Mathematics Interdisciplinary Research **8** (3) (2023) 247 – 260

Original Scientific Paper

An Upper Bound for Min-Max Angle of Polygons

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Abstract

Let S be a set of n points in the plane, $\nabla(S)$ the set of all simple polygons crossing S, γ_P the maximum angle of polygon $P \in \nabla(S)$ and $\theta = min_{P\in\nabla(S)}\gamma_P$. In this paper, we prove that $\theta \leq 2\pi - \frac{2\pi}{r.m}$ where m and r are the number of edges and inner points of the convex hull of S, respectively. We also propose an algorithm to construct a polygon with the upper bound on its angles. Constructing a simple polygon with the angular constraint on a given set of points in the plane can be used for path planning in robotics. Moreover, we improve our upper bound on θ and prove that this is tight for $r = 1$.

Keywords: Min-max angle, Upper bound, Sweep arc, Simple polygonization, Computational geometry.

2020 Mathematics Subject Classification: 68U05, 65D17, 68W01.

How to cite this article
S. Asaeedi, F. Didehvar and A. Mohades, An upper bound for min-max angle of polygons, *Math. Interdisc. Res.* **8** (3) (2023) 247-260.

1. Introduction

An optimal polygonization of a set of points in the plane is a classical problem in computational geometry and has been applied to many fields such as image processing [1, 2], pattern recognition [2, 3], geographic information system [4], etc. Considering a set S of points in the plane, there are different numbers of simple polygons on S. Enumerating and generating simple polygons on S have been the focus of many studies [5–9].

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Academic Editor: Abbas Saadatmandi
Received 28 June 2022, Accepted 6 April 2023
DOI: 10.22052/MIR.2023.246534.1363

The polygonization problem constructs a polygon with a special property on a given set of points. There are many variants of this problem. The minimum and maximum area polygonizations try to find a polygon with minimum and maximum areas, respectively. These problems are NP-complete, as shown by Fekete [10, 11]. TSP and max-TSP construct a polygon with minimum and maximum perimeters, respectively. There are many ongoing studies on approximation algorithms for minimum and maximum area polygonizations [12, 13], TSP [14, 15], and max-TSP [16]. Our problem is finding a polygon whose maximum angle is minimum over all possible polygons.

Moreover, the researchers have investigated some properties of angles in the above-mentioned problems. The Angular-Metric TSP [17] is the problem of finding a tour on S that minimizes the sum of the direction changes at each point. Fekete and Woeginger introduced the Angle-Restricted Tour problem [18]. For $A \subseteq (-\pi, \pi]$ set of angles, the Angle-Restricted Tour is the problem of finding a simple or non-simple polygon on S where all angles of the polygon belong to A. The concept of α-concave hull [19] refers to a simple polygon P with minimum area covering a set of points such that all angels of P are less than or equal to $\pi + \alpha$. The α-concave hull is a generalization of minimum area polygonization as well as a generalization of the convex hull.

Reflexivity, the smallest number of reflex vertices among all polygonizations of a set of points, is considered as a convexity measurement for the points. Arkin et al. [20] introduced the concept of reflexivity as $\rho(n)$ and presented lower and upper bounds for reflexivity of any set of n points. They proved that $\lfloor \frac{n}{4} \rfloor \leq \rho(n) \leq \lceil \frac{n}{2} \rceil$. Ackerman et al. [21] improved the upper bound and proposed an algorithm to compute a polygon with at most the number of reflex vertices in the time complexity of $O(n \log n)$. They showed that the reflexivity of any set of n points is at most $\frac{3}{7}n + O(1)$. Lien and Amato [22] proposed the convexity measurement $\frac{volume(P)}{volume(H_P)}$ for a polyhedra P such that H_P is the convex hull of P.

The last presented upper bound for min-max value of the angles in polygonization is $2\pi - \frac{2\pi}{2^{r-1}.m}$ where m and r are the number of edges and inner points of the convex hull of S, respectively [23]. Here we improved the upper bound to be $2\pi - \frac{2\pi}{r.m}$.

