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Improved Adaptive Stabilization Controller for an UPOs of Chaotic Systems with an Optimal Principle by TDFC Method

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Abstract

In this paper, we investigate an improved method for stabilizing a class of uncertain chaotic nonlinear dynamical system. Our approach follows techniques of optimal principle for time-delayed feedback control and adaptive tracking control theory for stabilizing unstable periodic orbits in a chaotic bounded attractor. The uncertain parameters expressed in the system can be separated. Analysis and proof are presented using the Lyapunov stability theorem. In particular, we use the adaptive control theory to design an adaptive law for the estimation of uncertain time-delayed controlled chaotic nonlinear dynamical systems. The predictions are presented by numerical simulation through the Rossler system to demonstrate theoretical results.

Keywords: Chaos, Nonlinear control, Adaptive control, Lyapunov stability.

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1. Introduction

The most characteristic of chaotic systems is high sensitivity to small perturbations regarding initial conditions. An important problem in nonlinear control is known

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as stabilizing unstable periodic orbits (UPOs) in chaos control [1]. An efficient scheme for stabilizing UPOs using small parameter perturbation has been proposed by Ott, Grebogy, and York (OGY)[2]. The OGY method is effective but requires unstable periodic solution information to be used as reference signals for tracking. Another efficient scheme for stabilizing UPOs using time-delayed feedback control (TDFC) was proposed by Pyragas in 1992, different from the OGY method [3, 4]. The Pyragas method does not require detailed UPOs information and is more flexible and robust to noise, in comparison with the OGY approach. This requires a solution of UPOs embedded in chaotic attractors[5, 6]. A scalar time delay constant as the period of the UPOs must be stabilized to use the time-delayed state as a tracking UPOs embedded in chaotic attractors. The approach is simple and has succeeded in a chaotic control system[7, 8]. This method is in the form of feedback proportional to the difference between the state of the chaotic system and this past on period state at times t and $t - T$, where T denotes the time delay for the control parameter and is adjusted to match the period of the UPOs to be stabilized. Many other classical nonlinear control techniques for stabilizing UPOs such as adaptive chaos control [9, 10], synchronization method [11, 12], parameter identification [13, 14], robust LMI approach [15], fuzzy adaptive sliding mode[16], stabilization controller [17] and optimal time delay control [18, 19] have been proposed so far for stabilizing UPOs.

This paper proposes an adaptive stabilizing control of chaotic systems with optimal principles for time-delayed feedback control. An improved adaptive stabilization controller is then designed to stabilize the UPOs solution embedded in the chaotic attractors.

The organization of this paper is as follows. Section 2, describes some preliminary knowledge about chaotic nonlinear dynamical systems. Section 3, describes an optimal principle for a time-delayed feedback control problem. Section 4, addresses the adaptive synchronization scheme and derives the proposed control strategy for a chaos nonlinear dynamical system with unknown parameters. In Section 5, the adaptive stabilization controller scheme for the delayed Rossler system is solved. Section 6 draws conclusions.

2. Description of the system

Chaos control study analysis and design of chaotic nonlinear dynamical systems. In analysis assumes a closed circuit is designed and the behavior of a system is determined to stabilize one of the UPOs in the embedded of attractor chaotic by TDFC tools. In design assumes that controlled chaotic nonlinear dynamical systems have desired behavior of closed circuit system and a sesign controller for this.

We consider uncertain chaotic nonlinear dynamical systems described by the following differential equation,

$$\dot{x} = f(t, x(t), \theta). \tag{1}$$

Where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$ is the state vector, $\theta \in \mathbb{R}^n$ is a vector of unknown constant parameters and f satisfies the sufficient conditions for the existence and uniqueness of solutions (typically, continuity and locally Lipschitz in all arguments $x(t)$ and for each θ, uniformly in t). We consider a general controlled continues time uncertain chaotic system as follows:

$$\dot{x} = f(t, x, u, \theta). \tag{2}$$

Where $u \in \mathbb{R}^m$ corresponds to the control inputs and f still satisfies the sufficient conditions for the existence and uniqueness of solutions. (typically, continuity and locally Lipschitz in all arguments and for each x, θ and also u, uniformly in t). Let $\Omega \in \mathbb{R}^n$be a chaotic bounded attractor. Suppose that $\bar{x}(t)$ is an unstable periodic orbit solution embedded in a chaotic bounded attracted set of the system (1).
We consider the time-delayed feedback control by following equation

$$u(t) = K(x(t) - x(t - \tau)), \tag{3}$$

with a proper delay-time $\tau > 0$.
System (3) is added to system (1) to form controlled uncertain chaotic nonlinear dynamical systems (2) such that the system orbit can track the $\bar{x}(t)$. On the other hand $\lim_{t\to\infty} \|x(t) - \bar{x}(t)\| = 0$, where $\|x(t) - \bar{x}(t)\| = [(x(t) - \bar{x}(t)]^T [(x(t) - \bar{x}(t)]$ is the Euclidean norm [1].
The goal is to find K with a proper known delay- time $\tau > 0$.

Assumption 2.1. *Let uniform Lipschitz condition $d_i > 0$ refer to the uniform Lipschitz constant for any $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \Omega$, there exist constants $l_i > 0$; $i = 1, 2, ..., n$ such that*

$$|k(x_i) - k(\hat{x}_i)| \leq \sqrt{l_i} \max_{1\leq j\leq n} \{|x_j - \hat{x}_j|\}, \quad i = 1, 2, ..., n. \tag{4}$$

It is held as long as the partial differential $\partial k_i/\partial x_i$ is bonded in Ω.

We first discuss the stabilization problem with the TDFC method when the target is an unstable fixed point $\tau = 0$ and the parameter of a chaotic system is known in Section 3 for analysis. Then we consider the case where the chaotic system contains unknown system and control parameters.

