^2 |\dot{\gamma}, \ddot{\gamma}|} \\ &= x(t) - \frac{D\langle T_{\bar{\theta}}, \dot{\gamma}\rangle}{2|\dot{\gamma}, \ddot{\gamma}|}. \end{aligned} \tag{111}$$

If we put $x_I(t) = x(t) - \mathcal{H}$ (i.e. we denote the fraction in Equation (111) by $\mathcal{H}$), then

$$\begin{aligned} y_I(t) &= m x_I(t) + h(t) = m(x(t) - \mathcal{H}) + h(t) \\ &= y(t) - m\mathcal{H}, \end{aligned} \tag{112}$$

and by substituting $m(t) = \frac{E}{D}$ and $\mathcal{H}$ in (112) we have

$$y_I(t) \quad = y(t) - m\mathcal{H} = y(t) - \frac{E\langle T_{\bar{\theta}}, \dot{\gamma}\rangle}{2|\dot{\gamma}, \ddot{\gamma}|}. \tag{113}$$

Finally, using numerator and denominator of (100) in Equations (111) and (113), the relation (110) will be obtained. □

Corollary 4.12. *Let* $J_{\bar{\theta}} = \begin{pmatrix} \cos\bar{\theta} \\ -\sin\bar{\theta} \end{pmatrix}$ *be the unit vector in the direction of incident ray* I*. Then the coordinates in (110) can be written as follows*

$$\begin{cases} x_I(t) = x(t) - \frac{|J_{\bar{\theta}}, \dot{\gamma}(t)|}{2|\dot{\gamma}(t), \ddot{\gamma}(t)|} \left|\dot{\gamma}(t), U_{(J_{\bar{\theta}}, \dot{\gamma}(t), \dot{\gamma}(t))}\right|, \\ y_I(t) = y(t) - \frac{|J_{\bar{\theta}}, \dot{\gamma}(t)|}{2|\dot{\gamma}(t), \ddot{\gamma}(t)|} \left\langle \dot{\gamma}(t), U_{(J_{\bar{\theta}}, \dot{\gamma}(t), \dot{\gamma}(t))}\right\rangle. \end{cases} \tag{114}$$

Proof. If we use the relations in (81), then from (103) we have

$$\begin{aligned} m(t) &= \frac{|T_{\bar{\theta}}, \dot{\gamma}(t)|\dot{y}(t) - \langle T_{\bar{\theta}}, \dot{\gamma}(t)\rangle \dot{x}(t)}{|T_{\bar{\theta}}, \dot{\gamma}(t)|\dot{x}(t) + \langle T_{\bar{\theta}}, \dot{\gamma}(t)\rangle \dot{y}(t)} \\ &= \frac{-\langle J_{\bar{\theta}}, \dot{\gamma}(t)\rangle \dot{y}(t) - |J_{\bar{\theta}}, \dot{\gamma}(t)|\dot{x}(t)}{-\langle J_{\bar{\theta}}, \dot{\gamma}(t)\rangle \dot{x}(t) + |J_{\bar{\theta}}, \dot{\gamma}(t)|\dot{y}(t)} \\ &= \frac{-\langle \dot{\gamma}(t), U_{(J_{\bar{\theta}}, \dot{\gamma}(t), \dot{\gamma}(t))}\rangle}{-\left|\dot{\gamma}(t), U_{(J_{\bar{\theta}}, \dot{\gamma}(t), \dot{\gamma}(t))}\right|} \\ &= \frac{\langle \dot{\gamma}(t), U_{(J_{\bar{\theta}}, \dot{\gamma}(t), \dot{\gamma}(t))}\rangle}{\left|\dot{\gamma}(t), U_{(J_{\bar{\theta}}, \dot{\gamma}(t), \dot{\gamma}(t))}\right|} = \frac{E}{D}. \end{aligned} \tag{115}$$

