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Approximate Convexity for Set-Valued Maps

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Abstract

In this paper, we extend the notion of approximate convexity to set-valued maps and obtain some relations between approximate convexity and approximate monotonicity of their normal subdifferential.

Keywords: Generalized convexity, Generalized monotonicity, Set-valued map, Generalized subdifferential.

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1. Introduction

The concept of convexity is very important in optimization theory as a local minimum for a convex function becomes a global minimum. In 1998 Jofre et al. [1] generalized the notion of convexity to ε-convexity and by using it Ngai et al. [2] presented the notion of approximate convexity which consists of several useful and interesting properties of convex functions. Daniilidis and Georgiev [3] proved that for locally Lipschitz functions, approximate convexity is equivalent to the submonotonicity of their Clarke's subdifferential. By using Clarke's subdifferential, Bhatia et al. [4] introduced several extensions of approximate convexity. Malmir and Barani [5], showed that under the locally Lipschitzian property, the set of lower-C^1 functions and the set of approximately convex functions are the same.
In the past years, characterizing and extending generalized convex functions to set-valued maps have been considered by many researchers, see [6–8] and references therein. In a recent paper, Durea and Strugariu [9] introduced a new technique to

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construct subdifferentials and directional derivatives for set-valued maps. In this paper, we present the definition of approximate convexity for set-valued maps and by using the concept of normal subdifferential in the sense of Murdukhovich we show that under some conditions it is equivalent by approximate monotonicity of its normal subdifferential. This paper is organized as follows. In Section 2, some preliminary results and notions used in the sequel are given. Section 3 is devoted to obtaining some relations between the approximate convexity of set-valued maps and the approximate monotonicity for their normal subdifferentials.

2. Preliminaries

Let X and Y be two Banach spaces. The norm in X and its dual space X^* will be denoted by $\| . \|$. Assume that S_X and B_X are the unit sphere and closed unit ball of X, respectively. Also, consider $K \subset Y$ to be a closed convex cone.

Definition 2.1. ([10]). A mapping $H : \Omega \subset X \rightrightarrows Y$ is called

(i) Lipschitz around $x_0 \in \Omega$ if there are $l \geq 0$ and a neighborhood U of x_0 such that

$$H(x) \subset H(x') + l \parallel x - x' \parallel B_Y, \quad \forall x, x' \in \Omega \cap U \subset X.$$

(ii) epi-Lipschitz around $x_0 \in \Omega$ if $\mathcal{E}_H(\cdot) := H(\cdot) + K$ is Lipschitz around this point.

H is locally epi-Lipschitz on Ω, if for every $x \in \Omega$, it is epi-Lipschitz around x.
Now, we present some definitions of subdifferential and coderivative of real and set-valued mapping.

Definition 2.2. ([10]). Let $h : X \to \mathbb{R}$. The limiting subdifferential of h at x_0 is given by

$$\partial_L h(x_0) := \{\xi \in X^* | (\xi, -1) \in N((x_0, h(x_0)); \text{epi}h)\},$$

where $N((x_0, h(x_0)); \text{epi}h)$ is the basic normal cone to epi$h$ at $(x_0, h(x_0))$.
The normal coderivative of $H : X \rightrightarrows Y$ at $(x_0, y_0) \in \text{gr}H$ is $D^*_N H(x_0, y_0) : Y^* \rightrightarrows X^*$ defined by

$$D^*_N H(x_0, y_0)(y^*) := \{x^* \in X^* | (x^*, -y^*) \in N((x_0, y_0); \text{gr}H)\}.$$

Also, the normal subdifferential [11] of H at the point $(x_0, y_0) \in \text{epi}H$ is given by $\partial H(x_0, y_0)(y^*) := D^*_N \mathcal{E}_H(x_0, y_0)(y^*)$.

Assume that $y^* \in Y^*$ and $H : X \rightrightarrows Y$. We associated to y^* and H a marginal function $h_{y^*} : X \to \bar{\mathbb{R}}$,

$$h_{y^*}(x) := \inf_{y \in H(x)} y^*(y),$$

and the set of minimum points

$$M_{y^*}(x) := \{y \in H(x) |\ h_{y^*}(x) = y^*(y)\}.$$

In next theorem, a relation between normal subdifferential of a map and limiting subdifferential of its marginal functions is given.