Javad et al. [24] presented the first polynomial time algorithm to compute the convex hull of imprecise points in the plane such that each imprecise point is modelled by a segment. Chan [25] presented output-sensitive algorithms to compute the convex hull of points in 2D or 3D. Jarvis [26] introduced the first hull on a set of points which allowed some concavities in the constructed shape. Edelsbrunner [27] introduced the concept of α-shape as a generalization of convex hull and Krasnoshchekov and Polishchuk [28] introduced Order-k α-shapes as a generalization of α-shape. Here we present an algorithm to cover a set of points in the plane. There are many studies for covering and separating point sets [29–31].

Finding a simple polygon on a set of points that do not construct sharp reflex angles is applicable to path planning in robotics. Assume that a robot with a turn

Table 1: Notations of symbols.

Notation	Description
S	A set of points in the plane
n	cardinality of S
s_i	ith point of S $(1 \leq i \leq n)$
CH	convex hull of S
m	number of vertices of CH
IP	inner points of CH
r	cardinality of IP
P	a simple Polygon crossing S
V_P	vertices of P
E_P	edges of P
c_j	jth vertex of CH $(1 \leq j \leq m)$
e_j	jth edge of CH $(1 \leq j \leq m)$
$\overline{s_i s_j}$	an edge of P with s_i and s_j as its end points $(1 \leq i, j \leq n,\ i \neq j)$
$\nabla(S)$	set of all simple polygons crossing S
α, β, γ, θ	angles between 0 and 2π

angle constraint is used to visit a set of points. If the turn angle limitation is less than or equal to the presented upper bound, we can plan a path for the robot to cover the points. In addition, this bound is useful for designing a path for a robot arm that does not have freedom of rotation.

The rest of the paper is as follows: in Section 2, notations and definitions are presented, in Section 3, the upper bound is derived and in Section 4, we conclude the paper by highlighting its achievements.

2. Preliminaries

Let $S = \{s_1, s_2, ..., s_n\}$ be a set of points in the plane and CH the convex hull of S. The vertices and edges of CH are denoted by $V_{CH} = \{c_1, c_2, ..., c_m\}$ and $E_{CH} = \{e_1, e_2, ..., e_m\}$, respectively. Furthermore, let $IP = \{a_1, a_2, ..., a_r\}$ be the inner points of CH where $r = n - m$. The points of IP are inside the CH but not on its boundary. We assume that $r > 0$, i.e., the points of S are not in the convex position. Table 1 shows other notations that are used in the rest of the paper. A polygon P crossing S is specified by a closed chain of vertices $P = (p_1, p_2, ..., p_n, p_1)$ such that $S = V_P = \{p_1, p_2, ..., p_n\}$.

Let $e = \overline{AB}$ be a line segment. The minor arc $\overset{\frown}{AB}$ with measure equal to α is denoted by s_e^{α}, and the major arc $\overset{\frown}{AB}$ with measure equal to $\beta = 2\pi - \alpha$ is denoted by S_e^{β}. We denote the minor and major segments on e by m_e^{α} and M_e^{β}, respectively (see Figure 1). Moreover, we use the concept of *Sweep Arc* in our algorithm which is defined in [23, Section 4].

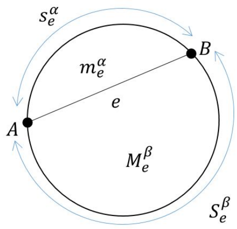

Figure 1: s_e^α, S_e^β, m_e^α and M_e^β are minor arc, major arc, minor segment and major segment on e, respectively.

Definition 2.1. ([23, Section 4]). Let $e = \overline{AB}$ be an edge of polygon P. A sweep arc on the edge e is a minor arc AB where $\widehat{AB}= 0$ and expands to the major arc AB where $\widehat{AB}= \pi$. The direction of expansion is to the inside of the polygon. Figure 2 depicts the sweep arc on the edge e.