Now, if we substitute the denominator and the numerator of this fraction as D and E in (111) and (113), respectively, then

$$\begin{aligned} x_I(t) = x(t) - \frac{\langle T_{\bar{\theta}}, \dot{\gamma}(t)\rangle}{2|\dot{\gamma}(t), \ddot{\gamma}(t)|} \left|\dot{\gamma}(t), U_{(J_{\bar{\theta}}, \dot{\gamma}(t), \dot{\gamma}(t))}\right|, \\ y_I(t) = y(t) - \frac{\langle T_{\bar{\theta}}, \dot{\gamma}(t)\rangle}{2|\dot{\gamma}(t), \ddot{\gamma}(t)|} \left\langle \dot{\gamma}(t), U_{(J_{\bar{\theta}}, \dot{\gamma}(t), \dot{\gamma}(t))}\right\rangle. \end{aligned} \tag{116}$$

Finally, the first relation in (81) gives us the coordinates in (114). □

Theorem 4.13. *Suppose that the reflected ray from* $\gamma(t_0)$ *intersects the reflected ray from* $\gamma(t)$ *at the intersection point* C*, whose coordinates are given by (82). If the parameter* t_0 *tends to* t*, then the coordinates of* C *tend to* $(x_I(t), y_I(t))$ *which is described by (114).*

Proof. In both relations of (82), for calculating the limits when $t_0 \to t$ there is only one $\frac{0}{0}$ indeterminacy that is

$$\lim_{t_0 \to t} \frac{|\dot\gamma(t),(\gamma(t)-\gamma(t_0))|}{|\dot\gamma(t_0),\dot\gamma(t)|^2}. \tag{117}$$

Using the L'Hospital's rule and the derivative of the numerator and denominator of the fraction with respect to t_0, we have

$$\lim_{t_0\to t} \frac{|\dot\gamma(t),(-\dot\gamma(t_0))|}{2|\ddot\gamma(t_0),\dot\gamma(t)|\ |\dot\gamma(t_0),\dot\gamma(t)|} = \lim_{t_0\to t} \frac{-1}{2|\dot\gamma(t),\ddot\gamma(t_0)|} = \frac{-1}{2|\dot\gamma(t),\ddot\gamma(t)|}, \tag{118}$$

in which we used the properties of determinants for changing the columns. Substituting this result in $\lim\limits_{t_0\to t}(x_C(t_0), y_C(t_0))$ will complete the calculations. □

4.6 Some examples of caustics for the case(2c)

We illustrate here the intersection of reflected rays (which were parallel before striking the curved reflector) by our formulation, as in 2.3. The intersection of such reflected rays is well known as the caustic of the curve. The above discussion shows that we can find the caustic of the curve as the limit of the intersection point of a discrete family of reflected rays described in case (2b).

Example 4.14. Let the reflector be a parabola with parametric equations $\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$, $t \in [-1, 1]$. Suppose that an orthotomic family of parallel rays strikes the reflector such that the incident angle with x-axis is $\bar\theta = 90°$. It is well known that in this case, all of the reflected rays from the parabola simultaneously reach focal point $\begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}$ of the parabola. We can verify this fact by the above discussion. Note that

$$\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}, \gamma'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}, T_{\bar\theta} = \begin{pmatrix} \sin 90 \\ \cos 90 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{119}$$

Now, from (66) we have

$$\frac{2}{2\,(t_0 - t)} = \frac{t - t_0}{(t^2 - t_0^2) - 2t\,(t - t_0)}, \tag{120}$$

which for each t_0 shows that all values of t satisfy this equation, i.e. for each value of t_0 all reflected rays pass through the interference point C, whose coordinates by (82) are

$$\begin{aligned} x_C(t) &= t_0 + \tfrac{(t-t_0)2t-\left(t^2-t_0^2\right)}{(2t-2t_0)^2}\,(-2t_0 - 2t_0) = 0, \\ y_C(t) &= t_0^2 + \tfrac{(t-t_0)2t-\left(t^2-t_0^2\right)}{(2t-2t_0)^2}\,(1 + 2t_0(-2t_0)) = \tfrac{1}{4}, \end{aligned} \tag{121}$$

