Mathematics Interdisciplinary Research **8** (4) (2023) 291 – 307

Original Scientific Paper

Row Stochastic Matrices and Linear Preservers of Matrix Majorization $T : \mathbb{R}_m \to \mathbb{R}_n$

Ahmad Mohammadhasani⋆, *Mehdi Dehghanian and Yamin Sayyari*

Abstract

A nonnegative square and real matrix R is a row stochastic matrix if the sum of the entries of each row is equal to one. Let x, $y \in \mathbb{R}_n$. The vector x is said to be matrix majorized by y and denoted by $x \prec_r y$ if $x = yR$ for some row stochastic matrix R. In the present paper, we characterize the linear preservers of matrix majorization $T : \mathbb{R}_m \to \mathbb{R}_n$.

Keywords: Linear preserver, Matrix majorization, m-Row stochastic matrix.

2020 Mathematics Subject Classification: 15A04, 15A21, 15A51.

How to cite this article
A. Mohammadhasani, M. Dehghanian and Y. Sayyari, Row stochastic matrices and linear preservers of matrix majorization $T : \mathbb{R}_m \to \mathbb{R}_n$, *Math. Interdisc. Res.* **8** (4) (2023) 291-307.

1. Introduction

The concept of majorization plays an important role in applied mathematics and linear algebra. Various extensions of this concept have also been studied (see [1–4]).

One can see the concepts of left and right majorization from each other by getting transpose on the equations, because if a matrix A is doubly stochastic then the matrix A^t is doubly stochastic too, where A^t is the transpose of the matrix A. But when we use the row stochastic matrices, we can not obtain the

⋆Corresponding author (E-mail: a.mohammadhasani53@gmail.com)
Academic Editor: Abbas Saadatmandi
Received 2 January 2023, Accepted 7 August 2023
DOI: 10.22052/MIR.2023.248765.1390

left and right majorizations from each other. So in this case the left and right concepts are investigated in different manners (see [5–13]).

Here, we focus on right and left matrix majorization. Dahl defined the right matrix majorization as follows [14].

Definition 1.1. A nonnegative square and real matrix A is a row stochastic matrix if the sum of the entries of each row is equal to one.

Definition 1.2. Let $A, B \in \mathbf{M}_{n,m}$. The matrix A is said to be right matrix majorized by B and write $A \prec_r B$, if $A = BR$ for some row stochastic matrix R. If $A \prec_r B \prec_r A$, we denote $A \sim_r B$.

In [4], M. Pería et al. introduced the left matrix majorization as follows:

Definition 1.3. Let $A, B \in \mathbf{M}_{n,m}$. The matrix A is said to be left matrix majorized by B and write $A \prec_l B$, if $A = RB$ for some row stochastic matrix R.

In [11], the authors did not completely find the linear preservers of left matrix majorization $T : \mathbb{R}^p \to \mathbb{R}^n$. In [12], the authors completely characterized the linear preservers of this relation $T : \mathbb{R}^p \to \mathbb{R}^n$. In [9], the authors completely characterized the linear preservers of right matrix majorization on matrices were studied. Also, in [2] the authors characterized the linear operators that strongly preserve the right matrix majorization.

In this paper, the structure of all linear operators $T : \mathbb{R}_m \to \mathbb{R}_n$, preserving right matrix majorization are characterized. Some of our notation is explained next.

Let $\mathbf{M}_{n,m}$ be the algebra of all n-by-m real matrices. Let $\mathbb{R}_n$ ($\mathbb{R}^n$) be 1-by-n (n-by-1) real vectors, and the notation $\mathcal{P}(m)$ for the collection of all m-by-m permutation matrices.

A matrix $R = [r_{ij}] \in \mathbf{M}_{n,m}$ is called a row stochastic matrix if $r_{ij} \geq 0$ and $\sum_{j=1}^{n} r_{ij} = 1$, for all i ($1 \leq i \leq n$). The collection of all $m \times m$-row stochastic matrices is denoted by $\mathcal{RS}(m)$. A matrix R is called standard row stochastic, if each row has exactly a nonzero entry, $+1$, and other entries are zero. The collection of all standard m-row stochastic m-by-m matrices is denoted by $\mathcal{R}(m)$. Clearly, $\mathcal{P}(m) \subseteq \mathcal{R}(m)$. The standard basis of $\mathbb{R}_n$ is denoted by $\{\varepsilon_1, \ldots, \varepsilon_n\}$, and $e = (1, 1, \ldots, 1)^t \in \mathbb{R}^n$. Span$\{S\}$ is denoted by the intersection of all subspaces of V that contain S, where V is a vector space over a field F. If S is nonempty, then

$$\text{Span}\{S\} = \left\{ \sum_{i=1}^{k} \alpha_i v_i | v_1, \ldots, v_k \in S, \alpha_1, \ldots, \alpha_k \in S, \text{and} \ \ k \in \mathbb{N} \right\}.$$

For each $\mathbf{a} \in \mathbb{R}_n$ define $A := \mathbf{a}^{(m)} \in \mathbf{M}_{n,m}$ the matrix which each its row is $\mathbf{a}$. Let $u \in \mathbb{R}$. Define

$$u^+ := \begin{cases} u, & \text{if } u \geq 0, \\ 0, & \text{if } u < 0, \end{cases}$$

and

$$u^- := \begin{cases} 0, & \text{if } u \geq 0, \\ -u, & \text{if } u < 0. \end{cases}$$

