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Applications of Q-hypergeometric and Hurwitz-Lerch Zeta Functions on Meromorphic Functions

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Abstract

A new subclass of meromorphic univalent functions by using the q-hypergeometric and Hurwitz-Lerch Zeta functions is defined. Also, by applying the generalized Liu-Srivastava operator on meromorphic functions, some geometric properties of the new defined subclass such as coefficient estimates, extreme points, convexity and connected set structure are investigated.

Keywords: Meromorphic function, Convolution, $\lambda-$Generalized, Hurwitz-Lerch Zeta function, Q-hypergeometric function.

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1. Introduction

The meromorphic functions bear the same relation to the entire functions as the rational functions do to the polynomials. A meromorphic function is a univalent function that is analytic in all but possibly a discrete subset of its domain, and at those singularities, like a polynomia, it must limit at infinity. By considering q-hypergeometric function, q-analogue of Liu-Srivastava operator, λ-generalized Hurwitz-Lerch Zeta function, see [1–4], and convolution structure a new subclass of meromorphic univalent function is defined, some geometric properties related to

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coefficient bounds, extreme points, convex family and connected sets are obtained. Of course, we will use the techniques used in [5], to show the family of convexity, and its related properties.
Let Σ denote the meromorphic functions f of the form

$$f(z) = Dz^{-1} + \sum_{n=0}^{+\infty} a_n z^n, \qquad D > 0, \tag{1}$$

which are univalent and analytic in the punctured open unit disk.
Let $\mathbb{U}^* = \{z \in \mathbb{C} : \ 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$. For the functions f_j $(j = 1, 2)$ introduced by

$$f_j(z) = Dz^{-1} + \sum_{n=1}^{+\infty} a_{n,j} z^n,$$

the convolution (or Hadamard product), see [6], of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = Dz^{-1} + \sum_{n=1}^{+\infty} a_{n,1} a_{n,2} z^n = (f_2 * f_1)(z).$$

Definition 1.1. Let $\alpha \in \mathbb{C} \setminus \{0, -1, -2, ...\}$ and $k \in \mathbb{N} \cup \{0\}$. Then $(\alpha)_0 = 1$, $(\alpha)_k = \alpha(\alpha + 1) \ldots (\alpha + k - 1)$ series

$$F(a, b, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \qquad (|z| < 1),$$

which is absolutely convergent and analytic on the unit disk, we call hypergeometric function.

This function was first used by Euler to solve the differential equation

$$z(1 - z)\chi^{''}(z) + (c - (a + b + 1)z)\,\chi^{'}(z) - ab\chi(z) = 0.$$

Some examples of functions, which according to the hypergeometric function, expressed:

Example 1.2.

$$F(1, 2, 1; z) := \frac{1}{(1 - z)^2}, \qquad F(1, b, b; z) := \frac{1}{(1 - z)},$$

$$zF(1, 1, 2; -z) := \log(1 + z), \qquad F(-n, b, b; -z) := (1 + z)^n,$$

$$F(\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}; \sin^2 z) := \cos z.$$

Also, the q-hypergeometric function ${}_r\Upsilon_s$ is defined by

$$\begin{aligned} &{}_r\Upsilon_s\left(x_1,\ldots,x_r;y_1,\ldots,y_s;q,z\right)\\ =& \sum_{n=0}^{+\infty}\frac{(x_1,q)_n\ldots(x_r,q)_n}{(q,q)_n(y_1,q)_n\ldots(y_s,q)_n}z^n\left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{1+s-r}, \end{aligned} \tag{2}$$

where x_i, y_j are complex numbers $(y_j \neq 0, -1, ...)$, $q \neq 0$, $r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $r > s + 1$, $z \in \mathbb{U}$ and

$$(w,q)_n = \begin{cases} (1-w)(1-wq)\ldots(1-wq^{n-1}), & n \in \mathbb{N},\\ 1, & n = 0.\end{cases}$$

By using the gamma function, we get

$$(q^w;q)_n = \frac{\Gamma_q(w+n)(1-q)^n}{\Gamma_q(w)}, \qquad n > 0.$$

Also, by a simple calculation, we conclude that $\lim\limits_{q\to 1}\left((q^w,q)_n\,/\,(1-q)^n\right)$ is equal to

$$(w)_n = w(w+1)\ldots(w+n-1),$$

where $(w)_n$ is the well-known Pochhammer symbol, see [7]. Moreover, we have:

$${}_r\Upsilon_s\left(x_1,\ldots,x_r;y_1,\ldots,y_s;q,z\right) = \sum_{n=0}^{+\infty}\frac{(x_1)_n\ldots(x_r)_n}{(y_1)_n\ldots(y_s)_n}\frac{z^n}{n!}.$$

If $0 < |q| < 1$, $r = s + 1$, then

$${}_r\Upsilon_s\left(x_1,\ldots,x_r;y_1,\ldots,y_s;q,z\right) = \sum_{n=0}^{+\infty}\frac{(x_1,q)_n\ldots(x_r,q)_n}{(q,q)_n(y_1,q)_n\ldots(y_s,q)_n}z^n.$$