Theorem 2.3. *([12]). Let $H : X \rightrightarrows Y$ be a map between two Asplund spaces. Suppose that $x_0 \in domH$, $y^* \in K^+$ and $y_0 \in M_{y^*}(x_0)$. If H is epi-Lipschitz around x_0, then $\partial h_{y^*}(x_0) \subseteq \partial H(x_0, y_0)(y^*)$.*

Definition 2.4. ([1]). Let $h : \Omega \subset X \to \mathbb{R}$. h is called approximately convex at $\bar{x} \in \Omega$, if for every $\alpha > 0$, there exists $\delta > 0$ such that for any $x, x' \in B(\bar{x}, \delta) \cap \Omega$ and any $0 \leq c \leq 1$, one has

$$h(cx + (1-c)x') \leq ch(x) + (1-c)h(x') + \alpha c(1-c) \parallel x - x' \parallel .$$

Definition 2.5. ([10]). $A \subset X$ is called sequentially normally compact (SNC) at $x \in A$ if for every sequence $(\varepsilon_n, x_n, x_n^*) \in [0, \infty[\times A \times X^*$ satisfying

$$\varepsilon_n \downarrow 0, x_n \to x, x_n^* \in \widehat{N}_{\varepsilon_n}(x_n; A), \quad \text{and} \quad x_n^* \overset{w^*}{\to} 0,$$

one has $||x_n^*|| \to 0$ as $n \to \infty$, where $\widehat{N}_\varepsilon(x; A)$ is the set of ε-normals to A at x. Also, X is called weakly compactly generated (WCG) if there exists a weakly compact set $A \subset X$ such that $X = \text{cl}(\text{span } A)$.

Note that, any separable Banach space is WCG. [10]

3. Main results

In this section, an extension of approximate convexity for set-valued maps and approximate monotonicity of their normal subdifferential are presented. Also, some relations between them are given. Throughout this section, X and Y are considered to be Asplund spaces.

Definition 3.1. Assume that $\Omega \subset X$ is a convex set and $H : \Omega \subset X \rightrightarrows Y$. H is called approximately K-convex at $\bar{x} \in \text{dom}H$ if for any $\alpha > 0$ there exists $\delta > 0$ (depending on $\bar{x}$ and α) such that for any $x, x' \in B(\bar{x}, \delta)$ and $0 \leq c \leq 1$, one has

$$cH(x) + (1-c)H(x') + \alpha c(1-c) \parallel x - x' \parallel e \subseteq H(cx + (1-c)x') + K,$$

for an $e \in \text{int}K$ with $\parallel e \parallel = 1$.

Definition 3.2. The mapping $\partial H : X \times Y \times Y^* \rightrightarrows X^*$ is called approximately K-monotone at $\bar{x} \in \text{dom}H$ if for every $\alpha > 0$, there exists $\delta > 0$ such that for any $x_j \in B(\bar{x}, \delta)$, $y^* \in S_{Y^*} \cap K^+$, $y_j \in M_{y^*}(x_j)$ and $x_j^* \in \partial H(x_j, y_j)(y^*), (j = 1, 2)$, one has

$$< x_1^* - x_2^*, x_1 - x_2 > \geq -\alpha \parallel x_1 - x_2 \parallel .$$

Remark 1. $H : \Omega \subset X \rightrightarrows Y$ is said to satisfy Condition $(AC)_1$ (resp. Condition $(AC)_2$) at $\bar{x} \in \text{dom}H$ if for any $\alpha > 0$ there exists $\delta > 0$ such that for every $x_j \in B(\bar{x}, \delta)$, $y^* \in S_{Y^*} \cap K^+$ and $y_j \in M_{y^*}(x_j)$, $(j = 1, 2)$, one has

$$< x^*, x_2 - x_1 > -\alpha \parallel x_2 - x_1 \parallel \leq y^*(y_2) - y^*(y_1),$$

for some (resp. any) $x^* \in \partial H(x_1, y_1)(y^*)$.

