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On the Maximal Graph of a Commutative Ring

Masoumeh Soleimani, Fatemeh Mahmudi and
Mohammad Hasan Naderi⋆

Abstract

Let R be a commutative ring with nonzero identity. Throughout this paper we explore some properties of two certain subgraphs of the maximal graph of R.

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1. Introduction

First, we state some definitions and notations that will be used in the paper. All rings are assumed to be commutative with $1 \neq 0$ and all graphs are assumed to be undirected and simple. As usual, the ring of integers, integers modulo n, for any positive integer $n \geq 2$, and the cardinality of a set A, for the set A, will be denoted by $\mathbb{Z}$, $\mathbb{Z}_n$ and $|A|$, respectively. Also, for the ring R, the set of all maximal ideals of R and the Jacobson radical of R will be denoted by $\mathrm{Max}(R)$ and $\mathrm{J}(R)$, respectively. A ring R is said to be $quasi-local$ if $\mathrm{Max}(R)$ is a singelton set; in this situation if $\mathrm{Max}(R) = \{M\}$, we will often write (R, M).

For a graph G and any two distinct vertices x and y, $x \neq y$, of G, the *complement* of G, the *degree* of x in G and the least length of a path are denoted by G^c, $\deg(x)$ and $\mathrm{d}(x, y)$, respectively. Given a graph G, a *dominating set*

⋆Corresponding author (E-mail: mh.naderi@qom.ac.ir)
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is a subset D of the vertex set G such that every vertex not in D is adjacent to at least one element of D. The number of vertices in the smallest dominating set G is denoted by $\gamma(G)$ and it is called the *domination number*. A connected graph G is called $semi-Eulerian$ if there exists an open trail containing every edge of the graph exactly once. A graph is clearly semi-Eulerian if and only if it has at most two vertices of odd order. A graph is said to be *split* if it can be partitioned in an independent set and a clique. A *pancyclic graph* is a graph that contains cycles of all possible lengths from three up to the number of vertices in the graph. In fact, pancyclic graphs are a generalization of Hamiltonian graphs. For other general concepts on graph theory, one may consult [1].

The comaximal graph of a ring R, denoted by $\Gamma(R)$, is a graph whose vertices are the elements of R and two vertices a and b are adjacent if and only if $Ra+Rb=R$. The graph $\Gamma(R)$ was introduced in [2] and then in [3], the authors studied some subgraphs of $\Gamma(R)$ such as $\Gamma_1(R)$, $\Gamma_2(R)$ and $\Gamma_2(R)-\mathrm{J}(R)$, where the vertex sets are the set of units of R, the set of nonunits of R, and the set of nonunits of R which are not in $\mathrm{J}(R)$, respectively. If R is a ring with at least two maximal ideals, for the sake of convenience, we denote $\Gamma_2(R)-\mathrm{J}(R)$ by $G(R)$. In [4], Gaur and Sharma studied the complement of $\Gamma(R)$, which is known as the maximal graph of the ring R. So the vertices of the maximal graph are the members of R and two distinct vertices a and b are adjacent if and only if there exists a maximal ideal of R that includes both a and b.

Throughout this paper, in continuation of [5, 6], we investigate some additional properties of the maximal graph. In Section 2, first we investigate the domination number of $(G(R))^c$ and $(\Gamma_2(R))^c$. Then we study when $(G(R))^c$ and $(\Gamma_2(R))^c$ are regular. Finally, we give some characterization of a ring R, such that $(G(R))^c$ is split. In Section 3, we obtain some results on $(\Gamma_2(R))^c$ and $(G(R))^c$ to be pancyclic. Finally, in Section 4, we get some necessary and sufficient conditions for $(G(R))^c$ and $(\Gamma_2(R))^c$ to be semi-Eulerian.

2. Domination number, regular, and split graph

In this section, we state some properties of two important subgraphs of the maximal graph, namely $(G(R))^c$ and $(\Gamma_2(R))^c$. In particular, we compute their domination number of them. Also we investigate rings R such that $(G(R))^c$ and $(\Gamma_2(R))^c$ are regular. At last, for any fields F_1 and F_2, we characterize rings $R=F_1\times F_2$, such that $((\Gamma_2(R))^c$ is split.
Firstly, we try to find out the domination number of $(\Gamma_2(R))^c$ and $(G(R))^c$.

Lemma 2.1. *For any commutative ring R with unity, $\gamma((\Gamma_2(R))^c)=1$.*

Proof. It is sufficient to set $D=\{a\}$ where $a\in\mathrm{J}(R)$. □

Lemma 2.2. *Let R be a commutative ring with identity, possessing at least two maximal ideals. Then $\gamma((G(R))^c)=2$.*

Proof. Let $|\mathrm{Max}(R)| = 2$ and $\mathrm{Max}(R) = \{N_1, N_2\}$. Set $D = \{a, b\}$, such that $a \in N_1 - \mathrm{J}(R)$ and $b \in N_2 - \mathrm{J}(R)$. If $(G(R))^c$ is disconnected, then $\gamma((G(R))^c) = 2$.