For all $x = (x_1, \ldots, x_n) \in \mathbb{R}_n$ we denote $tr(x) := \sum_{i=1}^n x_i$, $tr_+(x) := \sum_{i=1}^n x_i^+$, and $tr_-(x) := \sum_{i=1}^n x_i^-$.
Let $A = [a_{ij}] \in \mathbf{M}_{n,m}$. We say that $A \geq 0$ if $a_{ij} \geq 0$, for each i, j ($1 \leq i \leq n, 1 \leq j \leq m$). Define $|A| = [|a_{ij}|]$. The $\mathbf{i}$th row of A is denoted by $\mathbf{a_i^R}$. Also, the $\mathbf{j}$th column of A is denoted by $\mathbf{a_j^C}$, and $Col(A) = \{\mathbf{a_1^C}, \ldots, \mathbf{a_m^C}\}$. Define $c_+(A) = \{j : \mathbf{a_j^C} \geq 0\}$ and $c_-(A) = \{j : \mathbf{a_j^C} \leq 0\}$.
Let $[T]$ be the matrix representation of a linear operator $T : \mathbb{R}_m \to \mathbb{R}_n$ with respect to the standard basis. In this case, $Tx = xA$, where $A = [T]$.
A linear operator $T : \mathbb{R}_m \to \mathbb{R}_n$ preserves a relation $\sim$, if $x \sim y$ concludes that $Tx \sim Ty$.

This work continues in three further sections. Section 2 studies some conditions for $\prec_r$ and a linear operator T to preserve $\prec_r$ on $\mathbb{R}_m$. Section 3 characterizes the structure of all linear operators $T : \mathbb{R}_2 \to \mathbb{R}_n$ preserving matrix majorization. In Section 4 we obtain all linear preservers of $\prec_r$ from $\mathbb{R}_m$ to $\mathbb{R}_n$.

2. Matrix majorization on $\mathbb{R}_m$

In this section, we study some properties of the relation $\prec_r$.

Definition 2.1. For each $x \in \mathbb{R}_n$ define $\tilde{x} := tr_+(x)e_1 + tr_-(x)e_2$.

If $x \prec_r y \prec_r x$, we write $x \sim y$.

Lemma 2.2. *If $x \in \mathbb{R}_n$, then $x \sim \tilde{x}$.*

Proof. Let $x \in \mathbb{R}_n$ we defne the matrix R by

$$R = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix},$$

where

$$R_i := \begin{cases} e_1, & x_i \geq 0, \\ e_2, & x_i < 0. \end{cases}$$

We observe that $\tilde{x} = xR$, and $R \in \mathcal{RS}(n)$. So $\tilde{x} \prec_r x$.
Without loss of generality suppose that $x_1 \geq x_2 \geq \cdots \geq x_n$, where $x_1, \ldots, x_k > 0$, and $x_l, \ldots, x_n < 0$. Consider matrix S as follows

$$S = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{bmatrix},$$

where $S_1 := \left(\frac{x_1}{tr_+(x)}, \ldots, \frac{x_k}{tr_+(x)}, 0, \ldots, 0\right)$, $S_2 := \left(0, \ldots, 0, \frac{x_l}{tr_-(x)}, \ldots, \frac{x_n}{tr_-(x)}\right)$, and $S_i := \varepsilon_1$, for each i $(3 \leq i \leq n)$. In this case, $x = \tilde{x}S$ and $S \in \mathcal{RS}(n)$, and then $x \prec_r \tilde{x}$. Therefore, $\tilde{x} \sim x$. □

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}_n$. Define $\|x\| := \sum_{i=1}^{n} |x_i|$. Clearly, $\|.\|$ is a norm on $\mathbb{R}_n$. The following proposition provides a criterion for matrix majorization on $\mathbb{R}_n$.

Proposition 2.3. *Let $x, y \in \mathbb{R}_n$. Then the following conditions are equivalent:*
1) $x \prec_r y$,
2) $tr_+(x) + tr_-(x) = tr_+(y) + tr_-(y)$ *and* $tr_+(x) \leq tr_+(y)$,
3) $tr_+(x) + tr_-(x) = tr_+(y) + tr_-(y)$ *and* $tr_-(x) \geq tr_-(y)$,
4) $tr(x) = tr(y)$ *and* $\|x\| \leq \|y\|$.

Proof. As $x \prec_r y$ if and only if $\tilde{x} \prec_r \tilde{y}$, we can prove the statement. □

The following conclusion gives an equivalent condition for $\sim$ on $\mathbb{R}_n$.

Corollary 2.4. *Let $x, y \in \mathbb{R}_n$. Then the following statements are equivalent:*
1) $x \sim y$,
2) $tr_+(x) = tr_+(y)$ *and* $tr_-(x) = tr_-(y)$,
3) $tr(x) = tr(y)$ *and* $\|x\| = \|y\|$.

Theorem 2.5. *Suppose that $T : \mathbb{R}_m \to \mathbb{R}_n$ be a linear operator that preserve $\prec_r$ and $ker(T) \neq 0$. Then $Tx = x\mathbf{c}^{(m)}$ for some $\mathbf{c} \in \mathbb{R}_n$.*

Proof. Assume that $A \in \mathbf{M}_{n,m}$ is the matrix representation of the linear operator $T : \mathbb{R}_m \to \mathbb{R}_n$ with respect to the standard basis. So $Tx = xA$. Since T is not one-to-one, there is some $b = (b_1, \ldots, b_m) \in \mathbb{R}_m \setminus \{0\}$ such that $Tb = bA = 0$.