3. Time-delayed feedback control

In this section, a TDFC control method is introduced for stabilizing UPOs of the system (2) [1]. Let θ be a known parameter as θ^*, and controlled chaotic nonlinear dynamical system is following differential equation:

$$\dot{x}(t) = f(t, x(t), u(t), \theta^*) = f(t, x(t), \theta^*) + u(t), \tag{5}$$

$\bar{x}(t)$ is unstable periodic orbit solution then $u(t) \equiv 0$,

$$\dot{\bar{x}}(t) = f(t, \bar{x}(t), 0, \theta^*). \tag{6}$$

It is clear that the following equations hold.

$$\begin{aligned} \bar{x}(t) - \bar{x}(t-\tau) &= \int_{t-\tau}^{t} f(t, \bar{x}(t), 0, \theta^*) d(s) = 0, \qquad (7) \\ \frac{d}{dt}\bar{x}(t) - \frac{d}{dt}\bar{x}(t-\tau) &= f(t, \bar{x}(t), 0, \theta^*) - f(t, \bar{x}(t-\tau), 0, \theta^*) = 0, \\ \frac{d^2}{dt^2}\bar{x}(t) - \frac{d^2}{dt^2}\bar{x}(t-\tau) &= \frac{d}{dt} f(t, \bar{x}(t), 0, \theta^*) - \frac{d}{dt} f(t, \bar{x}(t-\tau), 0, \theta^*) = 0. \end{aligned}$$

For stabilizing UPOs of the system (2), regarding the minimization integral principle, the first, second, and higher order derivatives of $f(t, \bar{x}(t), 0, \theta^*)$ must be zero. We introduce the following performance index:

$$\begin{aligned} J &= \int_{t_0}^{\infty} [(\frac{1}{\tau}(\int_{t-\tau}^{t} f(t, \bar{x}(t), 0, \theta^*) d(s)))^T (\frac{1}{\tau}(\int_{t-\tau}^{t} f(t, \bar{x}(t), 0, \theta^*) d(s))) \qquad (8) \\ &+ (\frac{1}{\tau}(f(t, \bar{x}(t), 0, \theta^*) - f(t, \bar{x}(t-\tau), 0, \theta^*)))^T (\frac{1}{\tau}(f(t, \bar{x}(t), 0, \theta^*) \\ &- f(t, \bar{x}(t-\tau), 0, \theta^*) + (\frac{1}{\tau}(\frac{d}{dt} f(t, \bar{x}(t), 0, \theta^*)) \\ &- \frac{d}{dt} f(t, \bar{x}(t-\tau), 0, \theta^*)^T (\frac{1}{\tau}(\frac{d}{dt} f(t, \bar{x}(t), 0, \theta^*) \\ &- \frac{d}{dt} f(t, \bar{x}(t-\tau), 0, \theta^*)] dt. \end{aligned}$$

Subtracting (6) from (5) with $e(t) = x(t) - \bar{x}(t)$, gets error dynamical chaotic system. Since J is a convex function operating on the convex set Ω, minimizing it achieves a global minimum point. Furthermore, e(t) defines an error function that is also convex and contains the same convex feasible region as J. Therefore, minimizing e(t) yields the same global minimum as of J. Next, we derive the equations as follows,

$$\begin{aligned} \dot{e}(t) &= \dot{x}(t) - \dot{\bar{x}}(t) = f(t, x(t), u, \theta^*) - f(t, \bar{x}(t), 0, \theta^*) \qquad (9) \\ &= f(t, x(t), \theta^*) + u(t) - f(t, \bar{x}(t), \theta^*), \end{aligned}$$

Rewrite the error dynamical system:

$$u(t) = K(x(t) - x(t-T)), \tag{10}$$

$$\dot{e}(t) = F(t, x(t), \theta^*) + K(x(t) - x(t-\tau)), \tag{11}$$

where

$$F(t,x(t),\theta^*) = f(t,x(t),\theta^*) - f(t,\bar{x}(t),\theta^*) = f(t,\bar{x}(t)+e(t),\theta^*) - f(t,\bar{x}(t),\theta^*). \tag{12}$$

Firstly, without loss of generality, suppose $\bar{x}(t)$ is constant, let $\bar{x}(t) = 0$. Then

$$\dot{\mathrm{e}}(t) = F(t,e(t),\theta^*) + K(e(t) - e(t-\tau)). \tag{13}$$

Such that the controlled system orbit can track the $\bar{x}(t) = 0$ when $\lim_{t\to\infty} \|e(t)\| = 0$ and performance index J is minimized.
There is a parametric linearization $F(t,e(t),\theta^*)$ around the point:

$$\dot{\mathrm{e}}(t) = A(t)e(t) + v(e(t)) + K(e(t) - e(t-\tau)), \tag{14}$$

$$A = \frac{\partial F(t,e(t),\theta^*)}{\partial e}\ \Big|_{e(t)=0}\ , \tag{15}$$

where $v(t)$ is a high order term $e(t)$.

Theorem 3.1. Consider the error dynamical system (14), if there are two positive definite and symmetric constant matrices $P$and Q and a constant matrix K such that the Riccati polynomial matrix

$$P + PA + PKQ^{-1}K^TP + PK + K^TP + Q \tag{16}$$

is semi-negative definite (≤ 0), then $\lim_{t\to\infty} \|e(t)\| = 0$.