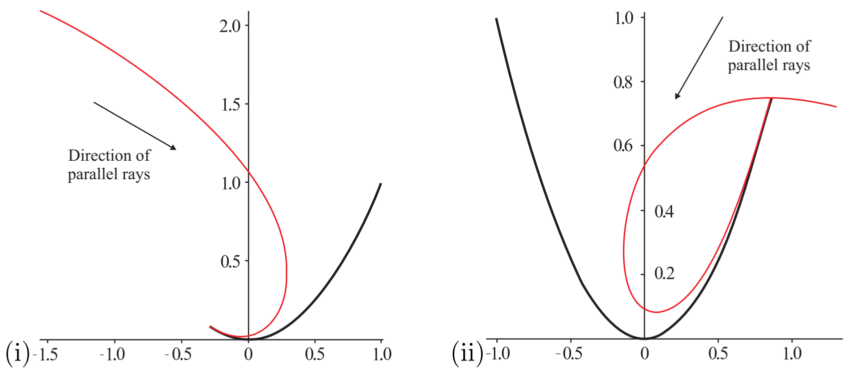

Figure 19: The reflector curve $\gamma(t) = (t, t^2)$ (black) and the image curve (red). The incident angle is $\bar{\theta} = 30°$ in (i), $\bar{\theta} = 120°$ in (ii).

that agrees with our knowledge about parabola. Note that if we use Equation (110), since $\ddot{\gamma}(t) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, then

$$\begin{aligned} x_I(t) &= t + \tfrac{-4t}{4} = 0, \\ y_I(t) &= t^2 + \tfrac{1-4t^2}{4} = \tfrac{1}{4}, \end{aligned} \tag{122}$$

that is the same result.

Remark 8. In the case of parallel rays, some rays may be in front of the reflector and some of them may strike behind it. Therefore, in order to the incident rays do not strike behind the reflector, it is needed to make some restrictions on the deffinition domain of the curve. In the following examples, we use the condition $\bar{\theta} \geq \tau = \arctan\left(\frac{\dot{y}(t)}{\dot{x}(t)}\right)$ for the beginning or the end of the domain interval of the curve.

Example 4.15. Consider the Parabola $\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$ in the Example 4.14 for which $\bar{\theta} = 30°$. With respect to the Remark 8, we restrict the domain of the curve to interval $t \in [\frac{-\sqrt{3}}{6}, 1]$. The intersection points of the reflected rays in this case are denoted in Figure 19 (i). However, if $\bar{\theta} = 120°$, then the domain of the curve should be restricted to $t \in [-1, \frac{\sqrt{3}}{2}]$, whose intersection points are denoted in Figure 19 (ii).

Example 4.16. If a family of parallel rays strikes the curve $\gamma(t) = \begin{pmatrix} t \\ t^3 \end{pmatrix}$, $t \in [0, 1]$, then the intersection points for $\bar{\theta} = 0°$ and $\bar{\theta} = 45°$ are illustrated in Figure 20 (i) and (ii), respectively.

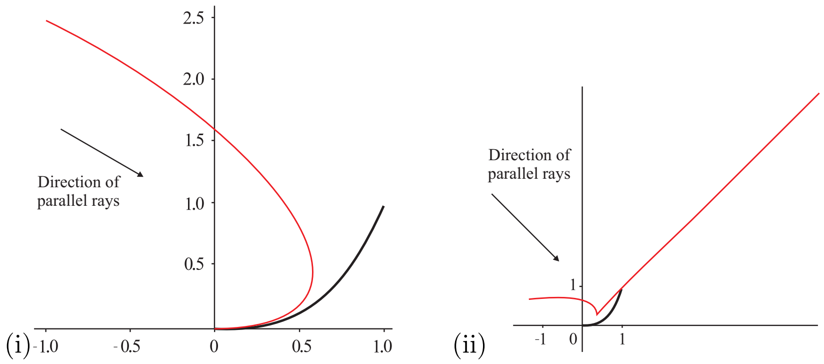

Figure 20: The reflector curve $\gamma(t) = (t, t^3)$ (black) and the image curve (red) the incident angle is $\bar{\theta} = 30°$ in (i), and $\bar{\theta} = 45°$ in (ii).