If $b_1 = \cdots = b_m$, Set $w_i = mb_1e_i$, for each $i = 1, \ldots, m$. For each $i = 1, \ldots, m$, we conclude that $w_i \prec_r b$, and then $Tw_i \prec_r Tb$. It implies that $Tw_i = 0$, and so $T\varepsilon_i = 0$. We deduce that $A = 0$. Choose $\mathbf{c} = 0$.
Let $b_i \neq b_j$ for some $i, j \in \{1, \ldots, m\}$ and $i \neq j$. For $t \neq s \in \{1, \ldots, m\}$ we have

$$\left(\sum_{k=1, k\neq i,j}^{m} b_k\right) e_1 + b_i e_t + b_j e_s \prec_r b.$$

Since T preserves $\prec_r$,

$$\left(\sum_{k=1,k\neq i,j}^{m} b_k\right)\mathbf{a_1^R} + b_i\mathbf{a_t^R} + b_j\mathbf{a_s^R} = 0,$$

for every $1 \leq t \neq s \leq m$. It follows that

$$\left(\sum_{k=1,k\neq i,j}^{m} b_k\right)\mathbf{a_1^R} + b_j\mathbf{a_t^R} + b_i\mathbf{a_s^R} = 0,$$

for each $t, s = 1, \ldots, m$ and $t \neq s$. Hence $(b_i - b_j)\mathbf{a_t^R} + (b_j - b_i)\mathbf{a_s^R} = 0$ and $\mathbf{a_r^R} = \mathbf{a_s^R}$ for each $t, s = 1, \ldots, m$ and $r \neq s$. Put $\mathbf{c} = \mathbf{a_1^R}$, we have $Tx = x\mathbf{c}^{(m)}$. □

The following conclusion is expressed in [15].

Lemma 2.6. *([15]). The set $\mathcal{RS}(n)$ is a convex set whose extreme points are $\mathcal{R}(n)$.*

Lemma 2.7. *Let $x, y \in \mathbb{R}_n$ be both nonnegative or nonpositive. Then the following statements are equivalent:*
1) $x \prec_r y$,
2) $x \sim y$,
3) $tr(x) = tr(y)$,
4) $\|x\| = \|y\|$.

Proof. By the use of Proposition 2.3, the proof is obvious. □

Lemma 2.8. *Let $T : \mathbb{R}_m \to \mathbb{R}_n$ preserve $\prec_r$. Then each column of A is nonnegative or nonpositive.*

Proof. Let $A = [a_{ij}]$ and $a_{tj}a_{sj} < 0$ for some t, s, j. Since $2e_t \sim_r e_t + e_s$ and T preserves $\sim_r$, we conclude that

$$2\mathbf{a_r^R} \sim \mathbf{a_r^R} + \mathbf{a_s^R}.$$

So $2\|\mathbf{a_r^R}\| = \|\mathbf{a_s^R} + \mathbf{a_r^R}\|$ and thus

$$\begin{aligned} 2\sum_{j=1}^{n} |a_{rj}| &= \sum_{j=1}^{n} |a_{rj} + a_{sj}| \\ &< \sum_{j=1}^{n} |a_{rj}| + \sum_{j=1}^{n} |a_{sj}| = 2\sum_{j=1}^{n} |a_{rj}|. \end{aligned}$$

It is a contradiction. Therefore we deduce that each column of A is nonnegative or nonpositive. □

Suppose that $T : \mathbb{R}_m \to \mathbb{R}_n$ preserves $\prec_r$. Since $\varepsilon_i \sim \varepsilon_j$, for each $1 \leq i, j \leq m$, we observe that $T\varepsilon_i \sim T\varepsilon_j$, and then $tr_+(\mathbf{a_i^R}) = tr_+(\mathbf{a_j^R})$, $tr_-(\mathbf{a_i^R}) = tr_-(\mathbf{a_j^R})$, and $tr(\mathbf{a_i^R}) = tr(\mathbf{a_j^R})$. Now define $tr_+(A) := tr_+(\mathbf{a_1^R})$, $tr_-(A) := tr_-(\mathbf{a_1^R})$, and $tr(A) := tr(\mathbf{a_1^R})$.

Lemma 2.9. *Let $T : \mathbb{R}_m \to \mathbb{R}_n$ preserve $\prec_r$. Then $|T|$ preserves $\prec_r$.*

Proof. First, we prove that for each $x \in \mathbb{R}_m$

$$tr(|T|(x)) = tr(x)(tr_+(A) - tr_-(A)), \tag{1}$$

$$\||T|(x)\| = \|T(x)\|. \tag{2}$$

Lemma 2.8 ensures that

$$\begin{aligned} tr(|T|(x)) &= \sum_{j=1}^{n} x.\left|\mathbf{a_j}^{\mathbf{C}^t}\right| = \sum_{j\in c_+(A)} x.\left|\mathbf{a_j}^{\mathbf{C}^t}\right| + \sum_{j\in c_-(A)} x.\left|\mathbf{a_j}^{\mathbf{C}^t}\right| \\ &= \sum_{j\in c_+(A)} x.\mathbf{a_j}^{\mathbf{C}^t} - \sum_{j\in c_-(A)} x.\mathbf{a_j}^{\mathbf{C}^t} = x.\sum_{j\in c_+(A)} \mathbf{a_j}^{\mathbf{C}^t} - x.\sum_{j\in c_-(A)} \mathbf{a_j}^{\mathbf{C}^t} \\ &= \sum_{i=1}^{m} x_i \sum_{j\in c_+(A)} a_{ij} - \sum_{i=1}^{m} x_i \sum_{j\in c_-(A)} a_{ij} = tr(x)(tr_+(A) - tr_-(A)). \end{aligned}$$

This shows that (1) holds. Also,

$$\begin{aligned} \||T|(x)\| &= \sum_{j=1}^{n} \left|x.\mathbf{a_j}^{\mathbf{C}^t}\right| = \sum_{j\in c_+(A)} \left|x.\mathbf{a_j}^{\mathbf{C}^t}\right| + \sum_{j\in c_-(A)} \left|-x.\mathbf{a_j}^{\mathbf{C}^t}\right| \\ &= \sum_{j\in c_+(A)} \left|x.\mathbf{a_j}^{\mathbf{C}^t}\right| + \sum_{j\in c_-(A)} \left|x.\mathbf{a_j}^{\mathbf{C}^t}\right| = \sum_{j=1}^{n} \left|x.\mathbf{a_j}^{\mathbf{C}^t}\right| = \|T(x)\|. \end{aligned}$$

Thus $\||T|(x)\| = \|T(x)\|$.