Now, let $|\mathrm{Max}(R)| \geq 3$ and $\mathrm{Max}(R) = \{N_1, ..., N_k\}$, where $k \geq 3$. Set $D = \{a, b\}$, such that $a \in \cap_{i=1}^{k-1} N_i - N_k$ and $b \in \cap_{i=1, i\neq k-1}^{k} N_i - N_{k-1}$. For every vertex $x \in (G(R))^c$, there exists $1 \leq j \leq k$ such that $x \in N_j$. Three cases might be distinguished:
Case 1 : $1 \leq j \leq k-2$, so x connects to a and b.

Case 2 : $j = k-1$, so $x, a \in N_{k-1}$, then x connects to a.

Case 3 : $j = k$, so $x, b \in N_k$, thus x connects to b.

We claim that $\gamma((G(R))^c) = 2$. Let $D = \{c\}$, so for every vertex $x \in ((G(R))^c$, x is connected to c, therefore, $c \in \mathrm{J}(R)$, which is impossible by the hypothesis. □

Lemma 2.3. *Let R be a finite commutative ring with identity. Then*
(1) *If (R, M) is quasi-local, then $(\Gamma_2(R))^c$ is a $(|M| - 1)$-regular graph.*
(2) *If R has at least two maximal ideals, then $(\Gamma_2(R))^c$ is not a regular graph.*

Proof. It is clear. □

Lemma 2.4. *Let $R = R_1 \times R_2$ where (R_1, M_1) and (R_2, M_2) are finite quasi -local rings. Then $(G(R))^c$ is a regular graph if and only if $\dfrac{|R_1|}{|M_1|} = \dfrac{|R_2|}{|M_2|}$.*

Proof. Let $N_1 = R_1 \times M_2$ and $N_2 = M_1 \times R_2$, so $\mathrm{Max}(R) = \{N_1, N_2\}$. For every vertex $x \in N_1 - \mathrm{J}(R)$, $\deg(x) = |R_1||M_2| - |M_1||M_2| - 1$ and for every vertex $x \in N_2 - \mathrm{J}(R)$, $\deg(y) = |M_1||R_2| - |M_1||M_2| - 1$. So $\deg(x) = \deg(y)$ if and only if $\dfrac{|R_1|}{|M_1|} = \dfrac{|R_2|}{|M_2|}$. Therefore, $(G(R))^c$ is a regular graph if and only if $\dfrac{|R_1|}{|M_1|} = \dfrac{|R_2|}{|M_2|}$. □

Lemma 2.5. *Let R be a ring with $|\mathrm{Max}(R)| \geq 3$. Then $(G(R))^c$ is not a regular graph.*

Proof. It is obvious. □

Suppose that F_1 and F_2 are two fields. In the next lemma, we classify the rings $R = F_1 \times F_2$, for which $(\Gamma_2(R))^c$ is a split graph. We need the following theorem.

Theorem 2.6. ([7, Theorem 6.7]). *Let G be an undirected graph with degree sequence $d_1 \geq d_2 \geq ... \geq d_n$, and let $k = \max\{i \mid d_i \geq i-1\}$. Then G is a split graph if and only if $\sum_{i=1}^{k} d_i = k(k-1) + \sum_{i=k+1}^{n} d_i$.*

Lemma 2.7. *Let F_1 and F_2 be fields such that $|F_1| \geq |F_2|$ and let $R = F_1 \times F_2$. Then $(\Gamma_2(R))^c$ is a split graph if and only if $|F_2| = 2$.*

Proof. Let $|F_1| = k_1$ and $|F_2| = k_2$. With keeping the notation of Theorem 2.6, we have:

$$\begin{aligned} d_1 &= k_1 + k_2 - 2 \geq d_2 = k_1 - 1 \geq ... \geq d_{k_1} \\ &= k_1 - 1 \geq d_{k_1+1} = k_2 - 1 \geq ... \geq d_{k_1+k_2-1} = k_2 - 1. \end{aligned}$$

Since $k_1 \geq k_2$, we have $m = k_1$ and $(\Gamma_2(R))^c$ is a split graph if and only if $\Sigma_{i=1}^{k_1} d_i = k_1(k_1-1) + \Sigma_{i=k_1+1}^{k_1+k_2-1} d_i$, so $k_1+k_2-2+(k_1-1)^2 = k_1(k_1-1)+(k_2-1)^2$ if and only if $k_2 = 2$. □

Theorem 2.8. *Let $R = F \times R$, where F is a field and (R, M) is a quasi-local ring that is not a field. Then $(\Gamma_2(R))^c$ is not a split graph.*