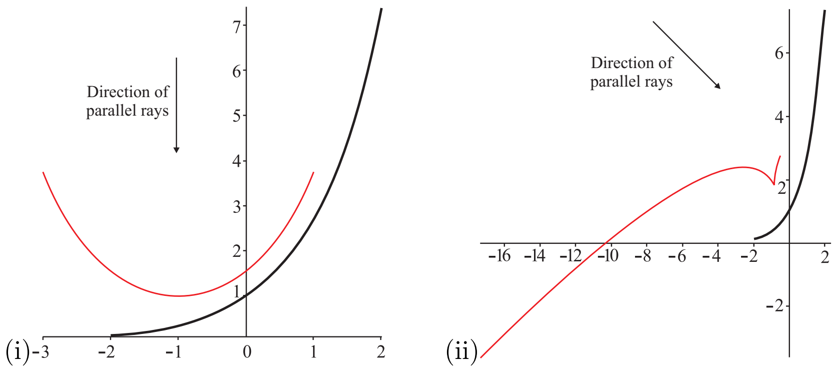

Figure 21: The reflector curve $\gamma(t) = (t, e^t)$ (black) and the image curve (red). The incident angle is $\bar{\theta} = 90°$ in (i), and $\bar{\theta} = 45°$ in (ii).

Example 4.17. Suppose that the reflector is $\gamma(t) = \begin{pmatrix} t \\ e^t \end{pmatrix}$, $t \in [-2, 2]$. The intersection points for $\bar{\theta} = 90°$ and $\bar{\theta} = 45°$ are denoted in Figure 21 (i) and (ii), respectively.

Example 4.18. Suppose that the reflector is a part of the Epicycloid $\gamma(t) = \begin{pmatrix} 6\cos(t) - 2\cos(3t) \\ 6\sin(t) - 2\sin(3t) \end{pmatrix}$. If $\bar{\theta} = 120°$, then from the Remark 8, we restrict the domain interval of the curve to $t \in [-2.617993878, -1.047197551]$. However, if $\bar{\theta} = -30°$, then the interval $[0.2617993880, 1.832595715]$ may be considered as the domain of the curve. These two cases are shown in Figure 22 (i) and (ii), respectively.

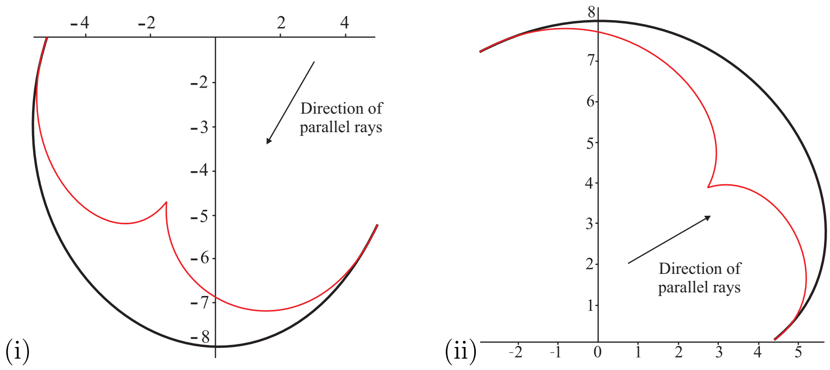

Figure 22: The reflector curve $\gamma(t) = (6\cos t - 2\cos(3t), 6\sin t - 2\sin(3t))$ (black) and the image curve (red). The incident angle is $\bar{\theta} = 120°$ in (i), and $\bar{\theta} = -30°$ in (ii).

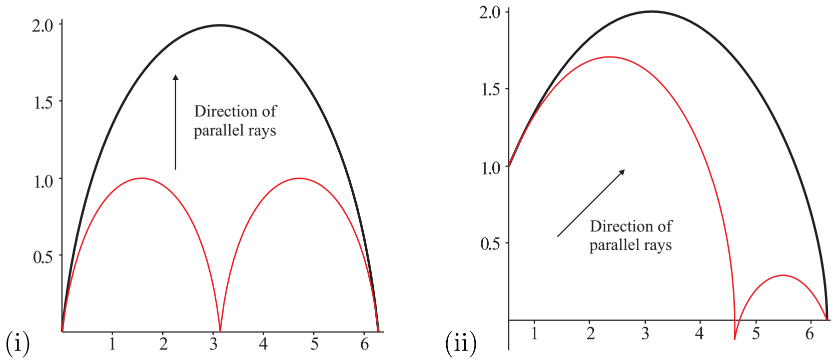

Figure 23: The reflector curve $\gamma(t) = (t - \sin(t), 1 - \cos(t))$ (black) and the image curve (red). The incident angle is $\bar{\theta} = -90°$ in (i), and $\bar{\theta} = -45°$ in (ii).

