Mathematics Interdisciplinary Research **8** (4) (2023) 359 – 367

Technical Paper

Characteristic Functions Assignment by Adding Perturbation

Aref Shahmansoorian⋆ *and Peyman Ahmadi*

Abstract

This paper presents a method for characteristic function assignment on rational functions by adding perturbation in systems with irrational characteristic functions. Generally, in systems with irrational characteristic loci, commutative compensator designs are not possible. Irrational characteristic functions have different forms. In our previous work, mentioned in the introduction section, a form of these characteristic functions was presented. Another form of irrational characteristic functions is considered in this paper. This approach is not based on the transfer function inverting, and characteristic loci are not used directly in the design process. The efficacy of the proposed approach is investigated through two numerical examples.

Keywords: Characteristic function assignment, Commutative compensator, Irrational characteristic function.

2020 Mathematics Subject Classification: 93C05, 93C35, 93D15.

How to cite this article
A. Shahmansoorian and P. Ahmadi, Characteristic functions assignment by adding perturbation, *Math. Interdisc. Res.* **8** (4) (2023) 359-367.

1. Introduction

A characteristic function method is an effective tool for analyzing closed-loop multivariate systems. This method is an extension of the Nyquist stability method and is a successful method for analyzing the stability of multivariate systems [1–3].

⋆Corresponding author (E-mail: shahmansoorian@eng.ikiu.ac.ir)
Academic Editor: Abbas Saadatmandi
Received 7 March 2023, Accepted 8 August 2023
DOI: 10.22052/MIR.2023.252619.1395

Commutative compensators are proposed to assign characteristic functions [2, 3]. The main drawback of this method is that the characteristic functions are not usually rational, leading to an unrealizable controller. The unrealizable transfer function can be approximated in a particular frequency range to have a rational compensator [4]. It is noted that the applied approximation is valid only in the same limited frequency range. In Reference [5] a method to find a pre-compensator is presented by an optimization problem which is solved by a symmetric matrix spectral decomposition or a singular value decomposition of a real matrix, whether normalization is obtained at one or more frequencies. In Reference [6] is stated that the first step in designing a controller for multivariable systems with a characteristic locus method is to design a static normalizing pre-compensator so that the pre-compensated system is as close as possible to a normal matrix in the vicinity of the crossover frequency. The eigenvalues' sensitivity to system perturbations is one of the main drawbacks of the characteristic locus approach [7, 8]. An approximation of eigenvectors with bi-causal power series is presents in Reference [9]. In [10], a normalizing of the pre-compensator for commutative compensator design is suggested. A method of characterization for a group of commutative controllers is investigated in [11], and a robust controller by characteristic locus method is designed for inter-connected large-scale systems [12]. In [13], minimizing perturbations between system inputs and outputs at each frequency is investigated to provide robust ability against perturbations. A method to prepare transfer functions with irrational characteristic functions to design a compensator is proposed in Reference [14]. In this approach, to have a transfer function with rational characteristic functions, a perturbation is applied to it. Also, it is demonstrated that the multivariable system stabilizing can be reduced to a diagonal system stabilising.

In Reference [14], the method of characteristic function assignment is presented only for systems with a specific form of characteristic functions. An extension to Reference [14] is provided in this paper and proposes a method for characteristic function assignment in a new class of multivariable systems. In many similar methods, the inverse of the transfer function is used in the assignment process. This article method is not inverse-based. Also, the characteristic loci are not used directly in the design process.

The remainder of this paper is organized as follows. The characteristic functions assignment is presented in Section 2. Two illustrative numerical examples are provided in Section 3. Finally, conclusions are given in Section 4.