Proof. Let $(\Gamma_2(R))^c$ be a split graph. Let $|F| = f$, $|R| = r$ and $|M| = m$. It suffices to consider the following cases:

Case 1 : $f \leq \dfrac{r}{m}$. With the notation as in Theorem 2.6, we have

$$\begin{aligned} d_1 &= r + fm - m - 1 \geq ... \geq d_m = r + fm - m - 1 \geq d_{m+1} \\ &= r - 1 \geq ... \geq d_r = r - 1 \geq d_{r+1} = fm - 1 \geq ... \geq d_{r+fm-m} = fm - 1, \end{aligned}$$

and $k = r$, so

$$\sum_{i=1}^{r} d_i = r(r-1) + \Sigma_{i=r_2+1}^{r+fm-m} d_i.$$

Thus

$$m(r + fm - m - 1) + (r - m)(r - 1) = r(r - 1) + (fm - 1)(fm - m),$$

so $f^2m - fm - f + m + 1 = 0$, which is impossible.
Case 2 : $\dfrac{r}{m} \leq f$. With the notation as in Theorem 2.6, we have

$$\begin{aligned} d_1 &= r + fm - m - 1 \geq ... \geq d_m = r + fm - m - 1 \geq d_{m+1} \\ &= fm - 1 \geq ... \geq d_{fm} = fm - 1 \geq d_{fm+1} = r - 1 \geq ... \geq d_{fm+r-m} = r - 1, \end{aligned}$$

and $k = fm$, so

$$\sum_{i=1}^{fm} d_i = fm(fm-1) + \sum_{i=fm+1}^{fm+r-m} d_i.$$

Therefore,

$$m(r + fm - m - 1) + (fm - m)(fm - 1) = (fm)(fm - 1) + (r - m)(r - 1),$$

indeed $(r - m)(r - m - 1) = 0$, which is impossible. □

3. Pancyclic graph

In this section, we obtain some results on $(\Gamma_2(R))^c$ and $(G(R))^c$ to be pancyclic. We use the following two theorems, that are proved in [8, 9], and also we use Lemma 3.3, that the proof is easy.

Theorem 3.1. ([8, Theorem 1.34]). *Let G be a graph. If there exist vertices x, y on a Hamiltonian cycle C such that $d(x, y) = 2$ and $d(x) + d(y) \geq n + 1$, then G is pancyclic.*

Theorem 3.2. ([9, Corollary 3.7]). *If G is Hamiltonian with more than $\frac{n}{3}$ vertices of degree $\geq \frac{n+1}{2}$, then G is pancyclic.*

Lemma 3.3. *Let R be a ring.*

(1) *If $R = R_1 \times R_2$, where R_1 and R_2 are quasi-local rings, at least one of which is not field, then $(\Gamma_2(R))^c$ is Hamiltonian.*

(2) *If $R = R_1 \times R_2 \times R_3$, where R_1, R_2 and R_3 are quasi-local rings, then $(\Gamma_2(R))^c$ and $(G(R))^c$ are Hamiltonian.*

(3) *If $R = F_1 \times F_2 \times F_3$, where for every $1 \leq i \leq 3$, F_i is a filed and $|F_i|$ is even, then $(G(R))^c$ is Eulerian.*

(4) *Let $R = R_1 \times R_2 \times ... \times R_n$, where for every $1 \leq i \leq n$, R_i is a quasi -local ring and $n \geq 2$. Then $(\Gamma_2(R))^c$ is Eulerian if and only if for every $1 \leq i \leq n$, $|R_i|$ is odd.*

First, note that if F_1 and F_2 are fields and $R = F_1 \times F_2$, then there does not exist a cycle of length $|(\Gamma_2(R))^c|$ in $(\Gamma_2(R))^c$. Therefore, the next lemma is clear.

Lemma 3.4. *Let F_1 and F_2 be fields and $R = F_1 \times F_2$. Then $(\Gamma_2(R))^c$ is not a pancyclic graph.*

Now, in the following proposition, we show that $(\Gamma_2(R))^c$ is a pancyclic graph, where $R = R_1 \times R_2$ and (R_1, M_1) and (R_2, M_2) are quasi-local rings and at least one of them is not a field.

Proposition 3.5. *Let F_1 be a field, (R_2, M_2) be a quasi-local ring which is not a field, and $R = F_1 \times R_2$. Then $(\Gamma_2(R))^c$ is a pancyclic graph.*

Proof. Let $|F_1| = f_1$, $|R_2| = r_2$, $|M_2| = m_2$ and $|(\Gamma_2(R))^c| = n$. First note that by Lemma 3.3 (1), $(\Gamma_2(R))^c$ is Hamiltonian. Let $N_1 = (0) \times R_2$ and $N_2 = F_1 \times M_2$, so $\text{Max}(R) = \{N_1, N_2\}$.