Proof. Consider a quadratic Lyapunov function as follows:

$$V(t,e) = e(t)^T P\, e(t) + \int_{t-\tau}^{t} e(s)^T Q\, e(s) ds. \tag{17}$$

By time-differentiating the function V along the trajectory of (14), we obtain

$$\dot{V}(t,e) = \dot{\mathrm{e}}(t)^T Pe(t) + \mathrm{e}(t)^T P\,\dot{\mathrm{e}}(t) + e(t)Qe(t) - e(t-\tau)^T Q\, e(t-\tau), \tag{18}$$

with (13) and (14) :

$$\begin{aligned} \dot{V}(t,e) &= [F(t,e) + K(e(t) - e(t-\tau)]^T P\, e(t) \\ &+ e(t)P[F(t,e) + K(e(t) - e(t-\tau)] \\ &+ e(t)^T Qe(t) - e(t-\tau)^T Qe(t-\tau), \end{aligned} \tag{19}$$

or

$$\begin{aligned} \dot{V}(t,e) &= [Ae(t) + v(t) + K(e(t) - e(t-\tau))]^T Pe(t) \\ &+ e(t)^T P[Ae(t) + v + K(e(t) - e(t-\tau))] \\ &+ e(t)^T Qe(t) - e(t-\tau)^T Qe(t-\tau), \end{aligned} \tag{20}$$

with simplicity, we obtain

$$\begin{aligned}\dot{V}(t,e) &= e(t)^T A^T Pe(t) + v(t)^T Pe(t) + e(t)^T K^T Pe(t) - e(t-\tau)^T K^T pe(t)\\ &+ e(t)^T PAe(t) + e(t)^T Pv(t) + e(t)^T PKe(t) - e(t)^T PKe(t-\tau)\\ &+ e(t)^T Qe(t) - e(t-\tau)^T Qe(t-\tau), \end{aligned} \tag{21}$$

by factoring, we get

$$\begin{aligned}\dot{V}(t,e) &= e(t)^T[A^T P + K^T P + PA + PK + Q]e(t) + e(t)^T Pv(t) + v(t)^T Pe(t)\\ &- [e(t-\tau)^T K^T Pe(t) + e(t)^T PKe(t-\tau) + e(t-\tau)^T Qe(t-\tau)], \end{aligned} \tag{22}$$

Consider equality:

$$\begin{aligned} & \left\| Q^{1/2}e(t-\tau) + Q^{-1/2}K^T P\, e(t) \right\| \qquad (23)\\ = \;& [Q^{1/2}e(t-\tau) + Q^{-1/2}K^T P\, e(t)]^T [Q^{1/2}e(t-\tau) + Q^{-1/2}K^T P\, e(t)]\\ = \;& e(t-\tau)^T Q\, e(t-\tau) + e(t-\tau)^T K^T P\, e(t-\tau)\\ + \;& e(t)^T P\, K\, e(t-\tau) + e(t)^T P\, KQ^{-1}K^T P\, e(t), \end{aligned}$$

we have

$$\begin{aligned}\dot{V}(t,e) &= -[Q^{1/2}e(t-\tau) + Q^{-1/2}K^T Pe(t)]^T [Q^{1/2}e(t-\tau) + Q^{-1/2}K^T P\, e(t)]\\ &+ e(t)^T P\, v(t) + v(t)^T Pe(t) \qquad (24)\\ &+ e(t)^T [A^T P + PA + PKQ^{-1}K^T P + PK + K^T P + Q\,]\, e(t\,). \end{aligned}$$

Then $\dot{V}$ is negative, for small $e(t)$, and $e(t)^T P\, v(t) + v(t)^T Pe(t) \simeq 0$. □

4. Improve adaptive synchronization scheme

In this section, we consider the case when the chaotic nonlinear dynamical system contains uncertain parameters. The controller design in Section 2 cannot deliver the stabilizing force for a chaotic system (1).

$$\dot{x} = f(t,\tau,x(t),\theta) + K(x(t) - x(t-\tau)). \tag{25}$$

We consider the time-delayed feedback control with a proper delay-time $\tau > 0$ to be added to the system (1) to form of controlled uncertain chaotic nonlinear dynamical systems $\dot{x} = f(t,\tau,x(t),x(t-\tau),\theta)$. The idea is to introduce a model tracking (adaptive stabilization) strategy which can provide a wide spectrum of the signal so that the identification of the parameters is realized. In the following, we investigated the adaptive stabilization between the references systems for a class of special chaotic nonlinear dynamical system contain uncertain parameters can be written as:

$$\dot{x} = f(t,\tau,x(t),x(t-\tau),\theta) = T\,x(t) + C\,g(x(t)) + \Gamma\,x(t-\tau(t)). \tag{26}$$

Where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \Omega$, Ω is a chaotic bounded strange attractor set and $\Omega \in R^n$, is the state vector, $T = (t_{ij})_{n\times n}$, $\Gamma = (\gamma_{ij})_{n\times n}$are uncertain constant matrices of the linear section, $C = (c_{ij})_{n\times n}$is the uncertain constant matrix of the nonlinear section, $g(x) = (g_1(x), g_2(x), ..., g_n(x))^T$ is a nonlinear function section and $\tau = \tau(t) \geq 0$ is the time-varying delay. Without loss of generality, it is assumed that the nonlinear dynamical system structure (26) is known and time series for all variables are available as the output of (26).