Now suppose that $x, y \in \mathbb{R}_m$ and T preserves $\prec_r$. So $tr(x) = tr(y)$ and $T(x) \prec_r T(y)$, therefore $\|Tx\| \leq \|Ty\|$. The relations (1) and (2) ensure that $tr(|T|(x)) = tr(|T|(y))$ and $\||T|(x)\| \leq \||T|(y)\|$. It means that $|T|(x) \prec_r |T|(y)$ and $|T|$ preserves $\prec_r$. □

Lemma 2.10. *Let $\mathbf{a} \in \mathbb{R}_k$ and let $T : \mathbb{R}_m \to \mathbb{R}_n$ be a linear operator. Define $\tilde{T} : \mathbb{R}_m \to \mathbb{R}_{n+k}$ by $[\tilde{T}] := [A|\mathbf{a}^{(m)}]$. Then T preserves $\prec_r$ if and only if $\tilde{T}$ preserves $\prec_r$.*

Proof. Assume that $x \prec_r y$ and T preserves $\prec_r$. So $tr(x) = tr(y)$ and

$$\begin{aligned} \tilde{T}x \prec_r \tilde{T}y &\Longleftrightarrow tr(Tx) + tr(x)tr(\mathbf{a}) = tr(Ty) + tr(y)tr(\mathbf{a}) \\ &\Longrightarrow tr(Tx) = tr(Ty), \end{aligned} \tag{3}$$

and

$$\begin{aligned} \|\tilde{T}x\| \leq \|\tilde{T}y\| &\Longleftrightarrow \|Tx\| + |tr(x)|tr(|\mathbf{a}|) \leq \|Ty\| + |tr(y)|tr(|\mathbf{a}|) \\ &\Longleftrightarrow \|Tx\| \leq \|Ty\|. \end{aligned} \tag{4}$$

By the relations (3) and (4) and Proposition 2.3 the proof is easy. □

Lemma 2.11. *Let $T : \mathbb{R}_m \to \mathbb{R}_n$ be a linear operator and let $P \in \mathcal{P}(n)$. Define $T_P : \mathbb{R}_m \to \mathbb{R}_n$ by $[T_P] := AP$. Then T preserves $\prec_r$ if and only if T_P preserves $\prec_r$.*

Proof. Suppose that $x \prec_r y$. We have

$$tr(Tx) = \sum_{i=1}^{n} x.\mathbf{a_i}{\mathbf{C}}^t \text{ and } \|Tx\| = \sum_{i=1}^{n} \left| x.\mathbf{a_i}{\mathbf{C}}^t \right|.$$

Since AP is A which its rows have been interchanged, we dedeuce that T preserves $\prec_r$ if and only if T_P preserves $\prec_r$. □

3. Linear preservers of matrix majorization on $\mathbb{R}_2$

In this section, we characterize the linear preservers of matrix majorization $T : \mathbb{R}_2 \to \mathbb{R}_n$. We use the symbol P for the following matrix

$$P := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Lemma 3.1. *Let $u \in \mathbb{R}_2$ and $T : \mathbb{R}_2 \longrightarrow \mathbb{R}_n$ be a linear operator such that $T(uR) \prec_r Tu$ for every $R \in \mathbb{R}_2$, then $T(uR) \prec_r Tu$ for every $R \in \mathbb{SR}_2$.*

Proof. Let $R \in \mathbb{SR}_2$. By the use of Lemma 2.6 we have $R = \sum_{i=1}^{4} \lambda_i R_i$ for some $R_i \in \mathbb{R}_2$, $\lambda_i \geq 0$, $\sum_{i=1}^{4} \lambda_i = 1$. Hence,

$$T(uR) = T\left(u \sum_{i=1}^{4} \lambda_i R_i\right) = \sum_{i=1}^{4} \lambda_i T(uR_i).$$

Since $T(uR_i) \prec_r T(u)$ for all i, $(1 \leq i \leq 4)$, there exists some $S_i \in \mathcal{RS}(2)$ such that $T(uR_i) = T(u)S_i$. Thus,

$$T(uR) = \sum_{i=1}^{4} \lambda_i T(u)S_i = Tu\left(\sum_{i=1}^{4} \lambda_i S_i\right) = (Tu)S,$$

and $S \in \mathcal{RS}(2)$, so $T(uR) \prec_r Tu$ and thus T preserves $\prec_r$. □

For any real and nonnegative number a, the symbol $\frac{a}{0}$ equal to $+1$ if $a > 0$, and equal to 1 if $a = 0$.

Theorem 3.2. *Let $T : \mathbb{R}_2 \longrightarrow \mathbb{R}_n$ be a nonnegative linear operator and*

$$[T] := \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \end{bmatrix}.$$

Then T preserves $\prec_r$ if and only if the following conditions occur
1) $\{\frac{a_{1k}}{a_{2k}} : 1 \leq k \leq n\} = \{\frac{a_{2k}}{a_{1k}} : 1 \leq k \leq n\}$;
2) *for all $a \in \{\frac{a_{1k}}{a_{2k}} : 1 \leq k \leq n\}$, we have*

$$\sum_{\frac{a_{1k}}{a_{2k}}=a} a_{1k} = \sum_{\frac{a_{2k}}{a_{1k}}=a} a_{2k}. \tag{5}$$

Proof. $\Longrightarrow$) Let T be a nonnegative linear operator such that preserves $\prec_r$,

$$[T] := \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \end{bmatrix},$$

$\mathbf{a}_i$ be the ith row of the matrix $[T]$ and $\frac{\mathbf{a}_i}{\mathbf{a}_j} := \{\frac{a_{ik}}{a_{jk}} : 1 \leq k \leq n\}$ $(i, j = 1, 2)$.