If $m_2 \geq 3$, then we can choose vertices $x \in N_1 - \mathrm{J}(R)$ and $y \in N_2 - \mathrm{J}(R)$. Therefore, $d(x,y) = 2$, $\deg(x) = r_2 - 1$, $\deg(y) = f_1 m_2 - 1$ and $n = r_2 + f_1 m_2 - m_2$. Then

$$r_2 + f_1 m_2 - m_2 + 1 \leq r_2 - 1 + f_1 m_2 - 1.$$

Since $n+1 \leq \deg(x) + \deg(y)$, then $(\Gamma_2(R))^c$ is a pancyclic graph, by Theorem 3.1. Now suppose $m_2 = 2$. Then $(\Gamma_2(R))^c$ has $r_2 - 2$ vertices of degree $r_2 - 1$ in $N_1 - \mathrm{J}(R)$, only 2 vertices of degree $r_2 + 2f_1 - 3$ in $\mathrm{J}(R)$ and $2f_1 - 2$ vertices of degree $2f_1 - 1$ in $N_2 - \mathrm{J}(R)$. We have three cases:

Case 1 : $r_2 - 2f_1 \geq 1$, According to Theorem 3.2, we show that there exist more than $\frac{n}{3} = \frac{r_2 + 2f_1 - 2}{3}$ vertices of degree $\geq \frac{n+1}{2} = \frac{r_2 + 2f_1 - 1}{2}$. Clearly, for every vertex $x \in \mathrm{J}(R)$, $\deg(x) \geq \frac{r_2 + 2f_1 - 1}{2}$ and for every $y \in N_1 - \mathrm{J}(R)$, $\deg(y) = r_2 - 1 \geq \frac{r_2 + 2f_1 - 1}{2}$. Since $r_2 - 2f_1 \geq 1$, $r_2 - f_1 - f_1 \geq 1$ so $r_2 - f_1 \geq f_1 + 1 \geq 2 + 1 > -1$. Therefore, $2r_2 - 2f_1 > -2$, so $3r_2 \geq r_2 + 2f_1 - 2$, and $r_2 \geq \frac{r_2 + 2f_1 - 2}{3}$. By the above statement there exist $r_2 - 2 + 2$ vertices in $(N_1 - \mathrm{J}(R)) \cup \mathrm{J}(R) = N_1$. Thus $(\Gamma_2(R))^c$ is pancyclic.

Case 2 : $r_2 - 2f_1 \leq -1$. For every vertex $x \in \mathrm{J}(R)$, $\deg(x) \geq \frac{r_2 + 2f_1 - 1}{2}$ since $r_2 - 2f_1 \leq -1$, and for every $z \in N_2 - \mathrm{J}(R)$, $\deg(z) = 2f_1 - 1 \geq \frac{r_2 + 2f_1 - 1}{2}$. Also we have $2f_1 - r_2 \geq 1$ so $4f_1 - r_2 \geq 2f_1 + 1 \geq 5 > -2$, therefore, $4f_1 - r_2 > -2$ that is $6f_1 > r_2 + 2f_1 - 2$. It implies that $2f_1 > \frac{r_2 + 2f_1 - 2}{3}$. By the above statement there exist $2f_1 - 2 + 2$ vertices in $(N_2 - \mathrm{J}(R)) \cup (\mathrm{J}(R)) = N_2$. Thus $(\Gamma_2(R))^c$ is pancyclic.

Case 3 : $r_2 - 2f_1 = 0$. If $f_1 = 2$, then $R = \mathbb{Z}_2 \times \mathbb{Z}_4$, so $(\Gamma_2(R))^c$ is pancyclic by the definition. Now, let $f_1 \neq 2$. Since (R_2, M_2) is a quasi-local ring, then r_2 and f_1 are powers of 2, so, one may easily check that the required result holds. □

Lemma 3.6. *Let (R_i, M_i) be a quasi-local ring which is not a field for $1 \leq i \leq 2$, and $R = R_1 \times R_2$. Then $(\Gamma_2(R))^c$ is a pancyclic graph.*

Proof. Let $N_1 = M_1 \times R_2$ and $N_2 = R_1 \times M_2$ be maximal ideals of R and let $|R_i| = r_i$ and $|M_i| = m_i$, for $1 \leq i \leq 2$ and $n = |(\Gamma_2(R))^c|$. Clearly $|N_1| = m_1 r_2$, $|N_2| = r_1 m_2$ and $|\mathrm{J}(R)| = m_1 m_2$. Let $n_1 \in N_1 - \mathrm{J}(R)$ and $n_2 \in N_2 - \mathrm{J}(R)$ so $d(n_1, n_2) = 2$ and since $m_1 m_2 \geq 4$ we have

$$m_1 r_2 + r_1 m_2 - m_1 m_2 + 1 = n + 1 \leq \deg(n_1) + \deg(n_2) = m_1 r_2 - 1 + r_1 m_2 - 1.$$

On the other hand, by Lemma 3.3 (1), $(\Gamma_2(R))^c$ is Hamiltonian. So by Theorem 3.1, $(\Gamma_2(R))^c$ is a pancyclic graph. □

By Proposition 3.5 and Lemma 3.6, we can get the next theorem that is one of the main results of the section.