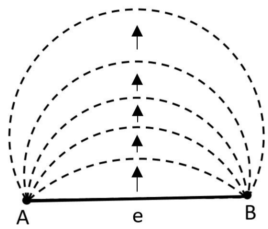

Figure 2: A sweep arc on the edge e [23, Figure 6].

3. Min-max angle

In this section, we present two upper bounds for θ. Let us first present a lemma followed by a theorem.

Lemma 3.1. *Let $l = \overline{c_1c_2}$ be a line segment and S a set of n points inside the $M_l^{\beta_{max}}$, where $\beta_{max} = 2\pi - \frac{4\pi}{m}$ for an integer number m. Assume that t points $\{s_1, s_2, ..., s_t\}$ are met by the sweep arc on l and $P = (c_1, s_1, s_2, ..., s_t, c_2, c_1)$ is a simple polygon such that all internal angles of $\hat{s}_i$ are greater than or equal to $\frac{2\pi}{t.m}$. Let x be $(t+1)th$ point met by the sweep arc. There exists an edge $\overline{ab}$ of P such that $\widehat{axb}$ is greater than or equal to $\frac{2\pi}{(t+1).m}$.*

Proof. We prove the lemma by induction on t. When $t = 0$, the chain $P = (c_1, s_1, s_2, ..., s_t, c_2, c_1)$ is a line segment of $\overline{c_1c_2}$. Therefore, we consider both cases $t = 0$ and $t = 1$ as the base cases.

Base case ($t = 0$) Let x be the first point that the sweeping arc meets. We construct the polygon by connecting x to c_1 and c_2. Since the maximum measure of the arc is β_{max}, the internal angle of $\gamma = \widehat{c_1xc_2}$ in the triangle $\triangle c_1xc_2$ is greater than or equal to $\frac{2\pi}{m}$ (see Figure 3).

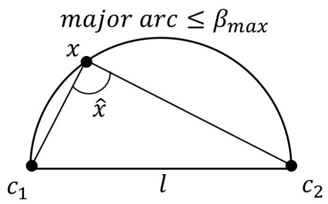

Figure 3: $\hat{x}$ is greater than $\frac{2\pi}{m}$ [23, Figure 11].

Base case ($t = 1$) Let s_1 be the first point that the sweeping arc meets and x the second one. In addition, let $e_1 = \overline{c_1s_1}$ and $e_2 = \overline{s_1c_2}$ be two edges of $P = (c_1, s_1, c_2, c_1)$. The edges e_1 and e_2 divide the sweeping arc into 3 parts; the arc B_1 where e_1 is visible but e_2 is not visible from all the points on it, the arc B_2 where e_2 is visible but e_1 is not visible from all the points on it, and finally the arc B_3 where e_1 and e_2 are both visible from all the points on it (see Figure 4).

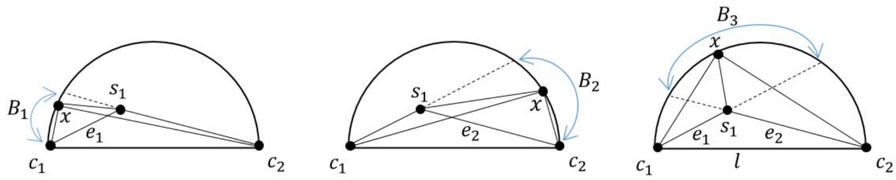

Figure 4: The edges e_1 and e_2 divide the sweeping arc into 3 parts: B_1, B_2 and B_3. [23, Figure 12].

1. If x is placed on B_1, the angle $\widehat{c_1xs_1}$ is greater than $\gamma = \widehat{c_1xc_2}$ and the angle γ is greater than or equal to $\frac{2\pi}{m}$. Hence, the angle $\widehat{c_1xs_1}$ is greater than $\frac{2\pi}{m}$. So, we consider the edge $\overline{c_1s_1}$ as the desired edge $\overline{ab}$ such that $\widehat{axb}$ is greater than or equal to $\frac{\pi}{m}$.