Example 4.19. Let the Cycloid $\gamma(t) = \begin{pmatrix} t - \sin(t) \\ 1 - \cos(t) \end{pmatrix}$ be the reflector. If $\bar{\theta} = -90°$, then the domain may be $[0, 2\pi]$. However, if $\bar{\theta} = -45°$, then the interval $[\frac{\pi}{2}, 2\pi]$ may be chosen as the domain. These cases are illustrated in Figure 23 (i) and (ii), respectively.

Example 4.20. Suppose that the reflector is a part of the Astroid $\gamma(t) = \begin{pmatrix} \cos^3(t) \\ \sin^3(t) \end{pmatrix}$. If a family of parallel rays strikes the reflector such that $\bar{\theta} = 90°$, then the interval domain may be $[0, \pi]$, But if $\bar{\theta} = 45°$, then the interval $[\frac{\pi}{2}, \pi]$ may be used as the domain. The intersection points for these cases are denoted in Figure 24 (i) and (ii), respectively.

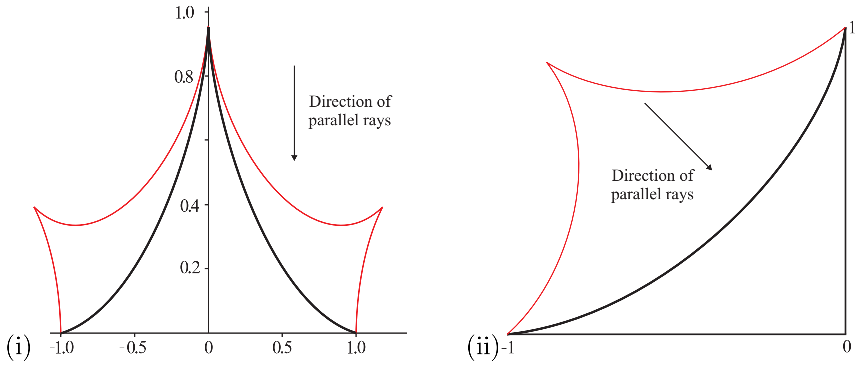

Figure 24: The reflector curve $\gamma(t) = (\cos^3(t), \sin^3(t))$ (black) and the image curve (red). The incident angle is $\bar{\theta} = 90°$ in (i), and $\bar{\theta} = 45°$ in (ii).

5. Curved mirror with self-intersection

When the rays are sent from a source point S and the reflector is a parametric curve with self-intersection we have the situation that a part of the curve shadows on another part of it. In order to find the first intersection of reflected rays, we can use Theorem 4.11 for finding the image curve (x_I, y_I) by formula (110) and then omit those parts of the image curve corresponding to that part of the mirror which is in the shadow. For this, we can proceed as follows:

- consider two line segments that begin at S and are tangent to that part of the curved mirror that shadows on another part of it;
- find that interval of changing parameter t of the curved reflector for which the obstacle part of the reflector is bounded by the tangents;
- obtain the intervals of t for which the corresponding parts of the reflector are in the shadow;
- put these intervals out of the computation of the image curve (x_I, y_I) described by (110) and draw the image curve for the remaining intervals of t.

Example 5.1. Suppose that the reflector is a parametric curve with self-intersection whose parametric equations are

$$x(t) = \cos t(1 + 2\sin t), \qquad y(t) = \sin t(1 + 2\sin t). \tag{123}$$

Let $S = (-1, 2.69)$ be the source point that is on the reflector for $t \approx 1.93$. Using Theorem 4.11 the image curve can be obtained as shown in Figure 25(i). Clearly, there is a part of the reflector that shadows on some other parts of it. This obstacle