It is concluded by the basic hypergeometric function which is given in (2), above series is absolutely convergent in $\mathbb{U}$; see [1]. Aldweby and Darus [1], also Challab et al. [2, 3] investigated the q-analogue of Liv-Srivastava operator associated with ${}_r\Upsilon_s\left(x_1,\ldots,x_r;y_1,\ldots,y_s;q,z\right)$ for $f \in \Sigma$ which presented by:

$$\begin{aligned} Q(z) &= Dz^{-1}{}_r\Upsilon_s\left(x_1,\ldots,x_r;y_1,\ldots,y_s;q,z\right) * f(z)\\ &= Dz^{-1} + \sum_{n=1}^{+\infty}\frac{\prod\limits_{i=1}^{r}(x_i,q)_{n+1}}{(q,q)_{n+1}\prod_{i=1}^{s}(y_i,q)_{n+1}}a_n z^n, \quad z \in \mathbb{U}^*. \end{aligned} \tag{3}$$

In [8, 9], Ghanim introduced the function $G_{u,a}$ by

$$G_{u,a} := (a+1)^u\left(\Phi(z,u,a) - a^u + \frac{1}{z(a+1)^u}\right),$$

for which Hurwitz-Lerch Zeta function $\Phi(z,u,a)$ is defined by

$$\Phi(z,u,a) := \sum_{n=0}^{+\infty} \frac{z^n}{(n+a)^u},$$

where $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $u \in \mathbb{C}$, $z \in \mathbb{U}$ and $\mathrm{Re}(u) > 1$ when $|z| = 1$; see, e.g., [10–12]. After a direct calculation, we get

$$G_{u,a} = z^{-1} + \sum_{n=1}^{\infty} \left(\frac{a+1}{a+n}\right)^u z^n, \qquad z \in \mathbb{U}^*.$$

Srivastava [13] introduced the generalized Hurwitz-Lerch Zeta function as follows:

$$\Phi_{\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_m}^{\eta_1,\dots,\eta_p,\gamma_1,\dots,\gamma_m}(z,u,a;b,\lambda) :=$$

$$\frac{1}{\lambda\Gamma(u)} \sum_{n=0}^{+\infty} \frac{\prod_{j=1}^{p} (\lambda_j)_n \eta_j}{(a+n)^u \prod_{j=1}^{m} (\mu_j)_n \gamma_j} H_{0,2}^{2,0}\left[(a+n)b^{1/\lambda} \middle| \overline{(u,1), \left(0, \frac{1}{\lambda}\right)}\right] \frac{z^n}{n!}, \quad (4)$$

$$(\min\{\mathrm{Re}(a), \mathrm{Re}(u)\} > 0, \mathrm{Re}\{b\} > 0, \lambda > 0).$$

Keep in mind that

$$(\lambda_j \in \mathbb{C}\ (j=1,\dots,p)\ \text{and}\ \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^-(j=1,\dots,m),\ \eta_j > 0\ (j=1,\dots,p),$$

$$\gamma_j > 0\ (j=1,\dots,m),\ 1 + \sum_{j=1}^{m} \gamma_j - \sum_{j=1}^{p} \eta_j \geq 0.)$$

H-function which was on the right-hand side of (4) is the well-known Fox's H-function [14], denoted by $H_{p,m}^{x,y}(z)$, define as below:

$$H_{p,m}^{x,y}\left[z\middle|_{(b_1,B_1),\dots,(b_m,B_m)}^{(a_1,A_1),\dots,(a_p,A_p)}\right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{j=1}^{x} \Gamma(b_j + B_j t) \prod_{j=1}^{y} (1 - a_j - A_j t)}{\prod_{j=y+1}^{p} \Gamma(a_j + A_j t) \prod_{j=x+1}^{m} \Gamma(1 - b_j - B_j t)} z^{-t} dt,$$

where

$$z \in \mathbb{C} \setminus \{0\}, |\arg(z)| < \pi.$$

We note that x, y, m and p all are integers that $1 \leq x \leq m$, $0 \leq y \leq p$, $a_j \in \mathbb{C}$, $A_j > 0$ $(j = 1,\dots,p)$ also $b_j \in \mathbb{C}$, $B_j > 0$ $(j = 1,\dots,m)$.
$\mathcal{L}$-Cantour is of the Mellin-Barnes type which separates the poles of the gamma

functions $\{\Gamma(tB_j + b_j)\}_{j=1}^{x}$ from the poles of the gamma functions $\{\Gamma(-A_jt - a_j + 1)\}_{j=1}^{y}$. Also, we see (4) is convergent for $|z| < W$, where

$$W := \left(\prod_{j=1}^{p} \eta_j^{-\eta_j}\right)\left(\prod_{j=1}^{m} \gamma_j^{\gamma_j}\right).$$

Now, by applying the Hadamard product, H-function, $\lambda-$generalized Hurwitz-Lernch Zeta, q-hypergeometric function and q-analogue of Liv-Srivastava operator. We consider the following operator which introduced by Challab et. al [4]:

$$\mathcal{K}^{x_r} f(z) \equiv \mathcal{K}^{u,a,\lambda,x_r,y_s}_{(\lambda_p)(\mu_r),b} f(z) : \Sigma \to \Sigma,$$

which has been defined by

$$\mathcal{K}^{x_r} f(z) = G^{u,a,\lambda}_{(\lambda_p)(\mu_r),b}(z) * Q(z),$$

where the Hadamard product (or convolution) of the analytical functions has been denoted by $*$, and $Q(z)$ is given in (3), also

