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The Borg's Theorem for Singular Sturm-Liouville Problem with Non-Separated Boundary Conditions

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Abstract

In this paper, we consider a Sturm-Liouville equation with non-separated boundary conditions on a finite interval. We discuss some properties of solutions of the Sturm-Liouville equation, where the potential function has a singularity in the finite interval. We also calculate eigenvalues and prove the uniqueness of Borg's Theorem of this boundary value problem.

Keywords: Non-separated boundary condition, Singularities, Eigenvalues, Inverse problems.

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1. Introduction

In this paper, we study the non-separated boundary value problem $L = L(q, a, b)$ of the form

$$-y''(x) + \frac{m(m+1)}{x^2} y(x) + q(x)y(x) = \lambda y(x), \qquad \mathrm{q}(x) \in L_2[0,1], \tag{1}$$

$$y(0) = 0, \qquad ay(1) + y'(1) + by'(0) = 0, \tag{2}$$

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where $q(x), a, b$ are real. Also, $ab = a + b$ and $m \geq 0$. The challenge of reconstructing an operator from its given spectral characteristics is termed an inverse problem in spectral analysis. The inverse spectral problems often appear in mathematics, electronics, quantum physics, and other branches of science. Sturm-Liouville equations without singularities were studied by many authors (see [1–5]).

The boundary value problems with one singular point in [6, 7] have been studied and authors have shown that the eigenvalues and norming constant uniquely determine the potential function. In [8], we deal with second-order differential operators having a singular potential. We provide a numerical algorithm for solving the inverse problem of spectral analysis and finding the potential function. In [9, 10], authors consider differential equations having singularities at the end-points of the interval, and investigate the inverse problems with separated boundary conditions.

Researchers in the past have made great efforts to obtain the uniqueness theorems for non-selfadjoint spectral problems with non-separated boundary conditions on a finite interval. In [11], the first result in this field was obtained. After that several uniqueness theorems for inverse selfadjoint problems with non-separated boundary conditions were proved in [12] and [13, 14]. However, the results obtained for non-selfadjoint problems were not direct generalizations of Borg's uniqueness theorem. Since 1949, many authors attempted to generalize Borg's result for non-selfadjoint problems with non-separated boundary conditions since 1949. But only in [11], a generalization of Borg's theorem for the case of inverse non-selfadjoint problems with general boundary conditions, including non-separated ones, was obtained. Freiling and Yurko in [15–17], studied inverse problems for some second-order differential equations with non-separated boundary conditions and showed that the operators can be recovered from their spectral characteristics. In [18], the authors investigated a boundary value problem when one of the non-separated boundary conditions contains a spectral parameter, and a uniqueness theorem is given along with an algorithm developed for solving the inverse problem. In [19], the authors stated that the Sturm-Liouville problem with general boundary conditions cannot be uniquely reconstructed from spectra. However, a class of non-separated boundary conditions is obtained for which two uniqueness theorems for the solution of the inverse Sturm-Liouville problem are proved.

The uniqueness theorem has been proved using Borg's theorem in various sources. One such example can be found in reference [20], where the theorem was applied based on the separable conditions of the problem. In reference [21], Hikmet outlines a problem in which the potential function has a singular point, and the conditions of the problem are also separable. In this case, the eigenvalues of the problem are first obtained, and then Borg's theorem is used to prove the uniqueness theorem.

In this paper, we are interested to study the spectral problems associated with singular differential equations with non-separated boundary conditions and prove the uniqueness theorem. In [16], the uniqueness theorem has been investigated for the inverse spectral problem with non-separated boundary conditions, but in this paper, we prove the uniqueness theorem using Borg's theorem. In Section 2, by using the Bessel function, we represent a solution of (1). This solution has features that have been investigated by the author in [20], and we obtain eigenvalues of the problem (3)-(4). In Section 3, we prove the Borg's theorem of the boundary value problem.