2. Characteristic functions assignment

Consider a 3×3 transfer function matrix in the following form

$$G(s) = \frac{1}{d(s)} G_1(s),$$

where $d(s)$ is the monic common denominator of all $G(s)$ entries and $G_1(s)$ is a polynomial matrix. Suppose the characteristic functions of $G_1(s)$ are as follows

$$\begin{cases} \lambda_1 & = p_1 + p_2 \sqrt[3]{p_3}, \\ \lambda_2 & = p_1 + p_2 \sqrt[3]{p_3} e^{j\frac{2\pi}{3}}, \\ \lambda_3 & = p_1 + p_2 \sqrt[3]{p_3} e^{-j\frac{2\pi}{3}}, \end{cases}$$

which p_1, p_2, and p_3 are polynomial functions of s, and p_3 does not have third-degree factors (or higher degree factors). The corresponding characteristic vectors have the following structure

$$V(s) = V_1(s) + V_2(s)\sqrt[3]{p_3(s)} + V_3(s)\sqrt[3]{p_3^2(s)},$$

where $V_1(s)$, $V_2(s)$ and $V_3(s)$ are vectors with polynomial entries. For λ_1 the equation $G_1 V = \lambda_1 V$ is considered. (Similar results are obtained for λ_2 and λ_3.)

$$G_1 \left[V_1(s) + V_2(s)\sqrt[3]{p_3(s)} + V_3(s)\sqrt[3]{p_3^2(s)} \right] = \left(p_1(s) + p_2(s)\sqrt[3]{p_3(s)}\right) \times \left[V_1(s) + V_2(s)\sqrt[3]{p_3(s)} + V_3(s)\sqrt[3]{p_3^2(s)} \right].$$

On both sides of the equality, rational and irrational terms are compared and give

$$\begin{cases} G_1 V_1 & = p_1 V_1 + p_2 p_3 V_3, \\ G_1 V_2 & = p_1 V_2 + p_2 V_1, \\ G_1 V_3 & = p_1 V_3 + p_2 V_2, \end{cases}$$

which can be rewritten as follows:

$$G_1 \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & 0 \\ 0 & p_1 & p_2 \\ p_2 p_3 & 0 & p_1 \end{bmatrix}. \tag{1}$$

We perturb G_1 such that $G_1 + \delta G_1$ has rational characteristic functions. It is clear that in this case $\frac{1}{d(s)} G_1 + \frac{1}{d(s)} \delta G_1$ also has rational characteristic functions. The characteristic functions of $G_1(s)$ are the same as the characteristic functions of

$$\Lambda(s) = \begin{bmatrix} p_1 & p_2 & 0 \\ 0 & p_1 & p_2 \\ p_2 p_3 & 0 & p_1 \end{bmatrix}.$$

If perturbation $\delta\Lambda(s)$ causes $\Lambda(s) + \delta\Lambda(s)$ have rational characteristic functions, then the following perturbation causes the characteristic functions of $G(s)$ to be rational.

$$\delta G = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \delta\Lambda(s) \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}^{-1} . \frac{1}{d(s)}. \tag{2}$$

The goal is to parameterize a type of perturbation so that the characteristic functions of $G(s) + \delta G(s)$ are rational. Consider $\delta\Lambda$ as follows:

$$\delta\Lambda = \begin{bmatrix} \delta_{11} & 0 & 0 \\ 0 & \delta_{22} & 0 \\ p_2\delta_3 & 0 & \delta_{33} \end{bmatrix}.$$

The perturbations of three diagonal entries of $\Lambda(s)$ are not assumed to be the same. If that were the case, it is impossible to assign the characteristic functions to the rational functions. Unless it is assumed that $\delta_3 = -p_3$, which is not acceptable. On the other hand, out of three parameters, only two are effective, so it is assumed that $\delta_{33} = 0$. (When the characteristic functions of the perturbed system become rational functions with δ_{11}, δ_{22} and δ_{33}, it is obvious that they will also become rational with $\delta_{11}-\delta_{33}$, $\delta_{22}-\delta_{33}$, and 0.) The characteristic functions of $\Lambda(s)+\delta\Lambda(s)$ are the roots of the following third-degree equation

$$(\lambda - p_1 - \delta_{11})(\lambda - p_1 - \delta_{22})(\lambda - p_1) = p_2^3(p_3 + \delta_3). \tag{3}$$

Now suppose our goal is that a characteristic function of $\Lambda+\delta\Lambda$ is $\lambda = p_1 + \frac{\delta_{11}+\delta_{22}}{3}$. Substitution this value of λ in (3), gives

$$\frac{(\delta_{22} - 2\delta_{11})(\delta_{11} - 2\delta_{22})(\delta_{11} + \delta_{22})}{27} = p_2^3(p_3 + \delta_3), \tag{4}$$