Theorem 3.7. *Let $R = R_1 \times R_2$, where (R_1, M_1) and (R_2, M_2) are quasi-local rings such that at least one of them is not a field. Then $(\Gamma_2(R))^c$ is a pancyclic graph.*

Now, by Lemma 3.3, we can get a different result of Lemma 3.4 for rings that have three maximal ideals.

Lemma 3.8. *Let F_1, F_2 and F_3 be fields and $R = F_1 \times F_2 \times F_3$. Then $(\Gamma_2(R))^c$ is a pancyclic graph.*

Proof. Let $N_1 = (0)\times F_2\times F_3$, $N_2 = F_1\times(0)\times F_3$ and $N_3 = F_1\times F_2\times(0)$ be maximal ideals of R and $n = |(\Gamma_2(R))^c|$ and $|F_i| = f_i$ for $1 \le i \le 3$. Let $x \in N_2-(N_1\cup N_3)$ and $y \in (N_1 \cap N_3) - N_2$, so $\deg(x) = f_1f_3 - 1$ and $\deg(y) = f_2f_3 + f_1f_2 - f_2 - 1$ and $n = f_2f_3+f_1f_3+f_1f_2-f_1-f_2-f_3+1$. Clearly $n+1 \le \deg(x)+\deg(y)$. By Lemma 3.3 (2), $(\Gamma_2(R))^c$ is Hamiltonian. Hence by Theorem 3.1, we obtain that $(\Gamma_2(R))^c$ is pancyclic. □

Lemma 3.9. *Let (R_1, M_1), (R_2, M_2) and (R_3, M_3) be quasi-local rings such that at least of them is not a field, and let $R = R_1 \times R_2 \times R_3$. Then $(\Gamma_2(R))^c$ is a pancyclic graph.*

Proof. Without loss of generality let $N_1 = M_1 \times R_2 \times R_3$, $N_2 = R_1 \times M_2 \times R_3$ and $N_3 = R_1 \times R_2 \times M_3$ be maximal ideals of R and $n = |(\Gamma_2(R))^c|$ and $|R_i| = r_i$, $|M_i| = m_i$ for $1 \le i \le 3$. Suppose that $x \in N_2-(N_1\cup N_3)$ and $y \in (N_1\cap N_3)-N_2$. It is obvious that $\deg(x) = r_1m_2r_3-1$ and $\deg(y) = m_1r_2r_3+r_1r_2m_3-m_1r_2m_3-1$. Also

$$n = m_1r_2r_3 + r_1m_2r_3 + r_1r_2m_3 - m_1m_2r_3 - m_1r_2m_3 - r_1m_2m_3 + m_1m_2m_3.$$

Since $m_2 \ge 2$ and $m_1r_3+r_1m_3-m_1m_3 \ge 2$, we have $n+1 \le \deg(x)+\deg(y)$. Now, by Lemma 3.3 (2), $(\Gamma_2(R))^c$ is Hamiltonian, so by Theorem 3.1, it is a pancyclic graph. □

The following theorem, that is an upshot of Lemma 3.8 and Lemma 3.9, is another main result of this section.

Theorem 3.10. *Let (R_1, M_1), (R_2, M_2) and (R_3, M_3) be quasi-local rings and $R = R_1 \times R_2 \times R_3$. Then $(\Gamma_2(R))^c$ is a pancyclic graph.*

Now we investigate when $(G(R))^c$ is pancyclic. First it should be noted that there does not exist a cycle of length $|(G(R))^c|$ in $(G(R))^c$, since $(G(R))^c$ is disconnected. Then we have the next lemma.

Lemma 3.11. *Let R be a commutative ring with identity with exactly two maximal ideals. Then $(G(R))^c$ is not a pancyclic graph.*

Proposition 3.12. *Let F_1, F_2 and F_3 be fields, $|F_1| \geq 3$ and $R = F_1 \times F_2 \times F_3$. Then $(G(R))^c$ is a pancyclic graph.*