Assumption 4.1. *For uniformly of function τ, let $\tau = \tau(t) \geq 0$ be the smooth function of time t and $\tau \in C^1$, there is a positive number $M > 0$ such that $\dot{\tau}(t) \leq M$.*

Lemma 4.2. *([20]). For any vectors $v_1, v_2 \in \mathbb{R}^n$and any positive definite matrix $A \in \mathbb{R}^{n\times n}$ the following inequality holds:*

$$2{v_1}^T v_2 \leq {v_1}^T A v_1 + {v_2}^T A^{-1} v_2. \tag{27}$$

Referring to the system (26) as the reference system, an auxiliary variable $\hat{x}(t) = (\hat{x}_1(t), \hat{x}_2(t), ..., \hat{x}_n(t))^T \in \mathbb{R}^n$is introduced as the estimation, and to get the estimation equation of system (26), we construct the following system as an identifier:

$$\dot{\hat{x}}(t) = \hat{T}(t)\,\hat{x}(t) + \hat{C}(t)g(\hat{x}(t)) + \hat{\Gamma}(t)\hat{x}(t-\tau) + \alpha(x, \hat{x}), \tag{28}$$

where $\hat{T} = (\hat{t}_{ij})_{n\times n}$, $\hat{C} = (\hat{c}_{ij})_{n\times n}$ and $\hat{\Gamma} = (\hat{\gamma}_{ij})_{n\times n}$ are the estimation of the uncertain parameters matrices $T = (t_{ij})_{n\times n}$, $C = (c_{ij})_{n\times n}$ and $\Gamma = (\gamma_{ij})_{n\times n}$ respectively, $\alpha(x, \hat{x}) = K(x(t) - \hat{x}(t))$ is a simple adaptive feedback controller with K updated adaptively according to some updated low, where estimation error as $e(t) = x(t) - \hat{x}(t)$ and subtracting (28) and (26) yield the error system as follows:

$$\begin{aligned} \dot{e}(t) \;&=\; \hat{T}(t)\hat{x}(t) + \hat{C}(t)g(\hat{x}(t)) + \hat{\Gamma}(t)\hat{x}(t-\tau) + Ke(t) \\ &-\; T\,x(t) - C\,g(x(t)) - \Gamma\,x(t-\tau(t)). \end{aligned} \tag{29}$$

Our task is to find a suitable adaptive low such that $\hat{x}(t)$ can track $x(t)$.

Theorem 4.3. If Assumption 2.1 and Assumption 4.1 are satisfied and there exists an arbitrary positive constants p_{ij}, q_{ij}, r_{ij} and λ_i , $(i, j = 1, 2, 3, ..., n)$ system (28) can tracking system (26) if one design:

$$\dot{\hat{t}}_{ij} = -p_{ij}e_i(t)\,\hat{x}_j, \qquad \dot{\hat{\gamma}}_{ij} = -q_{ij}e_i(t)\,\hat{x}_j(t-\tau), \qquad \dot{\hat{c}}_{ij} = -r_{ij}e_i(t)\,g_j(\hat{x}), \tag{30}$$

with the coupling $K = diag(k_1, k_2, ..., k_n)$ updated by

$$\dot{\hat{k}}_i = -\lambda_i e_i^2(t). \tag{31}$$

Proof. Construct a Lyapunov function of the form

$$\begin{aligned} V(e(t)) &= \frac{1}{2}e^T(t)e(t) + \frac{1}{2}\int_{t-\tau(t)}^{t} e^T(s)e(s)ds \\ &+ \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\lambda_i}(k_i+l_i)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{p_{ij}}(\hat{t}_{ij}-t_{ij})^2 \\ &+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{q_{ij}}(\hat{\gamma}_{ij}-\gamma_{ij})^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{r_{ij}}(\hat{c}_{ij}-c_{ij})^2. \end{aligned} \tag{32}$$

By differentiating the function V with respect to time along the trajectory of (29) we have

$$\begin{aligned} \dot{V}(e(t)) &= e^T(t)\dot{e}(t) + \frac{1}{2}e^T(t)e(t) - \frac{1}{2}(1-\dot{\tau}(t))(e^T(t-\tau(t))\,e(t-\tau(t)) \\ &- \sum_{i=1}^{n}(k_i+l_i)e_i^2(t) - \sum_{i=1}^{n}\sum_{j=1}^{n}(\hat{t}_{ij}-t_{ij})e_i(t)\hat{x}_j(t) \\ &- \sum_{i=1}^{n}\sum_{j=1}^{n}(\hat{\gamma}_{ij}-\gamma_{ij})e_i(t)\hat{x}_j(t-\tau(t)) - \sum_{i=1}^{n}\sum_{j=1}^{n}(\hat{c}_{ij}-c_{ij})e_i(t)g_j(\hat{x}). \end{aligned} \tag{33}$$

Substituting (29) into (33) yields

$$\begin{aligned} \dot{V}(e(t)) &= e^T(t)\,\hat{T}(t)\,\hat{x}(t) + e^T(t)\,\hat{C}(t)g(\hat{x}(t)) + e^T(t)\hat{\Gamma}(t)\hat{x}(t-\tau) + e^T(t)Ke(t) \\ &- e^T(t)\,T\,x(t) - e^T(t)C\,g(x(t)) - e^T(t)\,\Gamma\,x(t-\tau(t)) + \frac{1}{2}e^T(t)e(t) \\ &- \frac{1}{2}e^T(t-\tau(t))e(t-\tau(t)) + \frac{1}{2}\dot{\tau}(t)e^T(t-\tau(t)e(t-\tau(t)) - e^T(t)Ke(t) \\ &- e^T(t)Le(t) - e^T(t)\hat{T}(t)\hat{x}(t) + e^T(t)T\hat{x}(t) - e^T(t)\hat{\Gamma}(t)\hat{x}(t-\tau) \\ &+ e^T(t)\Gamma\hat{x}(t-\tau) - e^T(t)\hat{C}(t)g(\hat{x}(t)) + e^T(t)Cg(\hat{x}(t)), \end{aligned} \tag{34}$$

then

$$\begin{aligned} \dot{V}(e(t)) &= e^T(t)Te(t) + e^T(t)C(t)(g(\hat{x}(t)) - g(x(t))) + e^T(t)\Gamma e(t-\tau(t)) \\ &- e^T(t)L\,e(t) + \frac{1}{2}e^T(t)e(t) - \frac{1}{2}e^T(t-\tau(t)e(t-\tau(t)) \\ &+ \frac{1}{2}\dot{\tau}(t)e^T(t-\tau(t))e(t-\tau(t)). \end{aligned} \tag{35}$$