Equations (3) and (4) give

$$(\lambda - p_1 - \delta_{11})(\lambda - p_1 - \delta_{22})(\lambda - p_1) - \frac{(\delta_{22} - 2\delta_{11})(\delta_{11} - 2\delta_{22})(\delta_{11} + \delta_{22})}{27} = 0. \tag{5}$$

Dividing (5) by $\lambda - p_1 - \frac{\delta_{11}+\delta_{22}}{3}$ gives a quadratic equation whose roots are the other two characteristic functions $\Lambda + \delta\Lambda$.

$$\lambda^2 - \left(2p_1 + \frac{2\delta_{11} + 2\delta_{22}}{3}\right)\lambda + p_1^2 + \frac{2}{3}(\delta_{11} + \delta_{22})p_1 + \frac{(\delta_{11} - 2\delta_{22})(\delta_{22} - 2\delta_{11})}{9} = 0. \tag{6}$$

For the roots of (6) to be rational, the following polynomial must be a complete square.

$$\begin{aligned}\Delta &= \left(p_1 + \frac{\delta_{11} + \delta_{22}}{3}\right)^2 - \frac{1}{9}(\delta_{11} - 2\delta_{22})(\delta_{22} - 2\delta_{11}) \\ &= \frac{{\delta_{11}}^2 + {\delta_{22}}^2 - \delta_{11}\delta_{22}}{3} - p_1^2 - \frac{2}{3}(\delta_{11} + \delta_{22})p_1.\end{aligned}$$

For stablishing this condition and removing $p_2(s)$ from (4), Δ is parameterized as follows:

$$\Delta = \left(\delta_{11} - \frac{1}{2}\delta_{22}\right)^2 + \left(\frac{\sqrt{3}}{2}\delta_{22}\right)^2.$$

Now suppose that

$$\begin{cases} \delta_{11} - \frac{1}{2}\delta_{22} & = \frac{2ab}{a^2+b^2}p_2(s), \\ \frac{\sqrt{3}}{2}\delta_{22} & = \frac{a^2-b^2}{a^2+b^2}p_2(s), \end{cases}$$

which gives

$$\begin{cases} \delta_{11} & = \frac{1}{\sqrt{3}}\frac{a^2-b^2+2\sqrt{3}ab}{a^2+b^2}p_2(s), \\ \delta_{22} & = \frac{2}{\sqrt{3}}\frac{a^2-b^2}{a^2+b^2}p_2(s). \end{cases} \tag{7}$$

For δ_{11} and δ_{22} to be polynomial, the coefficients a and b must be constant, but generally, they can be polynomials in terms of s. The advantage of this parameterization is that both δ_{11} and δ_{22} are obtained in terms of $p_2(s)$ and, their size will be less than $|p_2(s)|$. Using (4), δ_3 is

$$\delta_3 = -p_3(s) + \frac{1}{27}\frac{(\delta_{22} - 2\delta_{11})(\delta_{11} - 2\delta_{22})(\delta_{11} + \delta_{22})}{p_2^3(s)}. \tag{8}$$

It is not hard to see that $\delta_3 = -p_3(s) + O(1)$. Using δ_{11}, δ_{22}, and δ_3, characteristic functions of $\Lambda + \delta\Lambda$ are

$$\begin{cases} \lambda_{1,2} & = p_1(s) + \frac{\delta_{11}+\delta_{22}}{3} \pm \frac{1}{\sqrt{3}}p_2(s), \\ \lambda_3 & = p_1(s) + \frac{\delta_{11}+\delta_{22}}{3}. \end{cases}$$

Using (7), it can be shown that for all λ's

$$|\lambda - p_1(s)| \leq \left(\frac{2}{3} + \frac{1}{\sqrt{3}}\right)|p_2(s)|.$$