By applying Lemma 2.10 and Lemma 2.11 we can assume that $a_{1j} \neq a_{2j}$, for each $j = 1, ..., n$. Let

$$\mathcal{A} := \frac{\mathbf{a_1}}{\mathbf{a_2}} \cup \frac{\mathbf{a_2}}{\mathbf{a_1}} = \{a_1, ..., a_p, a_p^{-1}, ..., a_1^{-1}\},$$

such that

$$0 \leq a_1 < \cdots < a_p < 1 < a_p^{-1} < \cdots < a_1^{-1} \leq \infty.$$

Assume that T preserves $\prec_r$. Since $e_1 \sim_r e_2$, $Te_1 \sim_r Te_2$. So $\mathbf{a_1} \sim_r \mathbf{a_2}$, thus $tr(\mathbf{a_1}) = tr(\mathbf{a_2})$. It is suficient to show that $a_j \in \frac{\mathbf{a_1}}{\mathbf{a_2}} \cap \frac{\mathbf{a_2}}{\mathbf{a_1}}$, for every $j = 1, ..., n$ and thus $a_j^{-1} \in \frac{\mathbf{a_1}}{\mathbf{a_2}} \cap \frac{\mathbf{a_2}}{\mathbf{a_1}}$. For every $j = 1, ..., p$, we define the open intervals $E_j \subseteq \mathbb{R}$ by

$$E_j := \begin{cases} (a_j, a_{j+1}), & \text{if } j < p, \\ (a_p, 1), & \text{if } j = p. \end{cases}$$

We see that $(x,-1) \sim_r (-1,x)$ for each $x \in \mathbb{R}$, so $T(x,-1) \sim_r T(-1,x)$ and $tr_+(T(x,-1)) = tr_+(T(-1,x))$ for all $x \in E_j$, $j = 1, ..., p$.

By induction on j we prove that

$$a_j \in \frac{\mathbf{a}_1}{\mathbf{a}_2} \cap \frac{\mathbf{a}_2}{\mathbf{a}_1}, \tag{6}$$

and

$$\sum_{\frac{a_{1k}}{a_{2k}}=a_j} \binom{a_{1k}}{a_{2k}} = \sum_{\frac{a_{2k}}{a_{1k}}=a_j} \binom{a_{2k}}{a_{1k}}, \tag{7}$$

thus

$$\sum_{\frac{a_{1k}}{a_{2k}}=a_j^{-1}} \binom{a_{1k}}{a_{2k}} = \sum_{\frac{a_{2k}}{a_{1k}}=a_j^{-1}} \binom{a_{2k}}{a_{1k}}, \tag{8}$$

for every$j = 1, ..., p$. Let $j = 1$. If $x \in E_1$ then

$$tr_+T(x,-1)) = \sum_{\frac{a_{2k}}{a_{1k}}=a_1} (a_{1k}x - a_{2k}), \tag{9}$$

and

$$tr_+T(-1,x)) = \sum_{\frac{a_{1k}}{a_{2k}}=a_1} (a_{2k}x - a_{1k}). \tag{10}$$

It implies that $a_1 \in \frac{\mathbf{a}_1}{\mathbf{a}_2} \cap \frac{\mathbf{a}_2}{\mathbf{a}_1}$, because if $a_1 \in \frac{\mathbf{a}_1}{\mathbf{a}_2}$ and $a_1 \notin \frac{\mathbf{a}_2}{\mathbf{a}_1}$ then $tr_+(T(-1,x)) > 0$ and $tr_+(T(x,-1)) = 0$, this is a contradiction. Similarly, $a_1 \notin \frac{\mathbf{a}_1}{\mathbf{a}_2}$ and $a_1 \in \frac{\mathbf{a}_2}{\mathbf{a}_1}$ yields a contradiction.

By the use of relations (9) and (10) and $tr_+(T(x,-1)) = tr_+(T(-1,x))$, we deduce that (7) holds for $j = 1$. Now assume that the conditions holds for $j < p$ and $x \in E_{j+1}$. So

$$\begin{aligned} tr_+(T(x,-1)) &= \sum_{i=1}^{j+1} \sum_{\frac{a_{2k}}{a_{1k}}=a_i} (a_{1k}x - a_{2k}) \\ &= \sum_{i=1}^{j} \sum_{\frac{a_{2k}}{a_{1k}}=a_i} (a_{1k}x - a_{2k}) + \sum_{\frac{a_{2k}}{a_{1k}}=a_{j+1}} (a_{1k}x - a_{2k}). \end{aligned} \tag{11}$$

Also,

$$
\begin{aligned}
tr_+(T(-1,x)) &= \sum_{i=1}^{j+1} \sum_{\frac{a_{1k}}{a_{2k}}=a_i} (a_{2k}x - a_{1k}) \\
&= \sum_{i=1}^{j} \sum_{\frac{a_{1k}}{a_{2k}}=a_i} (a_{2k}x - a_{1k}) + \sum_{\frac{a_{1k}}{a_{2k}}=a_{j+1}} (a_{2k}x - a_{1k}). \qquad (12)
\end{aligned}
$$

By induction hypothesis we have

$$\sum_{\frac{a_{2k}}{a_{1k}}=a_i} (a_{1k}x - a_{2k}) = \sum_{\frac{a_{1k}}{a_{2k}}=a_i} (a_{2k}x - a_{1k}),$$

for $i = 1, ..., j$. Thus

$$\sum_{i=1}^{j} \sum_{\frac{a_{2k}}{a_{1k}}=a_i} (a_{1k}x - a_{2k}) = \sum_{i=1}^{j} \sum_{\frac{a_{1k}}{a_{2k}}=a_i} (a_{2k}x - a_{1k}).$$