2. Solution and eigenvalues of $L(0, a, b)$

In this section, we consider the boundary value problem $L(0, a, b)$ as follows:

$$-y''(x) + \frac{m(m+1)}{x^2} y(x) = \lambda y(x), \tag{3}$$

$$y(0) = 0, \qquad ay(1) + y'(1) + by'(0) = 0. \tag{4}$$

Now, we obtain the eigenvalues of $L(0, a, b)$. We know that the eigenvalues of this problem are real (see [19]). By [20], we assume that the solution to problem (3)-(4), takes the form

$$\psi(x, \lambda) = \lambda^{\frac{-(m+1)}{2}} \theta_1(\mu x), \tag{5}$$

where $\mu = \sqrt{\lambda}$ and function $\theta_1(s)$ is a solution of

$$-\theta''(s) + \frac{m(m+1)}{s^2} \theta(s) = \theta(s), \qquad s \neq 0, \; Re(s) \geq 0, \; s = \sqrt{\lambda} x. \tag{6}$$

The integral form of the functions $\theta_1(s)$ and $\theta_1'(s)$, is based on Lemma 2.1 in [20], as follows

$$\theta_1(s) = \sin\left(s - \frac{m\pi}{2}\right) - \int_s^\infty \frac{m(m+1)}{t^2} \sin(s - t)\, \theta_1(t) dt, \tag{7}$$

$$\theta_1'(s) = \cos\left(s - \frac{m\pi}{2}\right) - \int_s^\infty \frac{m(m+1)}{t^2} \cos(s - t)\, \theta_1(t) dt. \tag{8}$$

By the Lemma 2.3 in [20], we have the following estimations

$$|\psi(x, \lambda)| \leq K \left(\frac{x}{1 + |\mu x|}\right)^{m+1} \exp(|\mathrm{Im}(\mu)| x),$$

$$|\psi'(x, \lambda)| \leq K \left(\frac{x}{1 + |\mu x|}\right)^{m} \exp(|\mathrm{Im}(\mu)| x),$$

where K is a constant.

Based on the content stated above, we obtain the eigenvalues of $L(0, a, b)$. According to $\psi(x, \lambda)$ in (5) and boundary conditions (4), we can write

$$a\psi(1) + \psi'(1) + b\psi'(0) = 0, \tag{9}$$

and by placing Equation (5) in (9), we have

$$a\theta_1(\mu) + \mu\theta_1'(\mu) + b\mu\theta_1'(0) = 0.$$

Now, we suppose $g(\mu)$ is as follows

$$g(\mu) = a\theta_1(\mu) + \mu\theta_1'(\mu) + b\mu\theta_1'(0). \tag{10}$$

By substituting (7) and (8) in (10), we have

$$\begin{aligned} g(\mu) &= a \sin\left(\mu - \frac{m\pi}{2}\right) + b\,\mu \cos\left(\frac{m\pi}{2}\right) - a\int_\mu^\infty \frac{m(m+1)}{t^2} \sin(\mu - t)\theta_1(t)dt \\ &+ \mu \cos\left(\mu - \frac{m\pi}{2}\right) - \mu\int_\mu^\infty \frac{m(m+1)}{t^2} \cos(\mu - t)\theta_1(t)dt \\ &- b\mu\int_0^\infty \frac{m(m+1)}{t^2} \cos(t)\ \theta_1(t)dt. \end{aligned} \tag{11}$$