Proof. Let $N_1 = (0) \times F_2 \times F_3$, $N_2 = F_1 \times (0) \times F_3$ and $N_3 = F_1 \times F_2 \times (0)$ be maximal ideals of R and $n = |(G(R))^c|$ and $|F_i| = f_i$ for $1 \leq i \leq 3$. Let $x \in N_2 - (N_1 \cup N_3)$ and $y \in (N_1 \cap N_3) - N_2$. Clearly $\deg(x) = f_1 f_3 - 2$ and $\deg(y) = f_2 f_3 + f_1 f_2 - f_2 - 2$ and $n = f_2 f_3 + f_1 f_3 + f_1 f_2 - f_1 - f_2 - f_3$. Since $f_1 \geq 3$, then $n+1 \leq \deg(x) + \deg(y)$. Then $(G(R))^c$ is Hamiltonian, by Lemma 3.3 (2). Hence by Theorem 3.1, $(\Gamma_2(R))^c$ is pancyclic. □

4. Semi-Eulerian

In this section, we explore some of the conditions under which $(\Gamma_2(R))^c$ and $(G(R))^c$ are semi-Eulerian. First, for any ring R with $|\text{Max}(R)| = 2$, we obtain a necessary and sufficient condition for $(\Gamma_2(R))^c$ to be semi-Eulerian.

Proposition 4.1. *Let F_1 and F_2 be fields and $R = F_1 \times F_2$. Then the following conditions are equivalent:*

(1) *$(\Gamma_2(R))^c$ is a semi-Eulerian graph.*

(2) *$|F_1|$ and $|F_2|$ are odd, $R = \mathbb{Z}_2 \times F_2$ where $|F_2|$ is odd or $R = \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Let $|F_1| = f_1$ and $|F_2| = f_2$.

(1) $\Rightarrow$ (2) Suppose $f_1 \geq 4$ is even and f_2 is odd, then we have exactly one vertex of degree $f_1 + f_2 - 2$ and $f_1 - 1 \geq 3$ vertices of degree $f_1 - 1$, which is a contradiction. Now suppose that f_1 and f_2 are even and $f_1 \geq 4$. So there exist at least $f_1 - 1$ vertices of degree $f_1 - 1$, which is a contradiction.

(2) $\Rightarrow$ (1) If f_1 and f_2 are odd, then by Lemma 3.3 (4), $(\Gamma_2(R))^c$ is Eulerian, as desired. If $R = \mathbb{Z}_2 \times F_2$, where f_2 is odd, then $(\Gamma_2(R))^c$ has $f_2 - 1$ vertices of degree $f_2 - 1$, and one vertex of either degrees 1 or f_2, so $((\Gamma_2(R))^c$ is semi-Eulerian. Now, if $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $(\Gamma_2(R))^c$ has one vertex of degree 2, and two vertices of degree 1, so $((\Gamma_2(R))^c$ is semi-Eulerian. □

Lemma 4.2. *Let $R = F_1 \times R_2$, where F_1 is a field and (R_2, M_2) is a quasi-local ring which is not a field. Then $(\Gamma_2(R))^c$ is Eulerian if and only if $(\Gamma_2(R))^c$ is semi-Eulerian.*

Proof. If $(\Gamma_2(R))^c$ is Eulerian, clearly it is semi-Eulerian. Conversely, assume $(\Gamma_2(R))^c$ is not Eulerian and $N_1 = (0) \times R_2$, $N_2 = F_1 \times M_2$, $|F_1| = f_1$, $|R_2| = r_2$ and $|M_2| = m_2$. There exist $r_2 - m_2$ vertices of degree $r_2 - 1$ in $N_1 - \text{J}(R)$ and m_2 vertices of degree $r_2 + f_1 m_2 - m_2 - 1$ in $\text{J}(R)$ and also $f_1 m_2 - m_2$ vertices of degree $f_1 m_2 - 1$ in $N_2 - \text{J}(R)$, so by Lemma 3.3 (4), three following cases may happen:

Case 1 : f_1 is even and r_2 is odd. Clearly all vertices in $N_2 - \text{J}(R)$ and $\text{J}(R)$ have odd degree, so $(\Gamma_2(R))^c$ is not semi-Eulerian.

Case 2 : f_1 and r_2 are even. So all vertices in $N_1 - \mathrm{J}(R)$ and $N_2 - \mathrm{J}(R)$ are of odd degree. Thus $(\Gamma_2(R))^c$ is not semi-Eulerian.

Case 3 : f_1 is odd and r_2 is even. Therefore, all vertices in $N_1 - \mathrm{J}(R)$ and $\mathrm{J}(R)$ have odd degree. So $(\Gamma_2(R))^c$ is not semi-Eulerian. □

Lemma 4.3. *Let (R_1, M_1) and (R_2, M_2) be quasi-local rings that are not fields and $R = R_1 \times R_2$. Then $(\Gamma_2(R))^c$ is Eulerian if and only if $(\Gamma_2(R))^c$ is semi-Eulerian.*