$$\begin{aligned} & G^{u,a,\lambda}_{(\lambda_p),(\mu_m),b}(z) \\ = \;& (a+1)^u \left[\Phi^{(1,\ldots,1;1,\ldots,1)}_{\lambda_1,\ldots,\lambda_p,\mu_1,\ldots,\mu_m}(z,u,a,b,\lambda) - \frac{a^{-u}}{\lambda\Gamma(u)} Y(a,b,u,\lambda) + \frac{(a+1)^{-u}}{z}\right] \\ = \;& \frac{D}{z} + \sum_{n=1}^{+\infty} \frac{\prod\limits_{j=1}^{p}(\lambda_j)_n}{\prod\limits_{j=1}^{m}(\mu_j)_n} \left(\frac{a+1}{a+n}\right)^u \frac{Y(a+n,b,u,\lambda)}{\lambda\Gamma(u)} \frac{z^n}{n!} \end{aligned}$$

gave the function $G^{u,a,\lambda}_{(\lambda_p),(\mu_m),b}(z)$, with

$$Y(a,b,u,\lambda) := H^{2,0}_{0,2}\left[ab^{1/\lambda}\middle|\overline{(u,1),\left(0,\frac{1}{\lambda}\right)}\right].$$

Now

$$\begin{aligned} & \mathcal{K}^{x_r} f(z) \qquad\qquad (5) \\ = \;& \frac{D}{z} + \sum_{n=1}^{+\infty} \frac{\prod\limits_{i=1}^{r}(x_i,q)_{n+1}\prod\limits_{j=1}^{p}(\lambda_j)_n}{(q,q)_{n+1}\prod\limits_{i=1}^{s}(y_i,q)_{n+1}\prod\limits_{j=1}^{m}(\mu_j)_n} \left(\frac{a+1}{a+n}\right)^u \frac{Y(a+n,b,u,\lambda)}{\lambda\Gamma(u)} a_n \frac{z^n}{n!}. \end{aligned}$$

It is convenient to write $\mathcal{K}^{x_r} f(z) = \mathcal{J}(z)$, see [14]. For $0 \le R \le 1$, $0 < S, T \le 1$, $k = 2l$; a function f of the form (1) belongs to $\mathcal{M}^{\mathcal{K}}(R,S,T)$ if it satisfies the inequality

$$\left|\frac{z^{k+2}\mathcal{J}^{(k+1)}(z)+D(k+1)!}{Rz^{k+2}\mathcal{J}^{(k+1)}(z)-D(k+1)!+D(1+R)S[(K+1)!]}\right|<T, \tag{6}$$

where $\mathcal{J}^{(j)}$ is the j-th derivative of $\mathcal{J}(z)=\mathcal{K}^{x_r}f(z)$.

2. Main results

In this section first we state coefficient estimates on the class $\mathcal{M}^k(R,S,T)$. For a given $x_0\in\mathbb{R}$ such that $0<x_0<1$, we define two subclasses of Σ and find some geometric properties of these subclasses. Also integral representation of $\mathcal{J}^{(k)}_{(z)}$ is obtained.

Theorem 2.1. *Let $f\in\Sigma$, then $f\in\mathcal{M}^k(R,S,T)$ if and only if*

$$\sum_{n=1}^{+\infty}\frac{(1+RT)\prod_{i=1}^{r}(a_i,q)_{n+1}\prod_{j=1}^{p}(\lambda_j)_n}{(n-k-1)!(q,q)_{n+1}\prod_{i=1}^{s}(\beta_i,q)_{n+1}\prod_{j=1}^{m}(\mu_j)_n}\left(\frac{a+1}{a+n}\right)^{v}\frac{Y(a+n,b,v,\lambda)}{\lambda\Gamma(v)}a_n\leq TD(R+1)(1-S)[(k+1)!]. \tag{7}$$

The result is sharp for $F(z)$ given by

$$F(z)=\frac{D}{z}+\frac{(n-k-1)!(q,q)_{1+n}\prod_{i=1}^{s}(\beta_i,q)_{1+n}\prod_{j=1}^{m}(\mu_j)_n(n+a)^v\lambda\Gamma(v)TD(R+1)(1-S)[(k+1)!]}{(1+RT)\prod_{i=1}^{r}(\alpha_i,q)_{1+n}\prod_{j=1}^{p}(\lambda_j)_n(a+1)^vY(a+n,b,v,\lambda)}z^{n+1}.$$

Proof. Let $f\in\mathcal{M}^k(R,S,T)$, then (5) holds true. So by replacing

$$\mathcal{J}^{(k+1)}(z)=(-1)^{k+1}D\frac{(k+1)!}{z^{k+2}}+\sum_{n=1}^{+\infty}\frac{\prod_{i=1}^{r}(\alpha_i,q)_{1+n}\prod_{j=1}^{p}(\lambda_j)_n}{(n-1-k)!(q,q)_{n+1}\prod_{i=1}^{s}(\beta_i,q)_{n+1}\prod_{j=1}^{m}(\mu_j)_n}\left(\frac{a+1}{a+n}\right)^{v}\frac{Y(a+n,b,v,\lambda)}{\lambda\Gamma(v)}a_nz^{n-(k+1)},$$

in (5), we have

$$\left|\frac{\sum\limits_{n=1}^{+\infty}\frac{\prod\limits_{i=1}^{r}(\alpha_i,q)_{1+n}\prod\limits_{j=1}^{p}(\lambda_j)_n\left(\frac{a+1}{a+n}\right)^v}{(n-1-k)!(q,q)_{1+n}\prod\limits_{i=1}^{s}(\beta_i,q)_{1+n}\prod\limits_{j=1}^{m}(\mu_j)_n}\frac{Y(a+n,b,v,\lambda)}{\lambda\Gamma(v)}a_nz^{n+1}}{-(R+1)(1-S)D[(k+1)!]+\sum\limits_{n=1}^{+\infty}\frac{R\prod\limits_{i=1}^{r}(\alpha_i,q)_{1+n}\prod\limits_{j=1}^{p}(\lambda_j)_n\left(\frac{1+a}{n+a}\right)^v}{(n-1-k)!(q,q)_{1+n}\prod\limits_{i=1}^{s}(\beta_i,q)_{1+n}\prod\limits_{j=1}^{m}(\mu_j)_n}\frac{Y(a+n,b,v,\lambda)}{\lambda\Gamma(v)}a_nz^{n+1}}\right|<T$$