According to the fundamental Lemma 4.2, we can write

$$\begin{aligned} & 2\,e^T(t)\,\Gamma e(t-\tau(t)) \le e^T(t-\tau(t))\,I\,e(t-\tau(t)) + e^T(t)\,\Gamma I^{-1}\Gamma^T e(t) \\ \le\; & e^T(t-\tau(t))e(t-\tau(t)) + e^T(t)\,\Gamma\Gamma^T e(t) \\ \Longrightarrow\; & e^T(t)\Gamma e(t-\tau(t)) - \frac{1}{2}e^T(t-\tau(t))e(t-\tau(t)) \le e^T(t)(\frac{1}{2}\Gamma\Gamma^T)e(t), \end{aligned} \tag{36}$$

Where $G(e) = g(\hat{x}) - g(x) = g(x+e) - g(x)$ and recalling lipschitz condition, we have the following inequality:

$$\begin{aligned} & 2\, e^T(t)\, CG\, e(t) \leq G^T(e(t))\, IG\,(e(t) + e^T(t))\, CI^{-1}C^T e(t) \\ \leq \quad & G^T(e(t))\, G\,(e(t) + e^T CC^T e(t)\,) \\ \Longrightarrow \quad & e^T(t)\, CG\, e(t) \ \leq \frac{1}{2} G^T(e(t)\, G\,(e(t) + e^T(\frac{1}{2}CC^T)\, e(t))). \end{aligned} \tag{37}$$

Substituting (37) into (35) yields

$$\begin{aligned} \dot{V}(e(t)) \quad & \leq \quad e^T(t)[T + \frac{1}{2}\Gamma\Gamma^T + \frac{1}{2}CC^T + (\frac{1}{2} - L)I]e(t) \\ & + \quad \frac{M}{2} e^T(t-\tau(t)e(t-\tau(t)) + \frac{1}{2}e^T(t)e(t) + \frac{1}{2}G^T e(t)Ge(t) \\ & \leq \quad e^T(t)[T + \frac{1}{2}\Gamma\Gamma^T + \frac{1}{2}CC^T + (\frac{b_1+1}{2} - L)I)]e(t) \\ & + \quad \frac{M}{2} e^T(t-\tau(t)e(t-\tau(t)). \end{aligned} \tag{38}$$

We use the following inequality from Assumption 4.1 in (38):

$$G^T(e(t))G(e(t)) = \sum_{i=1}^{n} (g_i(\hat{x}(t)) - g_i(x(t))^2 \leq nb_1 e^T(t)e(t) \ \leq e^T(t)b_1 e(t), \tag{39}$$

where $b_1 = \max\{l_1, l_2, ..., l_n\}$. By choosing

$$L = \lambda_{\max}[T + \frac{1}{2}\Gamma\Gamma^T + \frac{1}{2}CC^T] + \frac{b_1 + M + 1}{2} + 1. \tag{40}$$

One can use L to inequaltity (39) and can obtain $\dot{V}(e(t)) \leq 0$. We have $e(t) \to 0$ i.e, $\hat{x}(t) \to x(t)$. This completes the proof. □

5. Numerical simulation

In this section, we apply the effectiveness of the proposed method to adaptive stabilizing inherent UPOs in the Rossler system with uncertain parameter and time delay.
The proposed method for stabilizing the inherent UPOs in the Rossler system with unknown parameters:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ x_3 & 0 & -c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} + u. \tag{41}$$

The uncontrolled ($u = 0$) Rossler system exhibits a chaotic behavior if $a = b = 0.2$ and $c = 5.7$. We used the fourth-order Runge-Kutta method to solve the systems with a time step size 0.001. We let run until a periodic orbit of a predetermined length is located. The simulation of Rossler's system started at an

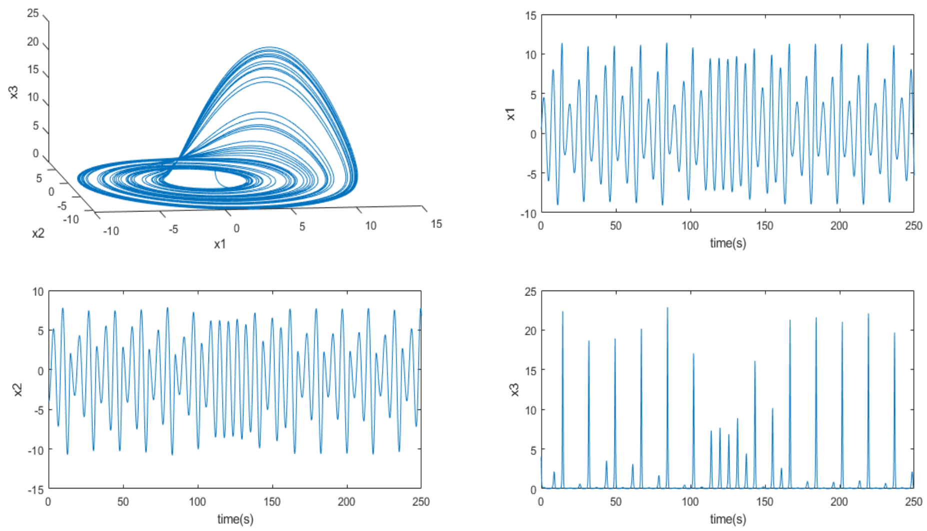

Figure 1: Chaotic orbits of the Rossler system and time response of states.

arbitrary initial condition targeting a UPO of a length near $\tau = 5.86$. Figure 1 shows the chaotic behavior of the Rossler system whit initial condition, $(x_1(0), x_2(0), x_3(0)) = (0.5, -4, 4)$ and time response Rossler system states.