Remark 1. This method, along with the previous work in this field (Reference [14]), can be applied to all systems with non-rational characteristic functions as $\{\lambda_Q(s) \pm \lambda_{Q'}(s)\sqrt{p(s)}\}$ or $\{p_1(s) + p_2(s)\sqrt[3]{p_3(s)}, p_1(s) + p_2(s)\sqrt[3]{p_3(s)}e^{j\frac{2\pi}{3}}, p_1(s) + p_2(s)\sqrt[3]{p_3(s)}e^{-j\frac{2\pi}{3}}\}$ if the characteristic functions are known. In the previous work (Reference [14]), the method of the characteristic function assignment was presented for a class of systems whose irrational characteristic functions are in the form of $\{\lambda_Q(s) \pm \lambda_{Q'}(s)\sqrt{p(s)}\}$. Also, for 3×3 plants, by using Cardano's formula for solving third-degree equations, the characteristic functions can be calculated.

3. Illustrative examples

In this section, the effectiveness of the proposed approach was demonstrated through two examples.

Example 3.1. Consider the transfer function given below

$$G(s) = \frac{1}{(s+2)^2}\begin{bmatrix} s & s & 0 \\ 0 & s & s \\ s^2 & 0 & s \end{bmatrix}.$$

Characteristic functions are

$$\frac{s + s\sqrt[3]{s}}{(s+2)^2}, \frac{s + se^{\pm j2\pi/3}\sqrt[3]{s}}{(s+2)^2}.$$

Now, we define

$$G_1 = \begin{bmatrix} s & s & 0 \\ 0 & s & s \\ s^2 & 0 & s \end{bmatrix}.$$

According to the notation in Section 2, we have $p_1(s) = p_2(s) = p_3(s) = s$. If we write the eigenvector corresponding to λ_1 as $V_1(s) + V_2(s)\sqrt[3]{p_3(s)} + V_3(s)\sqrt[3]{p_3^2(s)}$, then V_1, V_2 and V_3 are obtained as follows (That is one of the solutions, and there exist some others.)

$$\begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In (7), it is assumed that $a = b = 1$ that it gives $\delta_{11} = s$, $\delta_{22} = 0$. Substitution these values in Equation (8) gives $\delta_3 = -s - \frac{2}{27}$. By using Equation (2), we have

$$\delta G(s) = \frac{1}{(s+2)^2}\delta\Lambda(s) = \frac{1}{(s+2)^2}\begin{bmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{2}{27}s - s^2 & 0 & 0 \end{bmatrix}.$$

It can be seen that with the perturbation $\delta G(s)$, the characteristic functions of $G(s) + \delta G(s)$ are rational as follows:

$$\frac{4}{3(s+2)^2}, \frac{4s}{3(s+2)^2} + \frac{s}{\sqrt{3}(s+2)^2}, \frac{4s}{3(s+2)^2} - \frac{s}{\sqrt{3}(s+2)^2}.$$

Example 3.2. Consider the transfer function given below

$$G(s) = \frac{1}{(s+2)^3}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ s^3 + s^2 - 1 & -3s^2 & 3s \end{bmatrix},$$

Characteristic functions are

$$\frac{s+\sqrt[3]{s^2-1}}{(s+2)^3}, \frac{s+e^{\pm j2\pi/3}\sqrt[3]{s^2-1}}{(s+2)^3}.$$

Now, we define

$$G_1(s)=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ s^3+s^2-1 & -3s^2 & 3s \end{bmatrix}.$$

According to the notation in Section 2, we have $p_1(s)=s$, $p_2(s)=1$ and $p_3(s)=s^2-1$. If we write the eigenvector corresponding to λ_1 as $V_1(s)+V_2(s)\sqrt[3]{p_3(s)}+V_3(s)\sqrt[3]{p_3^2(s)}$, then V_1, V_2 and V_3 are obtained as follows (That is one of the solutions, and there exist some others.)

$$\begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ s^2 & 2s & 1 \end{bmatrix},$$

Equation (7) with $a=b=1$, gives $\delta_{11}=1$ and $\delta_{22}=0$. Substitution these values in Equation (8) gives $\delta_3=-s^2+\frac{25}{27}$. By using Equation (2), we have

$$\delta G(s)=\frac{1}{(s+2)^3}\begin{bmatrix} 1 & 0 & 0 \\ s & 0 & 0 \\ \frac{25}{27} & 0 & 0 \end{bmatrix}.$$