We compute each integral in (11) separately. The first one is calculated as follows:

$$\begin{aligned} & a\int_\mu^\infty \frac{m(m+1)}{t^2} \sin(\mu - t)\,\theta_1(t)dt \\ = & \ a\sin\mu\int_\mu^\infty \frac{m(m+1)}{t^2}\cos(t)\,\frac{m(m+1)}{t^2}\theta_1(t)dt \\ - & \ a\cos\mu\int_\mu^\infty \frac{m(m+1)}{t^2}\sin(t)\,\theta_1(t)dt \\ \le & \ C\left[\sin\mu\int_\mu^\infty \frac{1}{t^2}\theta_1(t)dt - \cos\mu\int_\mu^\infty \frac{1}{t^2}\theta_1(t)dt\right] \\ = & \ C\left[\frac{\sin\mu}{\mu} - \frac{\cos\mu}{\mu}\right] = O\left(\frac{1}{\mu}\right), \end{aligned}$$

then we get

$$a\int_\mu^\infty \frac{m(m+1)}{t^2}\sin(\mu - t)\,\theta_1(t)dt = O\left(\frac{1}{\mu}\right). \tag{12}$$

Now, we consider the second integral in (11). Using relations (7) and (12), we have

$$\begin{aligned} & -\mu\int_\mu^\infty \frac{m(m+1)}{t^2}\cos(\mu - t)\,\theta_1(t)dt \\ & = -\mu\left[\int_\mu^\infty \frac{m(m+1)}{t^2}\cos(\mu - t)\left(\sin\left(t - \frac{m\pi}{2}\right) - O\left(\frac{1}{t}\right)\right)dt\right] \\ & = \frac{-\mu}{2}\left[\int_\mu^\infty \frac{m(m+1)}{t^2}\left(\sin\left(\mu - \frac{m\pi}{2}\right) + \sin\left(2t - \mu - \frac{m\pi}{2}\right)\right)dt\right] + O\left(\frac{1}{\mu}\right), \end{aligned}$$

in this case, we can write

$$-\mu\int_{\mu}^{\infty}\frac{m(m+1)}{t^2}\cos(\mu-t)\,\theta_1(t)dt=\frac{-m(m+1)}{2}\sin\left(\mu-\frac{m\pi}{2}\right)+O\left(\frac{1}{\mu}\right). \tag{13}$$

By substituting (12) and (13) in (11), we get

$$g(\mu)=\mu\cos\left(\mu-\frac{m\pi}{2}\right)+b\mu\cos\left(\frac{m\pi}{2}\right)+\left[a-\frac{m(m+1)}{2}\right]\sin\left(\mu-\frac{m\pi}{2}\right)$$
$$-b\mu\int_0^{\infty}\frac{m(m+1)}{t^2}\cos(t)\,\theta_1(t)dt+O\left(\frac{1}{\mu}\right). \tag{14}$$

If $g(\mu)=0$, we can obtain eigenvalues of $L(0,a,b)$. When $g(\mu)=0$, we get

$$\cos\left(\mu-\frac{m\pi}{2}\right)+b\cos\left(\frac{m\pi}{2}\right)-b\int_0^{\infty}\frac{m(m+1)}{t^2}\cos(t)\,\theta_1(t)dt+O\left(\frac{1}{\mu}\right)=0. \tag{15}$$

To calculate the integral at (15), we split it into three parts

$$\int_0^{\infty}\frac{m(m+1)}{t^2}\cos(t)\,\theta_1(t)dt=\int_0^{\epsilon}\frac{m(m+1)}{t^2}\cos(t)\,\theta_1(t)dt$$
$$+\int_{\epsilon}^{1}\frac{m(m+1)}{t^2}\cos(t)\,\theta_1(t)dt+\int_1^{\infty}\frac{m(m+1)}{t^2}\cos(t)\,\theta_1(t)dt. \tag{16}$$

According to Lemma 2.1 of [20], the first integral in (16) is obtained. For the second integral in (16), $\theta_1(t)$ can be written as $\theta_1(t)=\sin\left(t-\frac{m\pi}{2}\right)+O\left(\frac{1}{t}\right)$, in this case we write

$$\int_0^{\infty}\frac{m(m+1)}{t^2}\cos(t)\,\theta_1(t)dt=(m+1)\epsilon^m+m(m+1)-\frac{m(m+1)}{2}\sin\left(\frac{m\pi}{2}\right)\left[\frac{1}{\epsilon}-1\right]. \tag{17}$$