Where u is an attached control term, the system (41) can be written as:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ x_3 & 0 & -c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix} \begin{bmatrix} x_1(t) - x_1(t-\tau) \\ x_2(t) - x_2(t-\tau) \\ x_3(t) - x_3(t-\tau) \end{bmatrix}. \end{aligned} \tag{42}$$

Thus using Theorem 3.1 and by using SDP3 toolbars of Mathlab software, we get a controller u with $(k_{11}, k_{22}, k_{33}) = (-0.12, -0.25, -0.1)$ stabilizing the system on UPO with a known period τ and known parameters. Figure 2, shows TDFC controlled period-one Rossler system and time response of the Rossler system with period one and Figure 3, shows time response of error states.

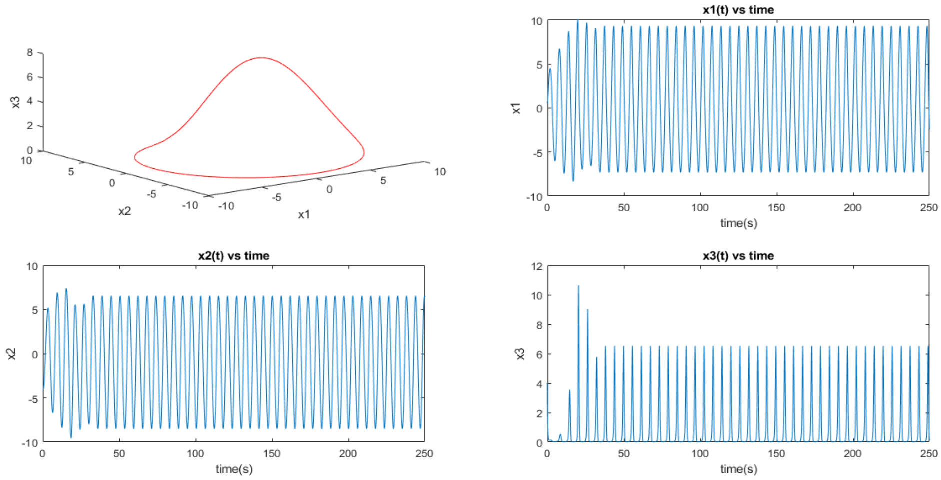

Figure 2: An orbits of the Rossler system and time response of states.

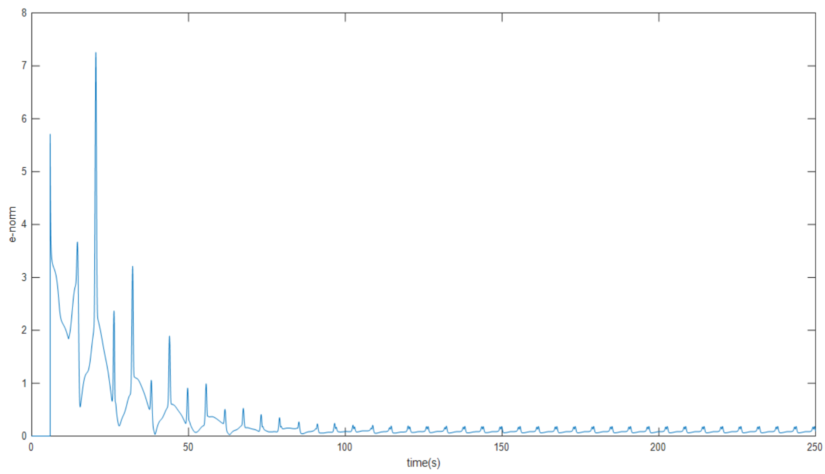

Figure 3: Time response of error states.

Let us rewrite the Rossler system in the standard of (26):

$$
\begin{aligned}
&\dot{x} = f(t,\tau,x(t),x(t-\tau)) = Tx(t) + Cg(x(t)) + \Gamma x(t-\tau(t)) \\
&T = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b & 1 \end{bmatrix}, \\
&g(x) = \begin{bmatrix} 0 \\ 1 \\ x_1 x_3 \end{bmatrix}, \; \Gamma = \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix}, \; x(t-\tau) = \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \\ x_3(t-\tau) \end{bmatrix}.
\end{aligned}
\tag{43}
$$

Here, system (43) with the unknown parameters $a, b, c, k_{11}, k_{22}, k_{33}$ is the reference model. To get the estimation of unknown parameters, we construct the following estimator:

$$
\begin{aligned}
&\dot{\hat{x}}(t) = \hat{T}(t)\hat{x}(t) + \hat{C}(t)g(\hat{x}(t)) + \hat{\Gamma}(t)\hat{x}(t-\tau) \; + \alpha(x,\hat{x}), \\
&\hat{T} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & \widehat{a} & 0 \\ 0 & 0 & -\widehat{c} \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \widehat{b} & 1 \end{bmatrix}, \quad \hat{\Gamma} = \begin{bmatrix} \hat{k}_{11} & 0 & 0 \\ 0 & \hat{k}_{22} & 0 \\ 0 & 0 & \hat{k}_{33} \end{bmatrix}.
\end{aligned}
\tag{44}
$$