It can be seen that with the perturbation $\delta G(s)$, the characteristic functions of $G(s)+\delta G(s)$ are rational as follows:

$$\frac{s+1/3}{(s+2)^3}, \frac{s+1/3+\sqrt{3}/3}{(s+2)^3}, \frac{s+1/3-\sqrt{3}/3}{(s+2)^3}.$$

4. Conclusion

This paper presented a characteristic function assignment by adding perturbation. In this approach, a perturbation is added to the transfer function such that the characteristic functions of the perturbed system are rational. The new transfer function with rational characteristic functions is suitable to apply various control objectives such as stabilizing, design of commutative compensator, and input-output decoupling. Our approach is not according to inverting the transfer function, and characteristic loci do not directly participate in the design process.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

References

[1] J. P. Corriou, *Process Control*, Springer, London, 2004.

[2] J. M. Maclejowski, *Multivariable Feedback Design*, Electronic Systems Engineering Series, 1989.

[3] S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control: Analysis and Design*, John Wiley and Sons, 2005.

[4] D. J. Cloudt and B. Kouvaritakis, Commutative controllers revisited: parallel computation, a new lease of life, *Int. J. Control* **45** (1987) 1335 – 1370, https://doi.org/10.1080/00207178708933812.

[5] M. V. Moreira and J. C. Basilio, Characteristic locus method robustness improvement through optimal static normalizing pre-compensation, *Int. J. Robust Nonlinear Control* **20** (2010) 371 – 386, https://doi.org/10.1002/rnc.1429.

[6] M. V. Moreira and J. C. Basilio, Static normalizing pre-compensator: the first step for addressing robustness in the design of multivariable controllers using the characteristic locus method, *In 2007 European Control Conference (ECC)*, IEEE, (2007) 1142 – 1148, https://doi.org/10.23919/ECC.2007.7068932.

[7] J. Doyle and G. Stein, Multivariable feedback design: concepts for a classical/modern synthesis, *IEEE Trans. Autom. Control* **26** (1981) 4 – 16, https://doi.org/10.1109/TAC.1981.1102555.

[8] I. Postlethwaite, Sensitivity of the characteristic gain loci, *Automatica* **18** (6) (1982) 709 – 712, https://doi.org/10.1016/0005-1098(82)90059-0.

[9] J. C. Basilio and B. Kouvaritakis, The use of rational eigenvector approximations in commutative controllers, *Int. J. Control* **61** (1995) 333 – 356, https://doi.org/10.1080/00207179508921906.

[10] J. C. Basilio and J. A. Sahate, A normalizing precompensator for the design of effective and reliable commutative controllers, *Int. J. Control* **73** (2000) 1280 – 1297, https://doi.org/10.1080/002071700421673.

[11] M. V. Moreira, J. C. Basilio and B. Kouvaritakis, Rational stabilising commutative controllers: parameterisation and characterisation of degrees of freedom, *Int. J. Control* **79** (2006) 1601 – 1612, https://doi.org/10.1080/00207170600867073.

[12] Y. K. Foo, Design of robust decentralised large-scale inter-connected control systems via characteristic locus method, *Int. J. Syst. Sci.* **44** (2013) 641 – 651, https://doi.org/10.1080/00207721.2011.617897.

[13] M. V. Moreira and J. C. Basilio, Design of normalizing precompensators via alignment of output-input principal directions, *Proceedings of the 44th IEEE Conference on Decision and Control*, (2005) 2170 – 2175, https://doi.org/10.1109/CDC.2005.1582483.

[14] A. Shahmansoorian and P. Ahmadi, Assignment of characteristic functions and a new method for stabilizing multivariable systems, *Circuits, Syst. Signal Process.* **40** (2021) 5983 – 5996, https://doi.org/10.1007/s00034-021-01762-1.

A. Shahmansoorian
Electrical Engineering Department,
Imam Khomeini International University,
Qazvin, I. R. Iran
e-mail: shahmansoorian@eng.ikiu.ac.ir

P. Ahmadi
Electrical Engineering Department,
Imam Khomeini International University,
Qazvin, I. R. Iran
e-mail: p.ahmadi@edu.ikiu.ac.ir