It is known that $\mathrm{Re}(z)\le|z|$ for all z. So

$$\mathrm{Re}\left\{\frac{\sum\limits_{n=1}^{+\infty}\frac{\prod\limits_{i=1}^{r}(\alpha_i,q)_{1+n}\prod\limits_{j=1}^{p}(\lambda_j)_n}{(n-1-k)!(q,q)_{1+n}\prod\limits_{i=1}^{s}(\beta_i,q)_{1+n}\prod\limits_{j=1}^{m}(\mu_j)_n}\left(\frac{a+1}{a+n}\right)^v\frac{Y(a+n,b,v,\lambda)}{\lambda\Gamma(v)}a_nz^{n+1}}{(R+1)(1-S)[(k+1)!]-\sum\limits_{n=1}^{+\infty}\frac{R\prod\limits_{i=1}^{r}(\alpha_i,q)_{n+1}\prod\limits_{j=1}^{p}(\lambda_j)_n}{(n-1-k)!(q,q)_{1+n}\prod\limits_{i=1}^{s}(\beta_i,q)_{1+n}\prod\limits_{j=1}^{m}(\mu_j)_n}\left(\frac{1+a}{n+a}\right)^v\frac{Y(n+a,b,v,\lambda)}{\lambda\Gamma(v)}a_nz^{n+1}}\right\}<T$$

By letting $z\to1^-$ through real values, we have

$$\sum_{n=1}^{+\infty}\frac{(1+RT)\prod\limits_{i=1}^{r}(\alpha_i,q)_{n+1}\prod\limits_{j=1}^{p}(\lambda_j)_n}{(n-k-1)!(q,q)_{n+1}\prod\limits_{i=1}^{s}(\beta_i,q)_{n+1}\prod\limits_{j=1}^{m}(\mu_j)_n}\left(\frac{a+1}{a+n}\right)^v\frac{Y(a+n,b,v,\lambda)}{\lambda\Gamma(v)}a_n$$

$\le TD(R+1)(1-S)[(k+1)!]$.

Conversely, let (7) hold, it is enough to show that

$$\begin{aligned}V &= \left|z^{k+2}\mathcal{J}^{(k+1)}(z)-D[(k+1)!]\right|\\ &- T\left|Rz^{k+2}\mathcal{J}^{(k+1)}(z)-D[(k+1)!]+(R+1)SD[(k+1)!]\right|<0.\end{aligned}$$

But, for $0<|z|=r$ and $k=2l(l\in\mathbb{N})$, we see that

$$V = \left| \sum_{n=1}^{+\infty} \frac{\prod_{i=1}^{r}(\alpha_i,q)_{1+n} \prod_{j=1}^{p}(\lambda_j)_n}{(n-1-k)!(q,q)_{1+n} \prod_{i=1}^{s}(\beta_i,q)_{1+n} \prod_{j=1}^{m}(\mu_j)_n} \left(\frac{a+1}{a+n}\right)^v \frac{Y(a+n,b,v,\lambda)}{\lambda\Gamma(v)} a_n z^{n+1} \right|$$

$$- T \left| D(R+1)(1-S)[(k+1)!] - \sum_{n=1}^{+\infty} \frac{\prod_{i=1}^{r}(\alpha_i,q)_{1+n} \prod_{j=1}^{p}(\lambda_j)_n R}{(n-1-k)!(q,q)_{1+n} \prod_{i=1}^{s}(\beta_i,q)_{1+n} \prod_{j=1}^{m}(\mu_j)_n} \left(\frac{a+1}{a+n}\right)^v \frac{Y(a+n,b,v,\lambda)}{\lambda\Gamma(v)} a_n z^{n+1} \right|$$

$$\leq \sum_{n=1}^{+\infty} \frac{\prod_{i=1}^{r}(\alpha_i,q)_{n+1} \prod_{j=1}^{p}(\lambda_j)_n}{(n-k-1)!(q,q)_{n+1} \prod_{i=1}^{s}(\beta_i,q)_{n+1} \prod_{j=1}^{m}(\mu_j)_n} \left(\frac{a+1}{a+n}\right)^v \frac{Y(a+n,b,v,\lambda)}{\lambda\Gamma(v)} |a_n| r^{n+1}$$
$$- TD(R+1)(1-S)[(k+1)!]$$
$$\sum_{n=1}^{+\infty} \frac{RT\prod_{i=1}^{r}(\alpha_i,q)_{n+1} \prod_{j=1}^{p}(\lambda_j)_n}{(n-k-1)!(q,q)_{n+1} \prod_{i=1}^{s}(\beta_i,q)_{n+1} \prod_{j=1}^{m}(\mu_j)_n} \left(\frac{a+1}{a+n}\right)^v \frac{Y(a+n,b,v,\lambda)}{\lambda\Gamma(v)} |a_n| r^{n+1}$$

$$\leq \sum_{n=1}^{+\infty} \frac{(1+RT)\prod_{i=1}^{r}(\alpha_i,q)_{n+1} \prod_{j=1}^{p}(\lambda_j)_n}{(n-k-1)!(q,q)_{n+1} \prod_{i=1}^{s}(\beta_i,q)_{n+1} \prod_{j=1}^{m}(\mu_j)_n} \left(\frac{a+1}{a+n}\right)^n \frac{Y(a+n,b,v,\lambda)}{\lambda\Gamma(v)} |a_n| r^{n+1}$$
$$- TD(R+1)(1-S)[(k+1)!].$$