2. If x is placed on B_2, the angle $\widehat{s_1xc_2}$ is greater than γ and the angle γ is greater than or equal to $\frac{2\pi}{m}$. Hence, the angle $\widehat{s_1xc_2}$ is greater than $\frac{2\pi}{m}$. So,

we consider the edge $\overline{s_1c_2}$ as the desired edge $\overline{ab}$ such that $\widehat{axb}$ is greater than or equal to $\frac{\pi}{m}$.

3. If x is placed on B_3, the maximum of $\widehat{c_1xs_1}$ and $\widehat{s_1xc_2}$ is greater than $\frac{\gamma}{2}$. Since γ is greater than $\frac{2\pi}{m}$, the maximum of $\widehat{c_1xs_1}$ and $\widehat{s_1xc_2}$ is greater than $\frac{2\pi}{2m}$. Hence, if $\widehat{c_1xs_1}$ is greater than $\widehat{s_1xc_2}$, the edge $\overline{c_1s_1}$ is considered as $\overline{ab}$, otherwise, the edge $\overline{s_1c_2}$ is considered as $\overline{ab}$.

Induction assumption Let y be kth point that the sweeping arc meets. There exists an edge $\overline{ab}$ of $P = (c_1, s_1, s_2, ..., s_{k-1}, c_2, c_1)$ such that $\widehat{ayb}$ is greater than or equal to $\frac{2\pi}{k.m}$.

Induction assumption Let x be $(k+1)$th point that the sweeping arc meets and $P = (c_1, s_1, s_2, ..., s_k, c_2, c_1)$ be the polygon such that all internal angles of $\hat{s}_i$ are greater than or equal to $\frac{2\pi}{k.m}$. We show that there exists an edge $\overline{ab}$ of P such that $\widehat{ayb}$ is greater than or equal to $\frac{2\pi}{(k+1).m}$. Here, three cases need to be examined:

1. All edges of P except $\overline{c_1c_2}$ are visible from x. Let $e_1 = \overline{c_1s_1}$, $e_2 = \overline{s_1s_2}, \ldots,$ and $e_{k+1} = \overline{s_kc_2}$ be edges of P, β_i the angle subtended by e_i at the point x and β_M the maximum one. When angle γ is greater than or equal to $\frac{2\pi}{m}$ and $\Sigma_{i=1}^{k+1}\beta_i = \gamma$, we have $\beta_M > \frac{2\pi}{(k+1).m}$. Let e be the edge that corresponds to β_M. So, the edge e is considered as $\overline{ab}$ such that $\widehat{axb}$ is greater than or equal to $\frac{2\pi}{(k+1).m}$ (see Figure 5).

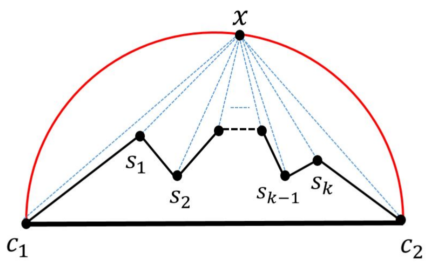

Figure 5: All edges e_i are visible from x.

2. There exists an edge $e = \overline{cd}$ of P such that both endpoints of e are not visible from x. We obtain a polygon P' from P by contracting [32] the edge e, i.e., $P' = P/e$. Since P' has $k+1$ vertex points, by induction assumption, there exists an edge $e' = \overline{ab}$ of P' such that $\widehat{axb}$ is greater than or equal to $\frac{2\pi}{k.m}$. The polygon P'' is obtained from P by removing the edge $\overline{ab}$ and adding two edges $\overline{ax}$ and $\overline{xb}$. Since the two end points of $e = \overline{cd}$ are invisible from x,

contracting and splitting e have no effect on the measure of the angle $\widehat{axb}$. Hence, the angle $\widehat{axb}$ in P'' is greater than or equal to $\frac{2\pi}{(k+1).m}$ (see Figure 6).