Lemma 3.3. *If $H : X \rightrightarrows Y$ is approximately K-convex, then for any $y^* \in S_{Y^*} \cap K^+$, h_{y^*} is a real-valued approximately convex function.*

Proof. Suppose that H is approximately K-convex at x_0. Then for every $\alpha > 0$ there exist $\delta > 0$ and $e \in \text{int}K$ ($\parallel e \parallel = 1$) such that for every $x, x' \in B(x_0, \delta)$ and $0 \leq c \leq 1$, we have

$$cH(x) + (1 - c)H(x') + \alpha c(1 - c) \parallel x - x' \parallel e \subseteq H(cx + (1 - c)x') + K.$$

Now, for all $y^* \in S_{Y^*} \cap K^+$, we obtain

$$\inf(y^*(H(cx+(1-c)x')+K)) \subseteq \inf(y^*(cH(x)+(1-c)H(x')+\alpha c(1-c) \parallel x-x' \parallel e)).$$

Since $y^* \in S_{Y^*}$ and $\parallel e \parallel = 1$, we get

$$h_{y^*}(cx + (1 - c)x') \leq ch_{y^*}(x) + (1 - c)h_{y^*}(x') + \alpha c(1 - c) \parallel x - x' \parallel .$$

It shows approximate convexity of h_{y^*} at x_0. □

Theorem 3.4. *Suppose that X is WCG, $\Omega \subset X$ is a closed subset that is SNC at $x \in \Omega$ and $H : \Omega \subset X \rightrightarrows Y$ is locally epi-Lipschitz. If H is approximately K-convex at $x \in \Omega$, then H satisfies Condition $(AC)_1$ at this point.*

Proof. By Lemma 3.3 and Lemma 3.2 [12] for every $y^* \in S_{Y^*} \cap K^+$, h_{y^*} is approximately convex at x and locally Lipschitz. Hence, for every $\alpha > 0$, there exists $\delta > 0$ such that for any $x_j \in B(x, \delta) \cap \Omega$, $(j = 1, 2)$ and $0 \leq c \leq 1$, one has

$$h_{y^*}(cx_1 + (1 - c)x_2) \leq ch_{y^*}(x_1) + (1 - c)h_{y^*}(x_2) + \alpha c(1 - c) \parallel x_1 - x_2 \parallel .$$

Therefore

$$\frac{h_{y^*}(cx_1 + (1 - c)x_2) - h_{y^*}(x_2)}{c} \leq h_{y^*}(x_1) - h_{y^*}(x_2) + \alpha(1 - c) \parallel x_1 - x_2 \parallel, \quad (1)$$

for any $c \in (0, 1)$. Since h_{y^*} is locally Lipschitz, $0 < \theta < 1$ can be found such that h_{y^*} is Lipschitz on an open set containing $[x_2, x_2 + c(x_1 - x_2)]$ for any $c \in [0, \theta]$. Now, by using Corollary 3.51 in [10] (mean value inequality), there exist $a_c \in [x_2, x_2 + c(x_1 - x_2)[$ and $x^*_c \in \partial h_{y^*}(a_c)$ such that

$$c < x^*_c, x_1 - x_2 > \leq h_{y^*}(x_2 + c(x_1 - x_2)) - h_{y^*}(x_2).$$

From relation (1), we have

$$< x_c^*, x_1 - x_2 > \leq h_{y^*}(x_1) - h_{y^*}(x_2) + \alpha(1-c) \parallel x_1 - x_2 \parallel .$$

By the locally boundedness of ∂h_{y^*} (see Corollary 1.81 in [10]), there are $k > 0$ and a neighborhood of x_2 such that for each u in this neighborhood and $x^* \in \partial h_{y^*}(u)$, we have $\parallel x^* \parallel \leq k$. Since, $a_c \to x_2$ when $c \to 0$, for c be sufficiently small $\parallel x_c^* \parallel \leq k$. Hence, without loss of generality we can suppose that $x_c^* \to x^*$ in w*-topology. Because the mapping $\partial h(\cdot)$ has closed graph (see Theorem 3.60 in [10]), we have $x^* \in \partial h_{y^*}(x_2)$ and

$$< x^*, x_1 - x_2 > \leq h_{y^*}(x_1) - h_{y^*}(x_2) + \alpha \parallel x_1 - x_2 \parallel .$$

Now, by using Theorem 2.3, H satisfies Condition $(AC)_1$. □

Lemma 3.5. *Let $H : \Omega \subset X \rightrightarrows Y$ satisfies Condition $(AC)_2$ at $\bar{x}$. Then ∂H is approximately K-monotone at this point.*

Proof. Suppose that H satisfies Condition $(AC)_2$. Therefore, for every $\alpha > 0$ there exists $\delta > 0$, such that for any $x_j \in B(\bar{x}, \delta)$, $y^* \in S_{Y^*} \cap K^+$, $y_j \in M_{y^*}(x_j)$ and $\xi_j \in \partial H(x_j, y_j)(y^*)$, $(j = 1, 2)$, we get

$$< \xi_1, x_2 - x_1 > -\alpha \parallel x_2 - x_1 \parallel \leq y^*(y_2) - y^*(y_1),$$

and

$$< \xi_2, x_1 - x_2 > -\alpha \parallel x_1 - x_2 \parallel \leq y^*(y_1) - y^*(y_2).$$

By adding these two relations, we obtain

$$< \xi_2 - \xi_1, x_2 - x_1 > \geq -2\alpha \parallel x_2 - x_1 \parallel .$$

It says that ∂H is approximately K-monotone at $\bar{x}$. □

In the next theorem, an extension of Theorem 3.5 in [13] is presented.