By letting $r \to 1^-$, since the last inequality holds for all $r(0 < r < 1)$, we conclule that $V \leq 0$, so the proof is complete. □

Theorem 2.2. *Let $f \in \mathcal{M}^k(R,S,T)$. Then*

$$J^{(k)}(z) = \int_0^z \frac{[(k+1)!]\,(T\Psi(u)(S+RS-1)-1)}{(1-T\Psi(u)R)\,u^{k+2}} du,$$

and

$$J^{(k)}(z) = z^{-k-2} \int_X \frac{[(k+1)!(Tx(S+RS-1)-1)}{(1-TxR)} d\mu(x),$$

where $\mu(x)$ is the probability measure on $X = \{x : \ |x| = 1\}$.

Proof. By (6), we have

$$\frac{z^{k+2}\mathcal{J}^{(k+1)}(z)+(k+1)!}{Rz^{k+2}\mathcal{J}^{(k+1)}(z)-(k+1)!+(1+R)S[(k+1)!]}=T\Psi(z),$$

where $|\Psi(z)|<1; z\in\mathbb{U}^*$. Then

$$z^{k+2}J^{(k+1)}(z)+(k+1)!-T\Psi(z)Rz^{k+2}J^{(k+1)}(z)+T\Psi(z)[(k+1)!](1-S-RS)=0,$$

or

$$z^{k+2}J^{(k+1)}(z)\left(1-T\Psi(z)R\right)=[(k+1)!]\left(T\Psi(z)(S+RS-1)-1\right).$$

After integration, we conclude that

$$J^{(k)}(z)=\int_0^z\frac{[(k+1)!](T\Psi(u)(S+RS-1)-1)}{(1-T\Psi(u)R)u^{k+2}}du.$$

For the second representation, we put $X=\{x:\ |x|=1\}$, and so

$$\frac{z^{k+2}J^{(k+1)}(z)+(k+1)!}{Rz^{k+2}J^{(k+1)}(z)-(k+1)!+(1+R)S[(k+1)!]}=Tx,$$

and after a simple calculation, we obtain the desired result. □

Two subclasses of Σ are defined and the above geometric properties of these classes are examined. For a given $x_0\in(0,1)$, let Σ_1 be a subclass of Σ provided that, $x_0f(x_0)=1$ and Σ_2 be a subclass of Σ provided that, $-x_0^2f'(x_0)=1$, also:

$$\mathcal{M}_\theta^k(R,S,T,x_0)=\mathcal{M}^k(R,S,T)\cap\Sigma_\theta,\qquad(\theta=1,2).$$

Theorem 2.3. *Let f be defined by (1). Then $f\in\mathcal{M}_1^k(R,S,T,x_0)$ if and only if*

$$\sum_{n=1}^{+\infty}\Bigg(\frac{(1+RT)\prod_{i=1}^{r}(\alpha_i,q)_{n+1}\prod_{j=1}^{p}(\lambda_j)}{(n-k-1)!(q,q)_{n+1}\prod_{i=1}^{s}(\beta_i,q)_{n+1}\prod_{j=1}^{m}(\mu_j)_n}\left(\frac{a+1}{a+n}\right)^v$$
$$\times\quad\frac{Y(n+a,b,,v,\lambda)}{\lambda\Gamma(v)T(R+1)(1-S)[(k+1)!]}+x_0^{n+1}\Bigg)a_n\le 1. \tag{8}$$

Proof. Since $f\in\mathcal{M}_1^k(R,S,T,x_0)$, we have

$$x_0f(x_0)=+\sum_{n=0}^{+\infty}a_nx_0^{n+1}=1.$$

Thus

$$D=1-\sum_{n=1}^{+\infty}a_nx_0^{n+1}.$$

By replacing this value of D in Theorem 2.1, we obtain

$$\sum_{n=1}^{+\infty} \frac{(1+RT)\prod_{i=1}^{r}(\alpha_i,q)_{1+n}\prod_{j=1}^{p}(\lambda_j)_n}{(n-k-1)!(q,q)_{1+n}\prod_{i=1}^{s}(\beta_i,q)_{1+n}\prod_{j=1}^{m}(\mu_j)_n}\left(\frac{a+1}{a+n}\right)^v \frac{Y(a+n,b,,v,\lambda)}{\lambda\Gamma(v)}a_n$$
$$\leq T(R+1)(1-S)[(k+1)!]\left(1-\sum_{n=1}^{+\infty}a_n x_0^{n+1}\right),$$