It shows that (7) for $j+1$ holds. Hence, the relation (5) holds.
Now we show that $a_{j+1} \in \frac{\mathbf{a}_1}{\mathbf{a}_2} \cap \frac{\mathbf{a}_2}{\mathbf{a}_1}$. Hence the induction argument is completed. Conversly, suppose that (1) and (2) hold, we prove T preserves $\prec_r$. The conditions (1) and (2) ensure that

$$\sum_{k=1}^{n} a_{1k} = \sum_{a\in\mathcal{A}} \left(\sum_{k\in\mathcal{K}_a} a_{1k} \right) = \sum_{a\in\mathcal{A}} \left(\sum_{k\in\mathcal{K}_a} a_{2k} \right) = \sum_{k=1}^{n} a_{2k}.$$

Since $A \geq 0$, $\mathbf{a}_1 \sim \mathbf{a}_2$. Let $u = (x, y) \in \mathbb{R}_2$ and $R \in \mathcal{R}(2)$, we prove that $T(uR) \prec_r Tu$ and thus $T(uD) \prec_r Tu$ for every $D \in \mathcal{RS}(2)$ by the Lemma 3.1. Observe that

$$\mathcal{R}(2) = \left\{ I, P, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

Let $u = (x, y) \in \mathbb{R}_2$ and $R \in \mathcal{R}(2)$, so $uR \sim_r u$. If $xy \geq 0$, since $A \geq 0$, we have $T(u) \sim_r T(uR)$ if and only if $tr(T(u)) \sim_r tr(T(uR))$. So in this case the proof is established. Now suppose that $xy < 0$. Since for $c \in \mathbb{R} - \{0\}$, $T(uR) \sim_r T(u)$ if and only if $T(cuR) \sim_r T(cu)$, so without loss of generality we can assume that $u = (x, -1)$ or $u = (-1, x)$, where $0 < x \leq 1$.

Suppose that $u = (x, -1)$, for some $0 < x \leq 1$ (similarly for the case $u = (-1, x)$). Since $tr(\mathbf{a_1}) = tr(\mathbf{a_2})$,

$$tr(T((x, -1)R)) = tr(T(x, -1)), \tag{13}$$

for every $R \in \mathcal{R}(2)$. We just have to prove that

$$tr_+(T((x, -1)R)) \leq tr_+(T(x, -1)). \tag{14}$$

Now we prove (14) in four cases.

- Case 1: If $R = I$, the proof is obvious.
- If $R = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, then

$$\begin{aligned} tr_+(T((x, -1)R)) &= tr_+(T(x-1, 0)) \\ &= tr_+((x-1)\mathbf{a_1}) = 0 \leq tr_+(T(x, -1)). \end{aligned}$$

- If $R = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, the proof is similar to the second case
- Let $R = P$ and $0 < x \leq 1$. So

$$x \in (0, a_1] \cup [a_1, a_2] \cup \ldots \cup [a_{p-1}, a_p] \cup [a_p, 1].$$

Observe that

$$tr_+(T(-1, x)) = \begin{cases} 0, & \text{if } x \in (0, a_1], \\ \sum_{i=1}^{j-1} \sum_{\frac{a_{1k}}{a_{2k}} = a_i} (a_{2k}x - a_{1k}), & \text{if } x \in [a_{j-1}, a_j], \\ \sum_{i=1}^{p} \sum_{\frac{a_{1k}}{a_{2k}} = a_i} (a_{2k}x - a_{1k}), & \text{if } x \in [a_p, 1], \end{cases}$$

and

$$tr_+(T(x, -1)) = \begin{cases} 0, & \text{if } x \in (0, a_1], \\ \sum_{i=1}^{j-1} \sum_{\frac{a_{2k}}{a_{1k}} = a_i} (a_{1k}x - a_{2k}), & \text{if } x \in [a_{j-1}, a_j], \\ \sum_{i=1}^{p} \sum_{\frac{a_{2k}}{a_{1k}} = a_i} (a_{1k}x - a_{2k}), & \text{if } x \in [a_p, 1], \end{cases}$$

where $1 < j < p$. On the other hand, from the conditions (1) and (2) of hypothesis we have

$$\sum_{\frac{a_{1k}}{a_{2k}}=a_j} \begin{pmatrix} a_{1k} \\ a_{2k} \end{pmatrix} = \sum_{\frac{a_{2k}}{a_{1k}}=a_j} \begin{pmatrix} a_{2k} \\ a_{1k} \end{pmatrix},$$

for every $j = 1, ..., p$. Thus,

$$\sum_{\frac{a_{1k}}{a_{2k}}=a_j} (a_{1k}x - a_{2k}) = \sum_{\frac{a_{2k}}{a_{1k}}=a_j} (a_{2k}x - a_{1k}),$$

for every $j = 1, ..., p$ and $x \in \mathbb{R}$. Hence,

$$\sum_{i=1}^{j} \sum_{\frac{a_{1k}}{a_{2k}}=a_j} (a_{1k}x - a_{2k}) = \sum_{i=1}^{j} \sum_{\frac{a_{2k}}{a_{1k}}=a_j} (a_{2k}x - a_{1k}),$$

for each $j = 1, ..., p$ and $x \in \mathbb{R}$. So, $tr_+(T(x, -1)) = tr_+(T(-1, x))$, and the relation (14) holds.