Theorem 3.6. *Let $H : X \rightrightarrows Y$ be locally epi-Lipschitz. If ∂H is approximately K-monotone, then H satisfies Condition $(AC)_2$.*

Proof. Let ∂H be approximately K-monotone at $\bar{x} \in X$. Hence, for every $\alpha > 0$ there exists $\delta > 0$ such that for every $x_j \in B(\bar{x}, \delta)$, $y^* \in S_{Y^*} \cap K^+$, $y_j \in M_{y^*}(x_j)$ and $\xi_j \in \partial H(x_j, y_j)(y^*)$, $(j = 1, 2)$, one has

$$< \xi_2 - \xi_1, x_2 - x_1 > \geq -\alpha \parallel x_2 - x_1 \parallel .$$

Let $z = x_2 + \frac{1}{2}(x_1 - x_2)$ and let $y^* \in S_{Y^*} \cap K^+$ be arbitrary. From Lemma 3.2 in [12], h_{y^*} is locally Lipschitz. Now, by using Corollary 3.51 in [10] (mean value inequality), there exist c_1, c_2 such that $0 < c_2 \leq \frac{1}{2} < c_1 \leq 1$, $\xi_1 \in \partial h_{y^*}(u_1)$, $\xi_2 \in \partial h_{y^*}(u_2)$ such that

$$h_{y^*}(x_1) - h_{y^*}(z) \geq \frac{1}{2} < \xi_1, x_1 - x_2 >, \tag{2}$$

and

$$h_{y^*}(z) - h_{y^*}(x_2) \geq \frac{1}{2} < \xi_2, x_1 - x_2 >, \tag{3}$$

where $u_1 = c_1x_1 + (1 - c_1)x_2$ and $u_2 = c_2x_1 + (1 - c_2)x_2$. Now, Theorem 2.3 implies $\xi_j \in \partial h_{y^*}(u_j) \subset \partial H(u_j, z_j)(y^*)$ that $z_j \in M_{y^*}(u_j)$, $(j = 1, 2)$. Since ∂H is approximately K-monotone at x and $u_1, u_2 \in B(x, \delta)$, we have

$$< \xi_1 - w, u_1 - x_2 >\geq -\alpha \parallel u_1 - x_2 \parallel,$$

for every $y_2 \in M_{y^*}(x_2)$ and $w \in \partial H(x_2, y_2)(y^*)$. Now, by using inequality (2), we get

$$h_{y^*}(x_1) - h_{y^*}(z) \geq \frac{1}{2}(< w, x_1 - x_2 > -\alpha \parallel x_1 - x_2 \parallel).$$

In a similar way, we can obtain

$$h_{y^*}(z) - h_{y^*}(x_2) \geq \frac{1}{2}(< w, x_1 - x_2 > -\alpha \parallel x_1 - x_2 \parallel).$$

By adding the latter two inequalities, we deduce that

$$h_{y^*}(x_1) - h_{y^*}(x_2) \geq< w, x_1 - x_2 > -\alpha \parallel x_1 - x_2 \parallel .$$

Hence, for every $x_j \in B(x, \delta)$, $y_j \in M_{y^*}(x_j)$, $j = 1, 2$ and $w \in \partial H(x_2, y_2)(y^*)$, we have

$$y^*(y_1) - y^*(y_2) \geq< w, x_1 - x_2 > -\alpha \parallel x_1 - x_2 \parallel .$$

It shows that H satisfies Condition $(AC)_2$. □

Theorem 3.7. *Let $H : X \rightrightarrows Y$ be a locally epi-Lipschitz map $\mathcal{E}_H$ with being a closed convex-valued. If H satisfies Condition $(AC)_2$, then it is approximately K-convex.*