or

$$\sum_{n=1}^{+\infty}\Bigg(\frac{(1+RT)\prod_{i=1}^{r}(\alpha_i,q)_{n+1}\prod_{j=1}^{p}(\lambda_j)_n}{(n-k-1)!(q,q)_{n+1}\prod_{i=1}^{s}(\beta_i,q)_{n+1}\prod_{j=1}^{m}(\mu_j)_n}\left(\frac{a+1}{a+n}\right)^v \times$$
$$\frac{Y(a+n,b,,v,\lambda)}{\lambda\Gamma(v)T(R+1)(1-S)[(k+1)!]}+x_0^{n+1}\Bigg)a_n \leq 1,$$

and we get the desired assertion. □

Theorem 2.4. *Suppose that f is defined as (1). So $f \in \mathcal{M}_2^k(R,S,T,x_0)$ if and only if*

$$\sum_{n=1}^{+\infty}\Bigg(\frac{(1+RT)\prod_{i=1}^{r}(\alpha_i,q)_{1+n}\prod_{j=1}^{p}(\lambda_j)_n}{(n-k-1)!(q,q)_{1+n}\prod_{i=1}^{s}(\beta_i,q)_{1+n}\prod_{j=1}^{m}(\mu_j)_n}\left(\frac{a+1}{a+n}\right)^v \frac{Y(a+n,b,,v,\lambda)}{\lambda\Gamma(v)T(R+1)(1-S)[(k+1)!]} \tag{9}$$
$$-nx_0^{n+1}\Bigg)a_n \leq 1$$

Proof. Since $-x_0^2 f'(x_0)=1$, we have

$$D = 1+\sum_{n=1}^{+\infty} n a_n x_0^{n+1}.$$

Substituting D in (7), we obtain (9). □

Corollary 2.5. *(i) If f be in the form (1)and in the class $\mathcal{M}_1^k(R,S,T,x_0)$, then*

$$a_n \leq \frac{\Phi}{W+\Phi x_0^{n+1}}, \tag{10}$$

where

$$\Phi = \left(\prod_{i=1}^{s}(\beta_i,q)_{n+1}\prod_{j=1}^{m}(\mu_j)_n(a+n)^v\lambda\Gamma(v)T(R+1)(1-S)[(k+1)!]\right) \times (n-k-1)!(q,q)_{n+1} \tag{11}$$

$$W = (1+RT)\prod_{i=1}^{r}(\alpha_i,q)_{1+n}\prod_{j=1}^{p}(\lambda_j)_n(1+a)^vY(n+a,b,v,\lambda), \tag{12}$$

(ii) If $f(z)$ be in the form (1)and in the class $\mathcal{M}_2^k(R,S,T,x_0)$, then

$$a_n \le \frac{\Phi}{W - n\Phi x_0^{n+1}},$$

where Φ and W are given in (2.5) *and* (2.6) *respectively.*

3. Convex and connected sets

In the last section we will show that $\mathcal{M}_\theta^k(R,S,T,x_0)$ for $\theta = 1,2$ are convex sets. Also, connected set conditions are investigated. The same properties were studied recently in [5].

Theorem 3.1. *Let f_j defined by*

$$f_j(z) = \frac{D_j}{z} + \sum_{n=1}^{+\infty} a_{n,j}z^n,$$

be in the class $\mathcal{M}_1^k(R,S,T,x_0)$. Then the function $F(z) = \sum\limits_{j=0}^{m} d_jf_j(z)$ $(d_j \ge 0)$ is also in the same class where $\sum\limits_{j=0}^{m} d_j = 1$.

Proof. By definition of $F(z)$, we have

$$F(z) = \sum_{j=0}^{m} d_j\left(\frac{D_j}{z} + \sum_{n=1}^{+\infty} a_{n,j}z^n\right) = \left(\sum_{j=0}^{m} d_jD_j\right)z^{-1} + \sum_{n=1}^{+\infty}\left(\sum_{j=0}^{m} d_ja_{n,j}\right)z^n.$$

Since $f_j \in \mathcal{M}_1^k(R,S,T,x_0)$ for $j = 0,1,\ldots,m$, by using (8), we have

$$\sum_{n=1}^{+\infty}\Bigg(\frac{(1+RT)\prod\limits_{i=1}^{r}(\alpha_i,q)_{1+n}\prod\limits_{j=1}^{p}(\lambda_j)}{(n-k-1)!(q,q)_{1+n}\prod\limits_{i=1}^{s}(\beta_i,q)_{1+n}\prod\limits_{j=1}^{m}(\mu_j)_n}\left(\frac{1+a}{n+a}\right)^u\frac{Y(n+a,b,,u,\lambda)}{\lambda\Gamma(u)T(R+1)(1-S)[(k+1)!]}$$
$$+\quad x_0^{1+n}\Bigg)a_{n,j} \le 1.$$

But

$$\begin{aligned}
&\sum_{n=1}^{+\infty}\Big(\frac{(1+RT)\prod_{i=1}^{r}(\alpha_i,q)_{1+n}\prod_{j=1}^{p}(\lambda_j)}{(n-k-1)!(q,q)_{1+n}\prod_{i=1}^{s}(\beta_i,q)_{1+n}\prod_{j=1}^{m}(\mu_j)_n}\left(\frac{1+a}{n+a}\right)^u\\
\times\ &\frac{Y(n+a,b,,u,\lambda)}{\lambda\Gamma(u)T(R+1)(1-S)[(k+1)!]}+x_0^{1+n}\Big)\sum_{j=0}^{m}d_ja_{n,j}\\
=\ &\sum_{j=0}^{m}d_j\Big(\sum_{n=1}^{+\infty}\Big(\frac{(1+RT)\prod_{i=1}^{r}(\alpha_i,q)_{1+n}\prod_{j=1}^{p}(\lambda_j)}{(n-k-1)!(q,q)_{1+n}\prod_{i=1}^{s}(\beta_i,q)_{1+n}\prod_{j=1}^{m}(\mu_j)_n}\left(\frac{1+a}{n+a}\right)^u\\
&\frac{Y(n+a,b,,u,\lambda)}{\lambda\Gamma(u)T(R+1)(1-S)[(k+1)!]}+x_0^{1+n}\Big)a_{n,j}\Big)\\
\leq\ &\sum_{j=0}^{m}d_j=1.
\end{aligned}$$