Defining the error state $e(t) = [e_1, e_2, e_3]^T = [\hat{x}_1 - x_1,\ \hat{x}_2 - x_2, \hat{x}_3 - x_3]^T$. The error dynamics are described by (2). Following the procedure proposed in Theorem 4.3, we can design the adaptive scheme $\alpha(x,\hat{x}) = K[\hat{x}_1 - x_1,\ \hat{x}_2 - x_2, \hat{x}_3 - x_3]^T$where K is $diag[k_{11}, k_{22}, k_{33}]$. From (30) and (31) we obtain the compensator and update law for parameter estimation as

$$
\begin{aligned}
&\dot{\hat{a}} = -e_2\hat{x}_2, && \dot{\hat{k}}_{11} = e_1(\hat{x}_1,\ (t-\tau) - x_1(t)), && \dot{k}_1 = -e_1^2, \\
&\dot{\hat{b}} = -e_2, && \dot{\hat{k}}_{22} = e_2(\hat{x}_2\ (t-\tau) - x_2(t)), && \dot{k}_2 = -e_2^2, \\
&\dot{\hat{c}} = e_3\hat{x}_3, && \dot{\hat{k}}_{33} = e_3(\hat{x}_3\ (t-\tau) - x_3(t)), && \dot{k}_3 = -e_3^2.
\end{aligned}
\tag{45}
$$

Some simulation were done. The true values of Rossler system parameters were taken as $a = b = 0.2$, $c = 5.7$, $k_{11} = 0.12$, $k_{22} = 0.25$, $k_{33} = 0.1$, respectively. The initial values of their estimates were $\hat{a}(0) = \hat{b}(0) = \hat{c}(0) = \hat{k}_{11}(0) = \hat{k}_{22}(0) = \hat{k}_{33}(0) = 0.01$ respectively. The initial state of the unknown system and estimator $(x_1(0), x_2(0), x_3(0)) = (0.5, -4, 4)$ and $(\hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0)) = (-5, 4, -4)$ respectively. The delay time $\tau = 5.86$ was used which is a fundamental period orbit of the Rossler system. Figures 4 and 5, present the time response of error between states of the unknown and the estimator. One can see the two systems are asymptotically synchronized. Figures 6 and 7 show the estimation of the unknown parameters. The values of estimates approach to true value asymptotically. These estimates are fed in turn to the TDFC controller u which ensures a controlled system converges to the inherent UPO as shown in Figure 8.

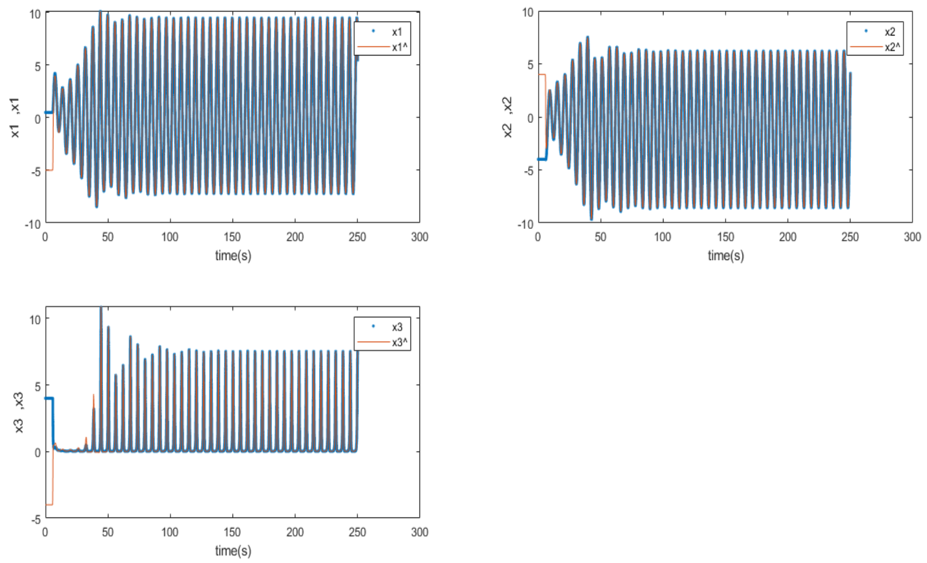

Figure 4: Time response between states of the unknown and the estimator.

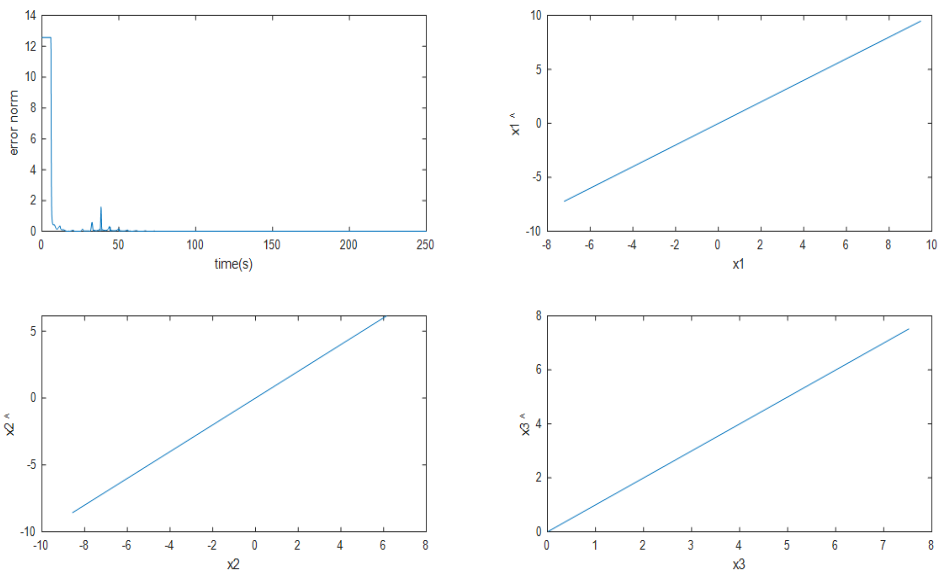

Figure 5: States of the unknown and the estimator.