Then we have

$$-\quad b\left[(m+1)\epsilon^m+m(m+1)-\frac{m(m+1)}{2}\sin\left(\frac{m\pi}{2}\right)\left(\frac{1}{\epsilon}-1\right)\right]$$
$$+\quad\cos\left(\mu-\frac{m\pi}{2}\right)+b\cos\left(\frac{m\pi}{2}\right)+O\left(\frac{1}{\mu}\right)=0. \tag{18}$$

In (18), we assume

$$A=\cos\left(\frac{m\pi}{2}\right)-\left[(m+1)\epsilon^m+m(m+1)-\frac{m(m+1)}{2}\sin\left(\frac{m\pi}{2}\right)\left(\frac{1}{\epsilon}-1\right)\right],$$

and we choose b so that $-1<bA<1$. Consequently, we obtain

$$\cos\left(\mu-\frac{m\pi}{2}\right)+\cos(M)+O\left(\frac{1}{\mu}\right)=0, \tag{19}$$

where $\cos(M) = bA$. With respect to (19), for n sufficiently large, there exist roots of the form

$$\mu_n^0 = \left(2n + \frac{m}{2}\right)\pi + M, \qquad n \in Z^+.$$

Let $\eta(x, \lambda, q)$ be the solution of (1) with the following conditions

$$\eta(0, \lambda, q) = 0, \qquad \eta(1, \lambda, q) = -b, \qquad \eta'(1, \lambda, q) = a, \qquad \eta'(0, \lambda, q) = 1. \tag{20}$$

Norming constant is defined as follows

$$k_n(q) = \varphi'(1, \lambda_n, q). \tag{21}$$

According to Lemmas 2.1 and 2.3 of [3], we consider $\varphi(x, \lambda, q)$ to be a solution for (1). By the Lemma 3.2 of the reference [20], we can conclude the estimate of solution $\varphi(x, \lambda, q)$, of the form

$$|\varphi(x, \lambda, q) - \psi(x, \lambda)| \leq C \left(\frac{x}{1 + |\mu x|}\right)^{m+1} \exp\left(|\mathrm{Im}(\mu)|x\right) E(x, \lambda), \tag{22}$$

$$|\varphi'(x, \lambda, q) - \psi'(x, \lambda)| \leq C \left(\frac{x}{1 + |\mu x|}\right)^{m} \exp\left(|\mathrm{Im}(\mu)|x\right) E(x, \lambda).$$

For notational convenience define

$$E(x, \lambda) = \exp\left(\int_0^x \frac{R(m, t)t|q(t)|}{1 + |\mu t|} dt\right) - 1, \tag{23}$$

from [20], $R(m, t) = 1$, for $m > 1$.

Lemma 2.1. *For a sufficiently large integer N, the functions $a\varphi(1, \lambda) + \varphi'(1, \lambda) + b\varphi'(0, \lambda)$ and $a\psi(1, \lambda) + \psi'(1, \lambda) + b\psi'(0, \lambda)$ have the same number of roots in half plane $Rel(\lambda) < [N + \frac{m}{2}]^2\pi^2, b \neq 0$.*

Proof. We assume

$$h(z) = a\psi(1, \lambda) + \psi'(1, \lambda) + b\psi'(0, \lambda),$$

$$h(z) + k(z) = a\varphi(1, \lambda) + \varphi'(1, \lambda) + b\varphi'(0, \lambda).$$