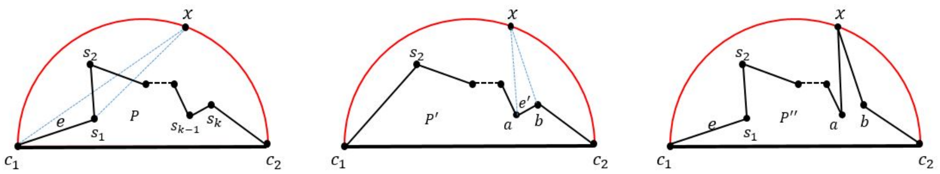

Figure 6: The edge e of P is invisible from x. Contracting e leads to constructing P' from P. The polygon P'' is obtained from P and P'.

3. There exists an edge $e = \overline{cd}$ of P such that one endpoint of e is not visible from x (see Figure 7). We obtain a polygon $P' = P/e$ from P by contracting the edge e. Since P' has $k+1$ vertex points, by induction assumption, there exists an edge $e' = \overline{ab}$ of P' such that $\widehat{axb}$ is greater than or equal to $\frac{2\pi}{k.m}$. The polygon P'' is obtained from P by removing the edge $\overline{ab}$ and adding two $\overline{ax}$ and $\overline{xb}$ edges.

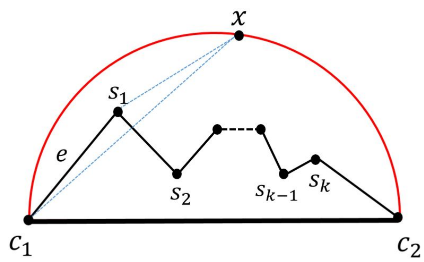

Figure 7: The vertex s_1 is visible and c_1 is invisible from x.

If either a or b in P'' is an endpoint of e, contracting and splitting e have an effect on the measure of the angle $\widehat{axb}$ (see Figure 8). In other words, the angle $\widehat{axb}$ in P'' is not equal to the angle $\widehat{axb}$ in P'. It is clear that the angle $\widehat{axb}$ in P'' is greater than the angle $\widehat{axb}$ in P'. Since the angle $\widehat{axb}$ in P' is greater than or equal to $\frac{2\pi}{k.m}$, the angle $\widehat{axb}$ in P'' is greater than or equal to $\frac{2\pi}{(k+1).m}$.

In addition, if neither a nor b in P'' is an endpoints of e, contracting and splitting e have no effect on the measure of the angle $\widehat{axb}$ (see Figure 9). In other words, the angle $\widehat{axb}$ in P'' is equal to the angle $\widehat{axb}$ in P'. Hence, the angle $\widehat{axb}$ in P'' is greater than or equal to $\frac{2\pi}{(k+1).m}$.

□

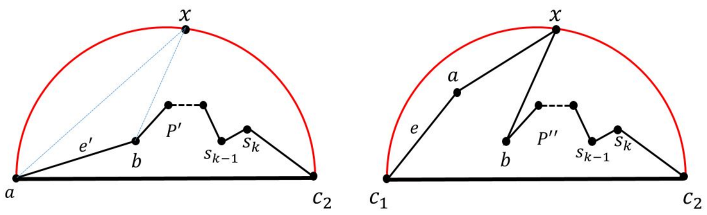

Figure 8: $\widehat{axb}$ in polygon P'' is greater than $\widehat{axb}$ in polygon P'.

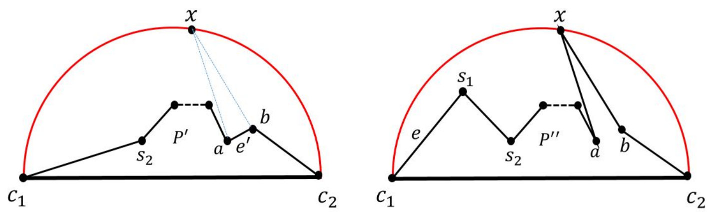

Figure 9: $\widehat{axb}$ in polygon P'' is equal to $\widehat{axb}$ in polygon P'.