Now, the proof is complete. □

Remark 1. By using the same techniques, we can prove the same result for $\mathcal{M}_2^k(R,S,T,x_0)$.

Corollary 3.2. *The classes $\mathcal{M}_\theta^k(R,S,T,x_0)$ for $\theta=1,2$ are convex sets.*

Theorem 3.3. *Let*

$$f_0(z)=z^{-1}, \tag{13}$$

and

$$f_n(z)=\frac{W+\Phi z^{n+1}}{z\left(W+\Phi x_0^{n+1}\right)},\qquad n\geq 1, \tag{14}$$

where Φ and W are given in (11) and (12), respectively. Then $f\in\mathcal{M}_1^k(R,S,T,x_0)$ if and only if it can be expressed by $f(z)=\sum_{n=0}^{+\infty}\xi_nf_n(z)$, where $\xi_n\geq 0$ and $\sum_{n=1}^{+\infty}\xi_n=1$.

Proof. Let

$$\begin{aligned} f(z) &= \sum_{n=0}^{+\infty} \xi_n f_n(z) = \xi_0 f_0(z) + \sum_{n=1}^{+\infty} \xi_n f_n(z) \\ &= \xi_0 f_0(z) + \sum_{n=1}^{+\infty} \xi_n \left[\frac{W + \Phi z^{n+1}}{z\left(W + \Phi x_0^{n+1}\right)} \right] \\ &= z^{-1} \left[\xi_0 + \sum_{n=1}^{+\infty} \xi_n \frac{W + \Phi z^{n+1}}{z\left(W + \Phi x_0^{n+1}\right)} \right] \\ &= z^{-1} \left[\xi_0 + \sum_{n=1}^{+\infty} \xi_n \frac{W}{W + \Phi x_0^{n+1}} \right] + \sum_{n=1}^{+\infty} \xi_n \frac{\Phi z^n}{W + \Phi x_0^{n+1}}. \end{aligned}$$

In view of Theorem 2.3 and by

$$\sum_{n=1}^{+\infty} \left(\frac{W}{\Phi} + x_0^{n+1} \right) \left(\xi_n \frac{\Phi}{W + \Phi x_0^{n+1}} \right) = \sum_{n=1}^{+\infty} \xi_n = 1 - \xi_0 \leq 1,$$

$$x_0 f(x_0) = \sum_{n=0}^{+\infty} \xi_n x_0 f_n(x_0) = \sum_{n=0}^{+\infty} \xi_n = 1,$$

we conclude that

$$f(z) = \mathcal{M}_1^k(R, S, T, x_0).$$

Conversely, suppose that $f(z) \in \mathcal{M}_1^k(R, S, T, x_0)$. Then inequality (2.4) is established. By setting

$$\xi_n = \frac{W + \Phi x_0^{n+1}}{\Phi} a_n, \qquad n \geq 1$$

and $\xi_0 = 1 - \sum_{n=1}^{+\infty} \xi_n$, we obtain the required result, so the proof is complete. □

Remark 2. With a similar way, we can prove the same theorem for $\mathcal{M}_2^k(R, S, T, x_0)$.

Remark 3. We note that the functions given by (13) and (14) are the extreme points of $\mathcal{M}_1^k(R, S, T, x_0)$. Now, we investigate convex family related to connected sets. Let I be a nonempty subset of $[0, 1]$. We define

$$\mathcal{M}_\theta^k(R, S, T, I) = \bigcup_{x_t \in I} \mathcal{M}_\theta^k(R, S, T, x_t), \qquad \theta = 1, 2. \tag{15}$$

According to Theorem 3.1 and its corollary, $\mathcal{M}_1^k(R, S, T, I)$ is convex family if I is a single points.

Theorem 3.4. *If $f \in \mathcal{M}_1^k(R, S, T, x_0) \cap \mathcal{M}_1^k(R, S, T, x_1)$, where x_0, x_1 are positive numbers and $x_0 \neq x_1$, then $f(z) = z^{-1}$.*

Proof. Suppose $f \in \mathcal{M}_1^k(R,S,T,x_0) \cap \mathcal{M}_1^k(R,S,T,x_1)$ also $f(z) = Dz^{-1} + \sum_{n=1}^{+\infty} a_n z^n$. Then

$$D = 1 - \sum_{n=1}^{+\infty} a_n x_0^{n+1} = 1 - \sum_{n=1}^{+\infty} a_n x_1^{n+1},$$

or

$$\sum_{n=1}^{+\infty} a_n \left(x_0^{n+1} - x_1^{n+1} \right) = 0.$$

Since $a_n \geq 0$, $x_0 > 0$ and $x_1 > 0$, so a_n is always zero. This concludes that $f(z) = z^{-1}$. □