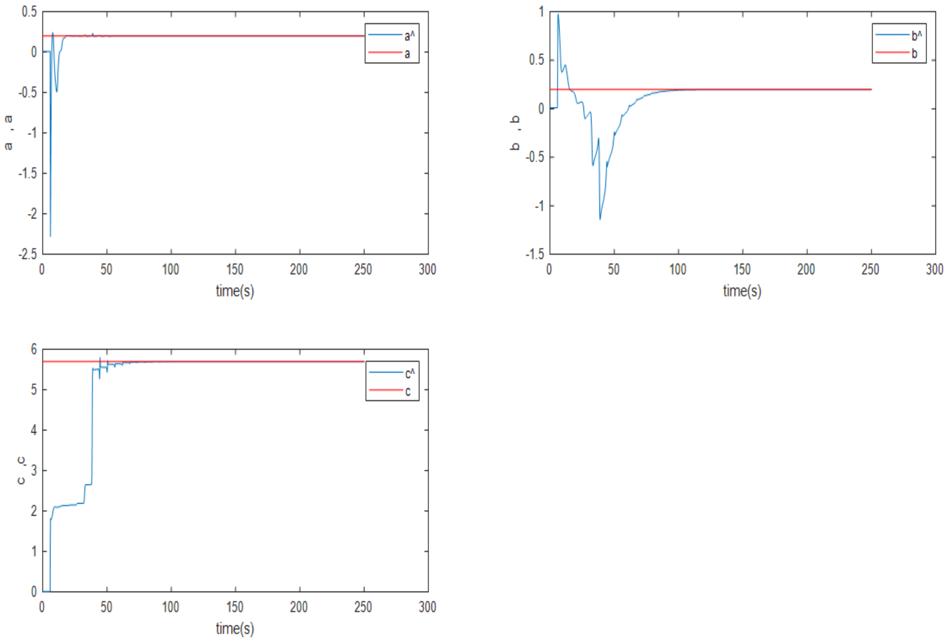

Figure 6: Estimation of the unknown parameters.

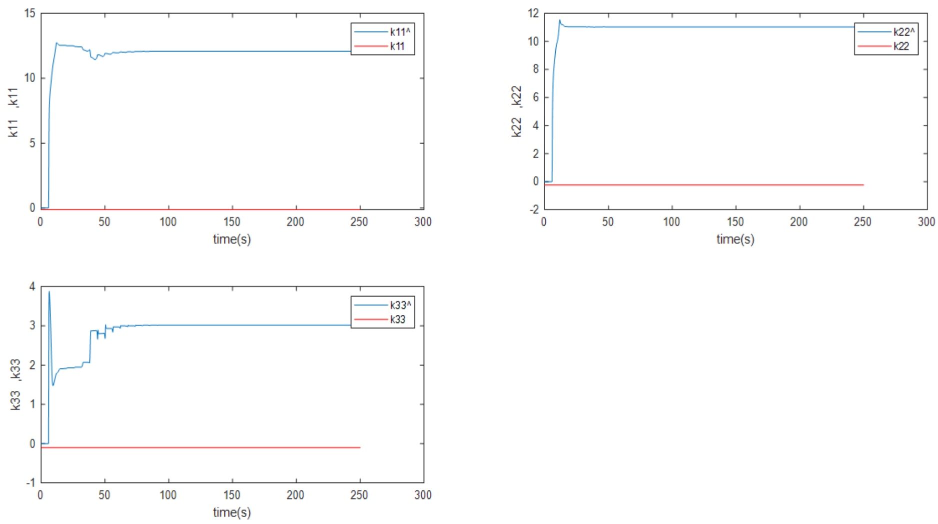

Figure 7: Estimation of the unknown parameters.

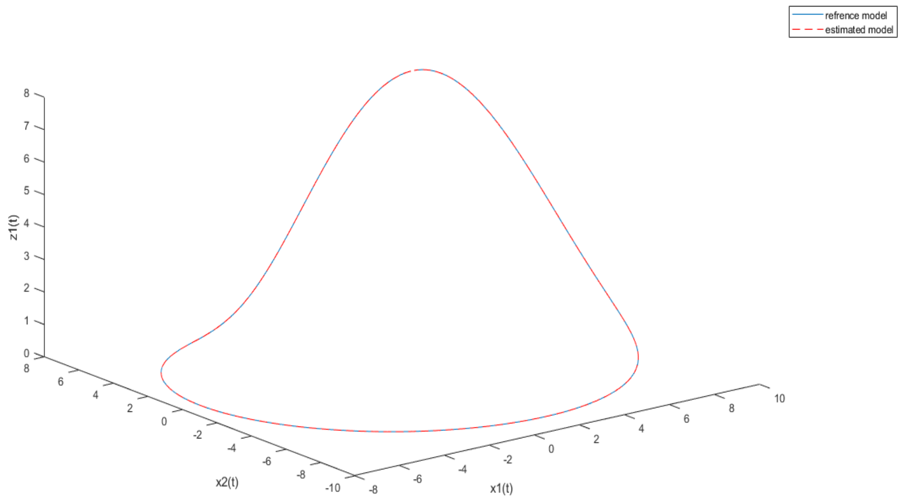

Figure 8: Values of estimates approach to true value asymptotically.

6. Conclusion

We have presented an adaptive stabilization controller for UPOs of an uncertain chaotic system with a TDFC controller. This paper deals with the problem of parameter identification and adaptive synchronization of the controlled TDFC chaotic system with uncertainties in system parameters and TDFC controller parameters. The main difference between the proposed method and other TDFC methods is that the feedback control law is not confined within a special format. To identify uncertain parameters of the controlled chaotic system with the TDFC method, unknown parameters can be identified when the two systems are synchronized. Then the proposed technique was utilized to estimate the unknown parameters in the model. By this method, the unknown parameters of the Rossler system were estimated exactly. Because of the limitation of the invariant principle that only guarantees the estimates of the unknown parameters to converge the largest invariant set containing in $\dot{V} = 0$. The effectiveness of the proposed scheme on adaptive stabilization is well demonstrated by the Rssler example.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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