Theorem 3.2. *Let $l = \overline{c_1c_2}$ be a line segment and S a set of n points inside the $M_l^{\beta_{max}}$, such that $\beta_{max} = 2\pi - \frac{4\pi}{m}$ for an integer number m (see Figure 10.a). There exists a chain $(s_1, s_2, ..., s_n)$ on S such that all internal angles of $\hat{s}_i$ in the polygon $(c_1, s_1, s_2, ..., s_n, c_2, c_1)$ are greater than or equal to $\frac{2\pi}{n.m}$ (see Figure 10.b).*

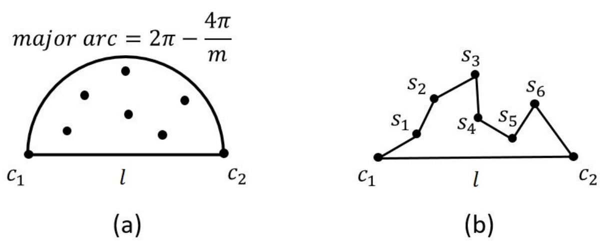

Figure 10: (a) S is a set of 6 points inside $M_l^{\beta_{max}}$. (b) $\forall 1 \leq i \leq 6$, $\hat{s_i} \geq \frac{2\pi}{6m}$ [23, Figure 10].

Proof. We prove Theorem 3.2 by constructing the polygon $(c_1, s_1, s_2, ..., s_n, c_2, c_1)$, using the following algorithm which is a modified version of the one originally presented in [23]:

Algorithm 1 (Modified Sweep Arc Algorithm)

1. Sweep the arc $\widehat{c_1c_2}$ from s_l^0 to $S_l^{\beta_{max}}$.
2. Let x_1 be the first point which is met by the sweep arc. Construct $P = (c_1, x_1, c_2, c_1)$ as the desired polygon.
3. Set $i = 2$.
4. Let $P = (c_1, s_1, s_2, ..., s_{i-1}, c_2, c_1)$ be the constructed polygon inside the sweep arc and x_i the ith point which is met by the sweep arc.
5. Assume that $e_1 = \overline{c_1s_1}$, $e_2 = \overline{s_1s_2}$, ... , and $e_i = \overline{s_{i-1}c_2}$ are the edges of P. If e_j is visible from x_i, set β_j =The angle subtended by e_j at the point x_i, otherwise set $\beta_j = 0$.
6. Let $\beta_M = \max_{1\leq j\leq i} \beta_j$ and $e = \overline{ab}$ be the edge that corresponds to β_M.
7. Remove the edge e from P and add two edges $\overline{ax_i}$ and $\overline{x_ib}$ to construct the desired polygon.
8. Set $i = i + 1$. If $i \leq n$, then go to 4, otherwise exit.

Based on Lemma 3.1, $\forall j \in \{1, 2, ..., i\}$ the angles $\hat{s_j}$ in P are greater than or equal to $\frac{2\pi}{i.m}$ in step 4 of the algorithm. Therefore, when $i = n$, the angles $\hat{s_j}$ in P will be greater than or equal to $\frac{2\pi}{n.m}$. □

It is proved in [23, Lemma 3] that all angles of the mentioned polygon P are greater than or equal to $\frac{2\pi}{2^{n-1}.m}$. Here, based on Theorem 3.2, we increase this bound to $\frac{2\pi}{n.m}$. This yields us the following corollaries:

Corollary 3.3. *Let S be a set of points in the plane, CH the convex hull of S and m and r the number of edges and inner points of CH, respectively. If we replace algorithm 1 of [23, Theorem 2, Step 2.a of Algorithm 2] by the modified sweep arc algorithm, the upper bound of $2\pi - \frac{2\pi}{r.m}$ is achieved for θ.*

Remark 1. Based on Corollary 3.3, in the case of $r = 1$, $2\pi - \frac{2\pi}{n-1}$ is an upper bound for θ over all simple polygons crossing S. It is noteworthy that this bound is tight in this case. The tightness is achieved when the inner point is at the center of a regular n-gons, as illustrated in Figure 11.