Using Rouche's theorem, we show that the functions $h(z)$, $h(z) + k(z)$ have the same number of roots. By (22) and (23), we have

$$|k(z)| \leq |a[\varphi(1, \lambda) - \psi(1, \lambda)]| + |\varphi'(1, \lambda) - \psi'(1, \lambda)| + |b[\varphi'(0, \lambda) - \psi'(0, \lambda)]|, \tag{24}$$

and $E(1,\lambda) = \exp\left(\int_0^1 \frac{t|q(t)|}{1+|\mu t|}dt\right) - 1$. Since by the Cauchy-Schwarz inequality

$$E(1,\lambda) \leq \frac{1}{|\mu|}\left(\int_0^1 |q(t)|^2 dt\right)^{\frac{1}{2}}\left(\int_0^1 dt\right)^{\frac{1}{2}},$$

then $E(1,\lambda) \leq \frac{1}{|\mu|}||q||$. Thus,

$$E(1,\lambda) < 1. \tag{25}$$

According to (24) and (25), we have

$$|k(z)| \leq C|\mu^{-(m+1)}| \exp(|\mathrm{Im}(\mu)|).$$

By Lemma 2.4 of [20], there is a constant K_1 such that

$$\begin{aligned} & \left|a\left[\psi(1,\lambda) - \mu^{-(m+1)}\sin\left(\mu - \frac{m\pi}{2}\right)\right] + \left[\psi'(1,\lambda) - \mu^{-m}\cos\left(\mu - \frac{m\pi}{2}\right)\right]\right| \\ + & \left|b\mu^{-m}\left[\cos\left(\frac{m\pi}{2}\right) - \int_0^\infty \cos(t)\,\frac{m(m+1)}{t^2}\theta_1(t)dt\right]\right| \\ \leq & K_1|\mu^{-(m+1)}|\exp(|\mathrm{Im}(\mu)|x) \end{aligned} \tag{26}$$

Due to the inequality (26), we can write

$$\left|a\mu^{-(m+1)}\sin\left(\mu - \frac{m\pi}{2}\right)\right| + \left|\mu^{-m}\cos\left(\mu - \frac{m\pi}{2}\right)\right| > K_2|\mu^{-m}|\exp(|\mathrm{Im}(\mu)|),$$

and using Rouche's theorem, we conclude that the number of roots $h(z)$ and $h(z) + k(z)$, are the same. □

From [20], we have the following Lemma:

Lemma 2.2. *If $q \in L_2[0,1]$ then $\lambda_n = \lambda_n^0 + O(1)$, where λ_n and λ_n^0 are eigenvalues of problems L and $L(0,a,b)$, respectively.*

In the next section, we give a version of Borg's theorem which is a statement of the inverse problem of the singular Sturm-Liouville problem with nonseparated boundary conditions.

3. The Borg's theorem

In [22] and [2], Ambarzumyan and Borg respectively presented crucial findings about the inverse problem of a regular Sturm-Liouville operator. In the following years, results which are obtained in these works have been generalized to various versions for Sturm-Liouville operator. The Borg's theorem is a uniqueness theorem in inverse eigenvalue problems. Regarding the result obtained in the previous section, we prove the Borg's theorem in this section. Firstly, we consider the following theorem.

Theorem 3.1. *Suppose that for all $n \geq 1$, $\lambda_n(p,a,b) = \lambda_n(q,a',b')$ and $k_n(p) = k_n(q)$, then $p = q$.*

Proof. We consider the function $\eta(x,\lambda,q)$ defined in (20). If $\lambda_n = \lambda_n(q)$ then φ defined as follows

$$\varphi(x,\lambda_n,q) = \frac{\varphi'(1,\lambda_n,q)}{a}\eta(x,\lambda_n,q), \qquad a \neq 0.$$

Now, first we place φ in (1) and considering $m \geq 0$, we can write $\varphi''(x)+q(x)\varphi(x) = \lambda\varphi(x)$. Then we take the derivative from the both sides of the obtained equation with respect to λ, at a result φ becomes $\varphi = (-D^2+q-\lambda)\partial_\lambda\varphi$, and by multiplying φ on the both sides of this equation, we can write

$$\varphi^2 = -\varphi\partial_\lambda\varphi'' + \varphi''\partial_\lambda\varphi, \tag{27}$$

by integrating both sides of (27), we have

$$\int_0^1 \varphi^2(x,\lambda)dx = [\varphi'\partial_\lambda\varphi - \varphi\partial_\lambda\varphi'](1,\lambda) - [\varphi'\partial_\lambda\varphi - \varphi\partial_\lambda\varphi'](0,\lambda). \tag{28}$$