Proof. If $(\Gamma_2(R))^c$ is Eulerian, clearly it is semi-Eulerian. Conversely, assume $(\Gamma_2(R))^c$ is not Eulerian and $N_1 = M_1 \times R_2$ and $N_2 = R_1 \times M_2$ be maximal ideals of R. Suppose that $|R_1| = k_1$, $|R_2| = k_2$, $|M_1| = m_1$ and $|M_2| = m_2$. Then there exist $m_1k_2 - m_1m_2$ vertices of degree $m_1k_2 - 1$ in $N_1 - \mathrm{J}(R)$ and m_1m_2 vertices of degree $m_1k_2 - k_1m_2 - m_1m_2 - 1$ in $\mathrm{J}(R)$ and also $k_1m_2 - m_1m_2$ vertices of degree $k_1m_2 - 1$ in $N_2 - \mathrm{J}(R)$. If k_1 and k_2 are even and m_1 and m_2 are odd, then all vertices in $(\Gamma_2(R))^c - \mathrm{J}(R)$ have odd degree. Otherwise all vertices in $\mathrm{J}(R)$ have odd degree. Thus $(\Gamma_2(R))^c$ is not semi-Eulerian. □

Suppose that R is a ring with exactly two maximal ideals. The following theorem, which is a result of Proposition 4.1, Lemma 4.2 and Lemma 4.3, provides a necessary and sufficient condition for $(\Gamma_2(R))^c$ to be semi-Eulerian.

Theorem 4.4. *Let (R_1, M_1) and (R_2, M_2) be quasi-local rings and let $R = R_1 \times R_2$. Then the following statements are equivalent:*

(1) *$(\Gamma_2(R))^c$ is semi-Eulerian.*

(2) *$|R_1|$ and $|R_2|$ are odd, $R \cong \mathbb{Z}_2 \times R_2$ where $R_2 = \mathbb{Z}_2$ or R_2 is a field of odd order.*

Now we show that $(\Gamma_2(R))^c$ is Eulerian if and only if it is semi-Eulerian, where R is a ring with at least three maximal ideals.

Lemma 4.5. *Let $R = F_1 \times F_2 \times ... \times F_t$, where $t \geq 3$ and for every $1 \leq i \leq t$, F_i is a field. Then $(\Gamma_2(R))^c$ is Eulerian if and only if it is semi-Eulerian.*

Proof. If $(\Gamma_2(R))^c$ is Eulerian, It is semi-Eulerian. Conversely, assume $(\Gamma_2(R))^c$ is not Eulerian, $|F_i| = f_i$ and $\mathrm{Max}(R) = \{N_1, ..., N_t\}$, where $N_i = F_1 \times F_2 \times ... \times F_{i-1} \times (0) \times F_{i+1} \times ... \times F_t$ for $1 \leq i \leq t$. By Lemma 3.3 (4), two cases occur:

Case 1 : f_1 is even and f_i is odd for $2 \leq i \leq t$. So $\mathrm{J}(R) = \{x\}$ and $\deg(x) = \sqcap_{i=1}^{t} f_i - \sqcap_{i=1}^{t}(f_i - 1) - 1$ is odd. There exists $y \in (N_2 \cap N_3) - \cup_{i=1, i\neq 2,3}^{t} N_i$ such that $\deg(y) = |N_2 \cup N_3| - 1$ is odd.

If $t = 3$, then we have $x \in (N_1 \cap N_3) - N_2$ and $\deg(x) = |N_1 \cup N_3| - 1$ is odd, so there exists at least 3 vertices of degree odd. It implies that $(\Gamma_2(R))^c$ is not semi-Eulerian.

If $t \geq 4$, then there exist $z \in (N_2 \cap N_4) - \cup_{i=1,i\neq 2,4}^{t} N_i$ with odd degree, so the assertion is obtained.

Case 2 : At least f_1 and f_2 are even. We have $x \in N_1 \cap N_3 - \cup_{i=2,i\neq 3}^{t} N_i$, $y \in N_2 \cap N_3 - \cup_{i=1,i\neq 2,3}^{t} N_i$ and $z \in N_1 - \cup_{i=2}^{t} N_i$ with odd degree, so $(\Gamma_2(R))^c$ is not semi-Eulerian. □

Lemma 4.6. *Let $R = R_1 \times R_2 \times ... \times R_t$, where $t \geq 3$ and for every $1 \leq i \leq t$, (R_i, M_i) is a quasi-local ring and at least one of them is not field. Then $(\Gamma_2(R))^c$ is Eulerian if and only if $(\Gamma_2(R))^c$ is semi-Eulerian.*

Proof. It is sufficient to show that if $(\Gamma_2(R))^c$ is not Eulerian, then it is not semi-Eulerian. Assume that R_1 is not field and $N_i = R_1 \times ... \times R_{i-1} \times M_i \times R_{i+1} \times ... \times R_t$, $|R_i| = r_i$ and $|M_i| = m_i$, for $1 \leq i \leq t$. Then $\mathrm{Max}(R) = \{N_1, ..., N_t\}$. We have two following cases:

Case 1 : r_1 is even. Then $|N_2 \cap N_3 \cap ... \cap N_t - \mathrm{J}(R)| = r_1 m_2 ... m_t - m_1 m_2 ... m_t = (r_1 - m_1)(m_2 ... m_t) \geq 2$, since $r_1 - m_1 > 1$ and $m_2 ... m_t \geq 1$. Clearly, for every $x \in N_2 \cap N_3 \cap ... \cap N_t - \mathrm{J}(R)$, $\deg(x)$ is an odd number. On the other hand, for every $y \in N_2 - \cup_{i=1,i\neq 2}^{t} N_i$, $\deg(y) = |N_2| - 1 = r_1 m_2 r_3 ... r_t - 1$ is odd, so we have at least three vertices of odd degree, thus $(\Gamma_2(R))^c$ is not semi-Eulerian.

Case 2 : r_1 is odd and at least one of the numbers $r_2, ..., r_t$ is even. Assume that r_2 is even. We have $|N_1 \cap N_3 \cap .... \cap N_t - \mathrm{J}(R)| = m_1 r_2 m_3 ... m_t - m_1 m_2 ... m_t = m_1 m_3 ... m_t (r_2 - m_2) \geq 3$, since r_1 is odd and $1 \neq m_1 | r_1$ and $r_2 - m_2 \geq 1$, for every vertex $x \in N_1 \cap N_3 \cap .... \cap N_t - \mathrm{J}(R)$, $\deg(x)$ is odd. So we found at least three vertices of odd degree, thus $(\Gamma_2(R))^c$ is not semi-Eulerian. □

The next theorem, that follows from Lemmas 4.5 and 4.6, is one of the main results of this section.

Theorem 4.7. *Let $R = R_1 \times R_2 \times ... \times R_t$, where $t \geq 3$ and for every $1 \leq i \leq t$, (R_i, M_i) is a quasi-local ring. Then $(\Gamma_2(R))^c$ is Eulerian if and only if it is semi-Eulerian.*

We conclude this section with a characterization for$(G(R))^c$ to be semi-Eulerian.

Theorem 4.8. *Let $R = F_1 \times F_2 \times F_3$, where for every $1 \leq i \leq 3$, F_i is a field and $|F_i| = f_i$. Then the following statements are equivalent:*

(1) *$(G(R))^c$ is semi-Eulerian.*

(2) *f_i is even for $1 \leq i \leq 3$ or $R = \mathbb{Z}_3 \times F_2 \times F_3$ where f_2 and f_3 are even.*

Proof. Let $N_1 = (0) \times F_2 \times F_3$, $N_2 = F_1 \times (0) \times F_3$ and $N_3 = F_1 \times F_2 \times (0)$ be maximal ideals of R.

(1) $\Rightarrow$ (2) If (2) is not valid, then one of the following cases may happen:

Case 1 : If f_i is odd for $1 \leq i \leq 3$, then all vertices have odd degree, which is a contradiction.

Case 2 : If f_1 and f_2 are odd and f_3 is even, then there exist $f_1f_2 - f_1 - f_2 + 1 = f_1(f_2 - 1) - f_2 + 1 = (f_2 - 1)(f_1 - 1) \geq 2 \times 2 = 4$ vertices of degree $f_1f_2 - 2$ in $N_3 - (N_1 \cup N_2)$, which is a contradiction.

Case 3 : If f_1 is odd and $f_1 > 3$ also f_2 and also f_3 are even, then there exist $f_1 - 1 > 2$ vertices of degree $f_1f_3 + f_1f_2 - f_1 - 2$ in $(N_2 \cap N_3) - N_1$, which is a contradiction.

(2) $\Rightarrow$ (1) If f_i is even for $1 \leq i \leq 3$, then by Lemma 3.3 (3), $(G(R))^c$ is Eulerian, so it is semi-Eulerian.

If $R = \mathbb{Z}_3 \times F_2 \times F_3$ where f_2 and f_3 are even, then by computing degrees of vertices we see that there exist $|\mathbb{Z}_3| - 1 = 2$ vertices of odd degree in $(N_2 \cap N_3) - N_1$ and other vertices have even order, so $(G(R))^c$ is semi-Eulerian. □

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Masoumeh Soleimani
Department of Mathematics, Faculty of Science,
University of Qom,
Qom, I. R. Iran
E-mail: m.soleimani99@gmail.com

Fatemeh Mahmudi
Department of Mathematics, Faculty of Science,
University of Qom,
Qom, I. R. Iran
E-mail: mahmodi.fateme64@gmail.com

Mohammad Hasan Naderi
Department of Mathematics, Faculty of Science,
University of Qom,
Qom, I. R. Iran
E-mail: mh.naderi@qom.ac.ir