□

Lemma 3.3. *Let $T : \mathbb{R}_2 \to \mathbb{R}_n$ be a linear operator. Then T preserves $\prec_r$ if and only if the following statements are true:*
1) *$|T|$ preserves $\prec_r$,*
2) *each column of A is nonnegative or nonpositive,*
3) *two rows of A are equivalent. i.e. $\mathbf{a_1} \sim \mathbf{a_2}$.*

Proof. We only prove the necessary condition. Let $x, y \in \mathbb{R}_2$ and $x \prec_r y$, so

$$\begin{aligned} tr(xA) &= \sum_{j=1}^{n} x.\mathbf{a_j^C} = x.\sum_{j=1}^{n} \mathbf{a_j^C} \\ &= x_1 \sum_{j=1}^{n} a_{1j} + x_2 \sum_{j=1}^{n} a_{2j} = x.tr(\mathbf{a_1}) + x.tr(\mathbf{a_2}) \\ &= tr(x)tr(\mathbf{a_1}) = tr(y)tr(\mathbf{a_1}) = tr(yA). \end{aligned} \quad (15)$$

On the other hand, $|T|$ preserves $\prec_r$, so $|T|x \prec_r |T|y$. Hence

$$\|xA\| = \|x|A|\| \leq \|y|A|\| = \|yA\|. \quad (16)$$

By the relations (15) and (16) we conclude that $Tx \prec_r Ty$. □

4. Linear preservers of matrix majorization on $\mathbb{R}_m$

In this section, we express linear preservers of matrix majorization $T : \mathbb{R}_m \to \mathbb{R}_n$.

Theorem 4.1. *Let $m \geq 3$ and $T : \mathbb{R}_m \to \mathbb{R}_n$ be a linear operator. Then T preserves $\prec_r$ if and only if*

$$[T] = [R|\mathbf{c^{(m)}}]P,$$

where in each column of matrix R there is only one non-zero entry, each two rows of R are equivalent, $P \in \mathcal{P}(n)$ and $\mathbf{c}$ is a vector.

Proof. Assume that T preserves $\prec_r$. If T is not one-to-one, then by Theorem 2.5 the proof is obvious. Now, let T is one-to-one. First, we prove the theorem for $m = 3$. According to Lemma 2.10 we can assume

$$[T] = R = A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \end{bmatrix},$$

where matrix A has no duplicate column. By Lemma 2.9, without loss of generality assume that $A \geq 0$. Define

- $\left\{\frac{a_{1j}}{a_{2j}+a_{3j}} | a_{2j} \neq a_{3j}, j = 1, \dots, n\right\} := \mathcal{A}_1 \subseteq [0, \infty)$,
- $\left\{\frac{a_{2j}}{a_{1j}+a_{3j}} | a_{1j} \neq a_{3j}, j = 1, \dots, n\right\} := \mathcal{A}_2 \subseteq [0, \infty)$,
- $\left\{\frac{a_{3j}}{a_{1j}+a_{2j}} | a_{1j} \neq a_{2j}, j = 1, \dots, n\right\} := \mathcal{A}_3 \subseteq [0, \infty)$.

If $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = \emptyset$, then the proof is obvious. Otherwise without loss of generality we assume that $\mathcal{A}_3 \neq \emptyset$ and

$$\frac{a_{3j_0}}{a_{1j_0} + a_{2j_0}} = \min\left\{\frac{a_{3j}}{a_{1j} + a_{2j}} | a_{1j} \neq a_{2j}, a_{3j} \neq 0, j = 1, \dots, n\right\}.$$

We define two vectors $u, v \in \mathbb{R}_3$ by

$$u := \begin{cases} \left(0, \frac{2a_{3j_0}}{a_{1j_0}+a_{2j_0}}, -1\right), & \text{if } a_{1j_0} < a_{2j_0}, \\ \left(\frac{2a_{3j_0}}{a_{1j_0}+a_{2j_0}}, 0, -1\right), & \text{if } a_{1j_0} > a_{2j_0}, \end{cases}$$

and

$$v := \left(\frac{a_{3j_0}}{a_{1j_0}+a_{2j_0}}, \frac{a_{3j_0}}{a_{1j_0}+a_{2j_0}}, -1\right).$$

Since $u \sim_r v$, $Tu \sim_r Tv$. Also, we have the following statements.

1. $\sum_{a_{3j}=0} a_{1j} = \sum_{a_{3j}=0} a_{2j}$.
2. If $a_{1j} = a_{2j}$, then $(Tu)_j = (Tv)_j = \left(\frac{2a_{3j_0}}{a_{1j_0}+a_{2j_0}}\right) a_{1j} - a_{3j}$.
3. If $a_{1j} \neq a_{2j}$, $a_{3j} \neq 0$, then $(Tv)_j \leq 0$, because

$$\begin{aligned}(Tv)_j &= \left(\frac{a_{3j_0}}{a_{1j_0}+a_{2j_0}}\right)(a_{1j}+a_{2j}) - a_{3j}\\ &\leq \left(\frac{a_{3j}}{a_{1j}+a_{2j}}\right)(a_{1j}+a_{2j}) - a_{3j} = 0.\end{aligned}$$

4. If $a_{1j} \neq a_{2j}$, $a_{3j} = 0$, then

$$(Tu)_j = \left(\frac{2a_{3j_0}}{a_{1j_0}+a_{2j_0}}\right) \max\{a_{1j}, a_{2j}\} > (Tv)_j.$$

From the recent statements and $(Tu)_{j_0} > 0$, we deduce that $tr_+(Tu) > tr_+(Tv)$, this is a contradiction. So, there is a maximum of one element nonzero in each column of the matrix $[T]$.
Now we prove the theorem for every $m > 3$. If the jth column of A has over a non-zero element, choose the vector $(a_{rj}, a_{sj}, a_{tj})^t$, where $(r < s < t)$ such that it has at least two non-zero elements and $(a_{rj}, a_{sj}, a_{tj})^t \notin \text{span}(e)$.