Proof. Suppose that H satisfies Condition $(AC)_2$ at $\bar{x}$. By using Theorem 2.3, we can deduce that h_{y^*} is a real-valued function satisfying Condition $(AC)_2$ at $\bar{x}$. Hence, for every $\alpha > 0$ there exists $\delta > 0$ such that for every $x_j \in B(\bar{x}, \delta)$, $(j = 1, 2)$, one has

$$< \xi, x_2 - x_1 > -\alpha \parallel x_2 - x_1 \parallel\leq h_{y^*}(x_2) - h_{y^*}(x_1), \quad \forall \xi \in \partial h_{y^*}(x_1).$$

Now, by applying the above inequality for x_1, $x_c = x_2 + c(x_1 - x_2)$ and also for x_2, x_c where $0 < c < 1$, we obtain

$$(1-c) < \xi, x_1-x_2 > -\alpha(1-c) \parallel x_1-x_2 \parallel\leq h_{y^*}(x_1)-h_{y^*}(x_c), \quad \forall \xi \in \partial h_{y^*}(x_c), \tag{4}$$

$$-c < \xi, x_1 - x_2 > -\alpha c \parallel x_1 - x_2 \parallel\leq h_{y^*}(x_2) - h_{y^*}(x_c), \quad \forall \xi \in \partial h_{y^*}(x_c). \tag{5}$$

By multiplying (4) by c and (5) by $(1 - c)$ and adding the resulting inequalities, we obtain

$$-2\alpha c(1 - c) \parallel x_1 - x_2 \parallel\leq ch_{y^*}(x_1) + (1 - c)h_{y^*}(x_2) - h_{y^*}(x_c),$$

and therefore

$$h_{y^*}(x_2 + c(x_1 - x_2)) \leq ch_{y^*}(x_1) + (1-c)h_{y^*}(x_2) + 2\alpha c(1-c)\|x_1 - x_2\|.$$

It means that h_{y^*} is approximately convex at $\bar{x}$.
Now, we suppose to the contrary that H is not approximately K-convex at $\bar{x}$. Therefore, there is $\alpha > 0$ such that for any $\delta > 0$ and $e \in \mathrm{int}K$, there exist $x_j \in B(\bar{x}, \delta)$, $y_j \in H(x_j)$, $(j = 1, 2)$, and $c \in [0, 1]$, such that

$$cy_1 + (1-c)y_2 + \alpha c(1-c) \parallel x_1 - x_2 \parallel e \notin H(cx_1 + (1-c)x_2) + K.$$

By using the separating theorem to the non-empty disjoint convex sets: $\{cy_1 + (1-c)y_2 + \alpha c(1-c)e \parallel x_1 - x_2 \parallel\}$ which is compact and $H(x_2 + c(x_1 - x_2)) + K$ which is convex and closed, a functional $\bar{y}^* \in Y^* \backslash \{0\}$ can be found such that

$$\begin{aligned} \bar{y}^*(cy_1 + (1-c)y_2 + \alpha c(1-c) \parallel x_1 - x_2 \parallel e) &< \inf \bar{y}^*(H(x_2 + c(x_1 - x_2)) + K) \\ &= \inf \bar{y}^*(H(x_2 + c(x_1 - x_2))) + \inf \bar{y}^*(K). \end{aligned}$$

Now, it can be easily verified that $\bar{y}^* \in K^+ \backslash \{0\}$ and thus $\inf \bar{y}^*(K) = 0$. Furthermore, without loss of generality, we may assume that $\bar{y}^*(e) = 1$. Therefore,

$$c\bar{y}^*(y_1) + (1-t)\bar{y}^*(y_2) + \alpha c(1-c) \parallel x_1 - x_2 \parallel < h_{\bar{y}^*}(x_2 + c(x_1 - x_2)). \quad (6)$$

Since $h_{\bar{y}^*}$ is approximately convex at $\bar{x}$, we obtain

$$h_{\bar{y}^*}(x_2 + c(x_1 - x_2)) \leq ch_{\bar{y}^*}(x_1) + (1-c)h_{\bar{y}^*}(x_2) + \alpha c(1-c) \parallel x_1 - x_2 \parallel.$$

Because $y_1 \in H(x_1)$ and $y_2 \in H(x_2)$, definition of marginal functions implies that

$$h_{\bar{y}^*}(x_2 + c(x_1 - x_2)) \leq c\bar{y}^*(y_1) + (1-c)\bar{y}^*(y_2) + \alpha c(1-c) \parallel x_1 - x_2 \parallel,$$

which is a contradiction with (6). □

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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