Theorem 3.5. *Assume that I belong to $[0,1]$. So $\mathcal{M}_1^k(R,S,T,I)$ is a convex family if and only if I is connected.*

Proof. Let I be connected and let $x_0, x_1 \in I$ with $x_0 \leq x_1$. It suffices to show that for

$$f(z) = Dz^{-1} + \sum_{n=1}^{+\infty} a_n z^n \in \mathcal{M}_1^k(R,S,T,x_0),$$
$$g(z) = Ez^{-1} + \sum_{n=1}^{+\infty} b_n z^n \in \mathcal{M}_1^k(R,S,T,x_1), \tag{16}$$

and $0 \leq \nu \leq 1$, there exists $x_2 (x_0 \leq x_2 \leq x_1)$ such that

$$h(z) = \nu f(z) + (1-\nu) g(z) \in \mathcal{M}_1^k(R,S,T,x_2).$$

By (16), we obtain

$$E = 1 - \sum_{n=1}^{+\infty} b_n x_1^{n+1}, \qquad D = 1 - \sum_{n=1}^{+\infty} a_n x_0^{n+1}.$$

Therefore, we get

$$\begin{aligned} H(z) &= zh(z) = z\left(\nu f(z) + (1-\nu)g(z)\right) \\ &= z\left(\nu D z^{-1} + \sum_{n=1}^{+\infty} \nu a_n z^n + (1-\nu)Ez^{-1} + \sum_{n=1}^{+\infty}(1-\nu)b_n z^n\right) \\ &= \nu D + \sum_{n=1}^{+\infty} \nu a_n z^{n+1} + (1-\nu)E + \sum_{n=1}^{+\infty}(1-\nu)b_n z^{n+1} \\ &= \nu\left(1 - \sum_{n=1}^{+\infty} a_n x_0^{n+1}\right) + (1-\nu)\left(1 - \sum_{n=1}^{+\infty} b_n x_1^{n+1}\right) \\ &+ \sum_{n=1}^{+\infty} \nu a_n z^{n+1} + \sum_{n=1}^{+\infty}(1-\nu)b_n z^{n+1} \\ &= 1 + \nu\sum_{n=1}^{+\infty}\left(z^{n+1} - x_0^{n+1}\right)a_n + (1-\nu)\sum_{n=1}^{+\infty}\left(z^{n+1} - x_1^{n+1}\right)b_n. \quad (17) \end{aligned}$$

Even though $H(x_0) \le 1$, $H(x_1) \ge 1$. Then x_2 exists so that $H(x_2) = 1$, where $x_2 \in [x_0, x_1]$. Hence

$$x_2 h(x_2) = 1. \quad (18)$$

As a result $h \in \Sigma_{(1)}$. Also from (17), (18) and (8), we infer that

$$\begin{aligned} &\sum_{n=1}^{+\infty}\left(\frac{W}{\Phi} + x_2^{n+1}\right)(\nu a_n + (1-\nu)b_n) \\ &= \nu\sum_{n=1}^{+\infty}\left(\frac{W}{\Phi} + x_2^{n+1}\right)a_n + (1-\nu)\sum_{n=1}^{+\infty}\left(\frac{W}{\Phi} + x_2^{n+1}\right)b_n \\ &\le \nu\sum_{n=1}^{+\infty}\left(\frac{W}{\Phi} + x_0^{n+1}\right)a_n + (1-\nu)\sum_{n=1}^{+\infty}\left(\frac{W}{\Phi} + x_1^{n+1}\right)b_n \\ &\le \nu + 1 - \nu = 1. \end{aligned}$$

Hence $h \in \mathcal{M}_1^k(R, S, T, x_2)$. Since x_0, x_1 and x_2 are arbitrary, the family $\mathcal{M}_1^k(R, S, T, I)$ is convex. Conversely, if I is not connected, then there exists x_0, x_1 and x_2 such that $x_0 < x_2 < x_1$ and $x_0, x_1 \in I$, but $x_2 \notin I$.

Let $f \in \mathcal{M}_1^k(R, S, T, x_0)$ and $g \in \mathcal{M}_1^k(R, S, T, x_1)$, $f(z)$ and $g(z)$ are not both equal to z^{-1}. Then for fixed x_2 and $0 \le \nu \le 1$, by (16), we obtain

$$H(\nu) = H(x_2, \nu) = 1 + \nu\sum_{n=1}^{+\infty}\left(z^{n+1} - x_0^{n+1}\right)a_n + (1-\nu)\sum_{n=1}^{+\infty}\left(z^{n+1} - x_1^{n+1}\right)b_n.$$

Since $H(x_2, 0) < 1$ and $H(x_2, 1) > 1$, there exists $\nu_0 (0 < \nu_0 < 1)$ such that $H(x_2, \nu_0) = 1$ or $x_2 h(x_2) = 1$, where $h(z) = \nu_0 f(z) + (1-\nu_0)g(z)$. Thus $h(z) \in$

$\mathcal{M}_1^k(R, S, T, x_2)$. From Theorem 3.4, we have $h(z) \notin \mathcal{M}_1^k(R, S, T, I)$. Since $x_2 \in I$ and $h(z) \neq z^{-1}$, this implies that the family $\mathcal{M}_1^k(R, S, T, I)$ is not convex which is a contradiction and the proof is complete. □

Remark 4. We note that by the same techniques where used in Theorem 3.4 and Theorem 3.5, we can prove the same results for the class $\mathcal{M}_2^k(R, S, T, I)$, and so the details were omitted.

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