The following corollary improved the upper bound to $2\pi - \frac{2\pi}{d.m}$ where d is *Depth of Angular Onion Peeling* on S which is defined in [23].

Definition 3.4. ([23]). Let us increase the measure of all sweeping arcs concurrently from 0 to the first hit (or β_{max}, if a sweeping arc does not hit any point). All the inner points that are hit by sweeping arcs form layer 1 of the points. The next layers are formed by deleting the points of the prior computed layer from inner points and continuous increasing of all sweeping arcs measures up to the next hit. The process continues until all inner points are hit. The process of peeling away the layers, described above, is defined as *Angular Onion Peeling* and the number of layers is called *Depth of Angular Onion Peeling* on these points.

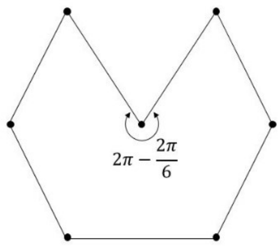

Figure 11: Maximum angle of each polygon crossing these points is equal to $2\pi - \frac{2\pi}{6}$ [23, Figure 9].

Corollary 3.5. *Let S be a set of points in the plane, CH the convex hull of S, m the cardinality of edges of CH and d the depth of angular onion peeling on S. If we replace algorithm 1 of [23, Theorem 3, Step 2.b of Algorithm 3] by the modified sweep arc algorithm, the upper bound $2\pi - \frac{2\pi}{d.m}$ is achieved for θ.*

Since the time complexity of the modified sweep arc algorithm is $O(r)$, those of both modified algorithms 2 and 3 are $O(n \log n + rm)$. Note that the modified algorithms 2 and 3 are those proposed in [23] in which the algorithm 1 is replaced by the modified sweep arc algorithm. Based on Corollary 3.3, the modified algorithm 2 constructs a polygon such that its internal angles are less than or equal to $2\pi - \frac{2\pi}{r.m}$. Based on Corollary 3.5, this bound is improved to $2\pi - \frac{2\pi}{d.m}$ using modified algorithm 3. When S is a set of n points in the plane and the convex hull of S has $n-1$ edges, the depth of angular onion peeling on S is equal to 1. Hence, the upper bound for θ is equal to $2\pi - \frac{2\pi}{1.(n-1)}$, which confirms Remark 1.

Computing α-concave hull on a set S of points is an NP-complete problem [19]. For all $\alpha > \theta$, α-concave hull crosses all points of S. So, the polygon computed by modified algorithm 3 is an α-polygon [19] which approximates α-concave hull of S. The following corollary shows the relation between α-concave hull and the computed upper bound.

Corollary 3.6. *Let S be a set of points in the plane, CH the convex hull of S, m the cardinality of edges of CH and d the depth of angular onion peeling on S, for all $\alpha > 2\pi - \frac{2\pi}{d.m}$, there always exists an α-concave hull P on S such that P crosses all points of S.*

Coverage path planning is a fundamental problem in the field of robotics. There are many limitation factors in order to plan a path for a robot to cover (or visit) all points of a set of points, such as robot rotation angle. The following corollary presents the essential relation between path planning in robotics and our upper bounds on θ.

Corollary 3.7. *Let S be a set of n points in the plane, CH the convex hull of S, m the cardinality of edges of CH and d the depth of angular onion peeling on S.*

If the robot rotation angle is greater than $2\pi - \frac{2\pi}{d.m}$*, there always exists a path for the robot to cover* S*. As stated before, this path can be found in* $O(n \log n + rm)$ *time complexity.*

4. Conclusion

The major problem investigated in this paper is that of finding a simple polygon with the angular constraint on a given set of points in the plane. We derived the upper bounds for min-max value of angles over all simple polygons crossing the given set of points. We also presented algorithms to compute the polygons thereby satisfying the derived upper bounds. In addition to the theoretical results, this bound is an important achievement in the field of robotics.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

Acknowledgment. The research of the first author is partially supported by the University of Kashan under grant number 991449/4.

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