Inserting $\lambda = \lambda_n(q)$ in (28) and using the conditions of L, we have

$$\int_0^1 \varphi^2(x,\lambda_n(q))dx = \varphi'(1,\lambda_n(q))\partial_\lambda\varphi(1,\lambda_n(q)) - \varphi(1,\lambda_n(q))\partial\varphi'(1,\lambda_n(q)). \tag{29}$$

By placing $\varphi(1,\lambda_n(q))$ in (29), we can write

$$\int_0^1 \varphi^2(x,\lambda_n(q))dx = \frac{\varphi'(1,\lambda_n(q))}{a}[a\partial_\lambda\varphi + \partial_\lambda\varphi'](1,\lambda_n(q)) \tag{30}$$

$$+\frac{b}{a}\varphi'(0,\lambda_n(q))\partial_\lambda\varphi'(1,\lambda_n(q)).$$

We define the following function

$$S(\lambda) = \frac{[\varphi(x,\lambda,q) - \varphi(x,\lambda,p)][\eta(x,\lambda,q) - \eta(x,\lambda,p)]}{a\varphi(1,\lambda,q) + \varphi'(1,\lambda,q) + b\varphi'(0,\lambda,q)}.$$

Now, we will look at the residues S. We define

$$\Delta(\lambda) := a\varphi(1,\lambda,q) + \varphi'(1,\lambda,q) + b\varphi'(0,\lambda,q),$$

so $\{\lambda_n\}$ are roots of the function $\Delta(\lambda)$. Then we define the residues function S

$$R_nS = \frac{[\varphi(x,\lambda_n,q) - \varphi(x,\lambda_n,p)][\eta(x,\lambda_n,q) - \eta(x,\lambda_n,p)]}{a\partial_\lambda\varphi(1,\lambda_n,q) + \partial_\lambda\varphi'(1,\lambda_n,q) + b\partial_\lambda\varphi'(0,\lambda_n,q)}.$$

By using (21) and with respect to the assumption of the theorem, we can write $\varphi'(1, \lambda_n, p) = \varphi'(1, \lambda_n, q)$. In this case

$$R_n S = \frac{[\varphi(x, \lambda_n, q) - \varphi(x, \lambda_n, p)]^2}{\int_0^1 \varphi^2(x, \lambda_n(q))dx + B},$$

where $B = \dfrac{b}{a}[\varphi'(1, \lambda_n, q)\partial_\lambda \varphi'(0, \lambda_n, q) - \varphi'(0, \lambda_n, q)\partial_\lambda \varphi'(1, \lambda_n, q)]$. The residues for when $B > 0$ are $R_n S \geq 0$.
By utilizing Lemmas 3.2 and 3.3 from [20], it can be asserted that $S(\lambda)$ is bounded. On the other hands in $S(\lambda)$, the roots of the denominator give us eigenvalues. In this case, if $\nu_n = \mu_n^0 + \frac{\pi}{4}$ then we get $\lim_{|\lambda|=\nu_n} \lambda S(\lambda) = 0$. Then, it can be concluded that the sum of the nonnegative residues R_n is zero so that, $\lim_{N\to\infty} \sum_{n=1}^{N} R_n = 0$. Since the eigenfunction for p and q are the same, therefore $p = q$ almost everywhere. □

Now, we will state and prove the main theorem of this section.

Theorem 3.2. ***(non-separated Borg's theorem).***
Suppose that for all $n \geq 1$, we have $\lambda_n(p, a_j, b_j) = \lambda_n(q, a_j, b_j)$ for $j = 1, 2$ and for linearly independent vectors (a_1, b_1) and (a_2, b_2). Then we conclude that $p = q$.