Define $S : \mathbb{R}_3 \to \mathbb{R}_m$ by $S(x, y, z) = (w_1, w_2, \ldots, w_m)$ such that $w_r = x, w_s = y, w_t = z, w_i = 0, i \notin \{r, s, t\}$. Observe that S preserves $\prec_r$, and so ToS preserves $\prec_r$. It is a contradiction. Because as we proved for $m = 3$, in the jth column of

$$[ToS] = \begin{bmatrix} a_{r1} & a_{r2} & \ldots & a_{rn}\\ a_{s1} & a_{s2} & \ldots & a_{sn}\\ a_{t1} & a_{t2} & \ldots & a_{tn}\end{bmatrix},$$

at least two non-zero elements exist.
Hence, according to Lemmas 2.10 and 2.11,

$$[T] = [R|\mathbf{c^{(m)}}]P,$$

where in each column of matrix $R_{n\times k} = [r_{ij}]$ there is only one non-zero entry, each two rows of R are equivalent, $P \in \mathcal{P}(n)$ and $\mathbf{c}$ is a vector. Let $x = (x_1, ..., x_n)$ be an arbitrary vector in $\mathbb{R}_n$. So,

$$tr(xR) = \sum_{j=1}^{k}\sum_{i=1}^{n} x_i r_{ij} = \sum_{i=1}^{n} x_i \sum_{j=1}^{k} r_{ij} = tr(R_1)tr(x), \tag{17}$$

and

$$\|xR\| = \sum_{j=1}^{k}\left|\sum_{i=1}^{n} x_i r_{ij}\right| = \|R_1\|\|x\|, \tag{18}$$

where R_1 is the first row of R. Thus if $x, y \in \mathbb{R}_n$ with $x \prec_r y$, then $tr(xR) = tr(yR)$ and $\|xR\| \leq \|xR\|$. Hence, according to Proposition 2.3 and Lemmas 2.10 and 2.11 the proof is complete. □

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

References

[1] T. Ando, Majorization, doubly stochastic matrices, and comparision of eigenvalues, *Linear Algebra Appl.* **118** (1989) 163 – 248, https://doi.org/10.1016/0024-3795(89)90580-6.

[2] L. B. Beasley, S. G. Lee and Y. H. Lee, A characterization of strong preservers of matrix majorization, *Linear Algebra Appl.* **367** (2003) 341 – 346, https://doi.org/10.1016/S0024-3795(02)00657-2.

[3] H. Chiang and C. K. Li, Generalized doubly stochastic matrices and linear preservers, *Linear Multilinear Algebra* **53** (2005) 1 – 11, https://doi.org/10.1080/03081080410001681599.

[4] F. D. M. Pería, P. G. Massey and L. E. Silvestre, Weak matrix majorization, *Linear Algebra Appl.* **403** (2005) 343 – 368, https://doi.org/10.1016/j.laa.2005.02.003.

[5] A. Armandnejad and A. Manesh, GUT-majorization and its linear preservers, *Electron. J. Linear Algebra* **23** (2012) 646 – 654, https://doi.org/10.13001/1081-3810.1547.

[6] A. Armandnejad, Right gw-majorization on $\mathbf{M}_{n,m}$, *Bull. Iran. Math. Soc.* **35** (2) (2009) 69 – 76.

[7] A. Armandnejad and A. Salemi, On linear preservers of lgw-majorization on $\mathbf{M}_{n,m}$, *Bull. Malaysian Math. Soc.* **35** (3) (2012) 755 – 764.

[8] M. Dehghanian and A. Mohammadhasani, A note on multivariate majorization, *J. Mahani Math. Res.* **11** (2) (2022) 119 – 126, https://doi.org/10.22103/JMMRC.2022.19004.1204.

[9] A. M. Hasani and M. Radjabalipour, On linear preservers of (right) matrix majorization, *Linear Algebra Appl.* **423** (2007) 255 – 261, https://doi.org/10.1016/j.laa.2006.12.016.

[10] A. Mohammadhasani, Y. Sayyari and M. Sabzvari, G-tridiagonal majorization on $M_{n,m}$, *Commun. Math.* **29** (2021) 395 – 405.

[11] F. Khalooei, M. Radjabalipour and P. Torabian, Linear preservers of left matrix majorization, *Electron. J. Linear Algebra* **17** (2008) 304 – 315, https://doi.org/10.13001/1081-3810.1265.

[12] F. Khalooei and A. Salemi, The structure of linear preservers of left matrix majorization on $\mathbb{R}^p$, *Electron. J. Linear Algebra* **18** (2009) 88 – 97, https://doi.org/10.13001/1081-3810.1296.

[13] Y. Sayyari, A. Mohammadhasani and M. Dehghanian, Linear maps preserving signed permutation and substochastic matrices, *Indian J. Pure Appl. Math.* **54** (2023) 219 – 223, https://doi.org/10.1007/s13226-022-00245-6.

[14] G. Dahl, Matrix majorization, *Linear Algebra Appl.* **288** (1999) 53 – 73, https://doi.org/10.1016/S0024-3795(98)10175-1.

[15] A. M. Hasani and M. Radjabalipour, The structure of linear operators strongly preserving majorizations of matrices, *Electron. J. Linear Algebra* **15** (2006) 260 – 268, https://doi.org/10.13001/1081-3810.1236.

Ahmad Mohammadhasani
Department of Mathematics,
Sirjan University of Technology,
Sirjan, I. R. Iran
e-mail: a.mohammadhasani53@gmail.com

Mehdi Dehghanian
Department of Mathematics,
Sirjan University of Technology,
Sirjan, I. R. Iran
e-mail: mdehghanian@sirjantech.ac.ir

Yamin Sayyari
Department of Mathematics,
Sirjan University of Technology,
Sirjan, I. R. Iran
e-mail: y.sayyari@sirjantech.ac.ir