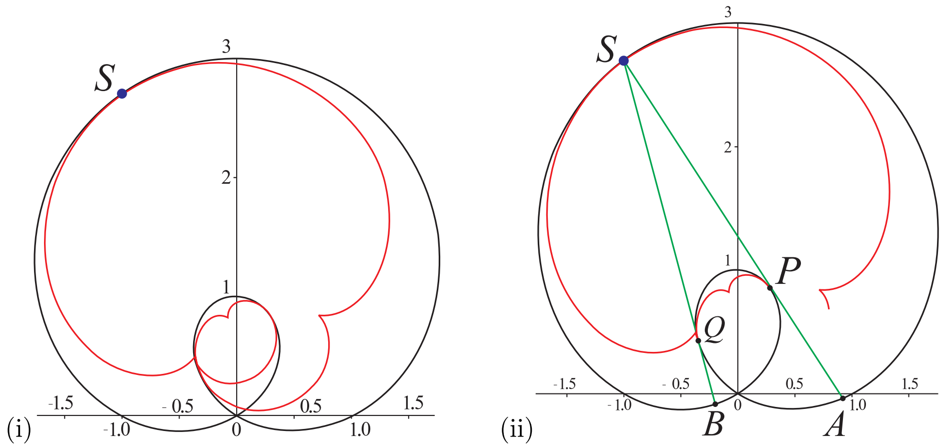

Figure 25: (i) The caustic of all the curve with self-intersection. (ii) The caustic of those parts of the curve which are not in the shadow (see Example 5.1 for details).

part can be bounded between two rays which are sent from S and are tangent to the reflector. The equation of a line which passes through S and is tangent to the reflector is as follows

$$y - y(t) = \frac{\dot{y}(t)}{\dot{x}(t)}(x - x(t)).$$

For values $x = -1$ and $y = 2.69$ we can solve this equation for parameter t and find values $t = -1.9$ and $t = -0.88$ which are respectively correspounding to pairs $P = (0.29, 0.85)$ and $Q = (-0.345, 0.42)$ as tangent points on the reflector. The line $\overleftrightarrow{SP}$ has another intersection with the reflector in point $A = (0.92, -0.037)$ when $t = -0.04$, and the line $\overleftrightarrow{SQ}$ intersects the reflector in point $B = (-0.2, -0.086)$ for $t = -2.74$. Therefore, we can separate the domain of parameter t of the reflector into the following intervals

$$I_1 = [-4.35, -2.74],\ I_2 = [-2.74, -1.9],\ I_3 = [-1.9, -0.88],\ I_4 = [-0.88, -0.04],\ I_5 = [-0.04, 1.93],$$

in which I_2 and I_4 correspond to those parts of the reflector which are in the shadow and the incident rays can not strike them. Omitting these two intervals from the domain of the reflector, in Figure 25(ii) for 3 other intervals, the corresponding parts of the image curve (x_I, y_I) and the tangent lines are shown.

Remark 9. When the incident rays are parallel and the reflector has self-intersection, the situation is similar to the above, but there is a difference that in the first step, we use the incident rays which are tangent to the mirror and are parallel to the vector $v = (x_v, y_v)$ showing the direction of the rays. For this, we should solve the equation $\dfrac{\dot{y}(t)}{\dot{x}(t)} = \dfrac{y_v}{x_v}$ for parameter t.

6. An idea for finding the shape of the mirror

Consider the situation that we have a source point and its image curve, which is formed by an unknown mirror. Is it possible to find the shape of the mirror? Here we briefly explain a theoretical idea to find the shape (or precisely the equation) of the mirror as a curve in the plane.

Suppose that we have a source point $S = (x_S, y_S)$ and the parametric equations of the image curve (x_I, y_I). Then it is possible to find the curve $(x(t), y(t))$ as the solution of the system of ordinary differential equations (110), which can be solved by numerical methods.

Moreover, if we have the image as the set of points (that make a shape similar to a curve) in the plane, then there are numerical methods (see for example [19–21]) to fit a B-spline curve to these points and thus it is possible to find a parametric curve as the image curve (x_I, y_I). After that, the system (110) can be used to find the mirror $(x(t), y(t))$ as the solution of this system. Here, we leave this investigation to whom likes the experimental researchs. It should be mentioned that by having the equation of the mirror, we have all the information about it, such as curvature. Therefore generalizing the results above and the proposed idea to parametric surfaces as reflectors in 3D space may be used to find the curvature of the space around the black holes in gravitational lensing. To whom is interested in this research it is useful to see [22] for fitting a B-spline surface to a given set of image points that make a shape similar to a surface in 3D space.

7. Future study

There are some questions that may be investigated:

(i) How the above results for parametric curves can be considered for implicit curves which are defined by some function f as $f(x, y) = 0$?
(ii) How these investigations may be used for mechanical wavefronts or in the dynamics of billiards?
(iii) How the above results can be extended to parametric surfaces as reflectors in 3-dimensional space and finding the curvature of the space, especially in gravitational lensing?

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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