Proof. We know that $a_j\varphi(1, \lambda) + \varphi'(1, \lambda) + b_j\varphi'(0, \lambda)$ is the entire function with an order of $\frac{1}{2}$. By Hadamard theorem [23], we understand that it has zeroes in the forms $\{\lambda_{n,j}\}$ which are the eigenvalues. Suppose that 0 is not an eigenvalue, then we have

$$a_j\varphi(1, \lambda) + \varphi'(1, \lambda) + b_j\varphi'(0, \lambda) = T_j \prod_n \left(1 - \frac{\lambda}{\lambda_{n,j}}\right), \tag{31}$$

where T_j are constants compared to λ. Now, we consider $b \neq 0$. Let $\xi_k = [2k\pi + m]^2$, so $\xi_k \geq 0$. On the other hand $\mu = \sqrt{\xi_k}$, so can say $Im\xi_k = 0$. With respect to Lemma 2.4 of [20], we write

$$\psi(x, \lambda) \sim \mu^{-(m+1)} \sin\left(\mu x - \frac{m\pi}{2}\right), \tag{32}$$

$$\psi'(x, \lambda) \sim \mu^{-m} \cos\left(\mu x - \frac{m\pi}{2}\right), \tag{33}$$

and by using Lemma 3.2 of [20], we get

$$\varphi(x, \lambda, q) \sim \psi(x, \lambda), \tag{34}$$

$$\varphi'(x, \lambda, q) \sim \psi'(x, \lambda). \tag{35}$$

For the case $x = 0$, $\psi'(0, \lambda)$ is as follows

$$\psi'(0,\lambda) = \mu^{-m}\theta_1'(0) = \mu^{-m}\left[\cos\left(\frac{m\pi}{2}\right) - \int_0^\infty \cos(t)\,\frac{m(m+1)}{t^2}\theta_1(t)dt\right], \quad (36)$$

then we see $\psi'(0, \lambda)$ is bounded. The goal is to obtain the following limit

$$\lim_{k\to\infty} \xi_k^{\frac{m}{2}}[a_j\varphi(1,\xi_k) + \varphi'(1,\xi_k) + b_j\varphi'(0,\xi_k)]. \quad (37)$$

By placing relations (32)-(36) in (37), we have

$$\begin{aligned} \lim_{k\to\infty} \xi_k^{\frac{m}{2}}[a_j\varphi(1,\xi_k) + \varphi'(1,\xi_k) + b_j\varphi'(0,\xi_k)] &= 1 + Cb_j \qquad (38) \\ &= T_j \lim_{k\to\infty} \xi_k^{\frac{m}{2}} \prod_n \left(1 - \frac{\xi_k}{\lambda_{n,j}}\right). \end{aligned}$$

We assume that p and q are distinct. According to (38), T_j values for $j = 1, 2$ are determined from the eigenvalues sequence. On the other hand, according to the assumption of the theorem, we can conclude that T_j are equal for the different j. We know when $p \neq q$, in this case, T_j values would be different, leading to a contradiction. As a result, we have $p = q$. □

Various methods for solving inverse spectral problems have been developed, including Borg's method, the transformations operator method, the spectral maps method, and others. In this study, we focused on uniqueness theorems and presented Borg's theorem for the Sturm-Liouville problem with a singular potential. We have also provided a simple and short proof of Borg's theorem.

4. Conclusion

This paper focused on the Sturm-Liouville problem where the potential function has a singular point. We established boundary conditions for the equation and identified the Sturm-Liouville problem with non-separated boundary conditions. We demonstrated that by utilizing the solutions presented for the equation and following the problem's terms, we could obtain eigenvalues. Additionally, we discussed the Borg's theorem. The key contribution of this paper is the novel result obtained for the Sturm-Liouville problem with non-separated boundary conditions.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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