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Gorenstein Homological Dimension of Groups Through Flat-Cotorsion Modules

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Abstract

The representation theory of groups is one of the most interesting examples of the interaction between physics and pure mathematics, where group rings play the main role. The group ring $\mathrm{R}\Gamma$ is actually an associative ring that inherits the properties of the group Γ and the ring of coefficients R. In addition to the fact that the theory of group rings is clearly the meeting point of group theory and ring theory, it also has applications in algebraic topology, homological algebra, algebraic K-theory and algebraic coding theory. In this article, we provide a complete description of Gorenstein flat-cotorsion modules over the group ring $\mathrm{R}\Gamma$, where Γ is a group and R is a commutative ring. It will be shown that if $\Gamma' \leqslant \Gamma$ is a finite-index subgroup, then the restriction of scalars along the ring homomorphism $\mathrm{R}\Gamma' \to \mathrm{R}\Gamma$ as well as its right adjoint $\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} -$, preserve the class of Gorenstein flat-cotorsion modules. Then, as a result, Serre's Theorem is proved for the invariant $\widetilde{\mathrm{Ghd}}_{\mathrm{R}}\Gamma$, which refines the Gorenstein homological dimension of Γ over R, $\mathrm{Ghd}_R\Gamma$, and is defined using flat-cotorsion modules. Moreover, we show that the inequality $\mathrm{GFCD}(\mathrm{R}\Gamma) \leqslant \mathrm{GFCD}(R) + \mathrm{cd}_R\Gamma$ holds for the group ring $\mathrm{R}\Gamma$, where $\mathrm{GFCD}(R)$ denotes the supremum of Gorenstein flat-cotorsion dimensions of all R-modules and $\mathrm{cd}_R\Gamma$ is the cohomological dimension of Γ over R.

Keywords: Group ring, Flat-cotorsion module, Gorenstein flat-cotorsion module, Gorenstein flat-cotorsion dimension.

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1. Introduction

Throughout the article, unless stated to the contrary, R will always denote a commutative ring with a unity, although some arguments are also valid without the assumption of commutativity. Let Γ be a group and let $R\Gamma$ denote the group ring of Γ over the ring R. In the theory of group rings, the methods of ring theory and group theory have an effective relationship and create an active research field which is replete with diverse problems and novel insights. There is a mutual relationship between the characteristics of a group and its group ring, where investigations lead to deep results and create interesting problems. The study of group rings was first done in the framework of the theory of representations of finite groups. Representation theory is a mathematical subject in which algebraic structures are studied through the representation of their elements as linear transformations of vector spaces [1]. Basically, a representation can describe algebraic objects more tangible via matrices and their algebraic actions, such as matrix addition and multiplication. Groups, associative algebras and Lie algebras are the most important algebraic objects that are described in this way, and group representation theory is the oldest one. In the theory of representations of groups, group elements can be represented by invertible matrices so that matrix multiplication plays the role of group operation [2]. In fact, the advantage of group representation theory is that it can reduce abstract algebra problems to understandable and concrete problems of linear algebra. For example, representing a group with a Hilbert space of infinite dimension makes it possible to apply analytical methods in group theory [3]. Representation theory is also important in physics, because it can be shown, for example, how the group symmetry of a physical system can affect the set of solutions to the equations that describe that system [4]. Representation theory, and in particular representations of the three-dimensional rotation group, is widely used in quantum mechanics. On the other hand, the study of representations of the rotation group of three-dimensional space can serve as a good introduction to the general theory of representations of Lie groups, see [5, 6].

An important problem that was considered in the representation theory of finite groups was the determination of projective modules over group rings. Chouinard showed that the projectivity of an $R\Gamma$-module A is equivalent to the fact that the restriction of A to any elementary Abelian subgroup of Γ is also projective [7]. Another way to recognize projective modules over group rings is given by Benson and Goodearl in [8]. According to their results, when Γ is a finite group, any flat $R\Gamma$-module that is projective as an R-module is also projective as an $R\Gamma$-module.

Group rings associated with infinite groups appeared in the texts of algebraic topology and group theory in the 1930's, and their applications were mainly considered two decades later. Nonetheless, the beginning of the 60's can be considered as the turning point of research on this type of rings, so that the flow of writing articles in this field increased significantly.

One of the important topics in the theory of group rings is the study of some characteristics of groups through their (co)homological properties, which nowadays is referred to as cohomological theory of groups. This theory, which has algebraic and topological roots, has a long history in group theory. So far, various (co)homological dimensions have been assigned to a group Γ, the most important of which is the cohomological dimension (respectively, homological dimension) of the group Γ over the ring R, which is denoted by $\mathrm{cd}_R\Gamma$ (respectively, $\mathrm{hd}_R\Gamma$), see 2.2. In [9] and [10], it is proved that a torsion-free group Γ satisfies $\mathrm{cd}_R\Gamma \leqslant 1$ if and only if it is free. Also, a theorem known as Serer's Theorem states that if $\Gamma' \leqslant \Gamma$ is a finite-index subgroup of a group Γ without R-torsion, then $\mathrm{cd}_R\Gamma = \mathrm{cd}_R\Gamma'$ ($\mathrm{hd}_R\Gamma = \mathrm{hd}_R\Gamma'$) [11]; recall a group Γ is without R-torsion, provided that the order of every element of Γ is either infinite or a unit in R. In 1953, Tate's cohomology for finite groups was introduced. The definition of this cohomology was based on the result which stated that the $\mathbb{Z}\Gamma$-module $\mathbb{Z}$, with trivial Γ-action, has a complete projective resolution. Inspired by Tate's observations, a class of finitely generated modules that had complete projective resolution was introduced by Auslander and Bridger in [12]. By using such modules, the G-dimension of finitely generated modules on the Noetherian rings was defined by himself and actually refined the projective dimension of these types of modules. Then, in order to generalize the notion of G-dimension for arbitrary modules over any ring, Enochs and Jenda introduced Gorenstein projective and Gorenstein injective modules in [13]. Based on these definitions, the theory of Gorenstein homological algebra formed, which in fact, the main idea of its formation was to look at Gorenstein projective and Gorenstein injective modules, respectively, instead of the projective and injective modules in classical homological algebra. In order to complete this theory, Gorenstein flat modules were also introduced, see [14]. According to a general principle, every result in classical homological algebra must have a parallel in Gorenstein homological algebra. Following this, from 2004 onwards, Holm, Bennis, Mahdou et al. introduced certain Gorenstein cohomological dimensions to which the methods and theories of classical homologica algebra (probably with some additional conditions) can be applied (for example see [15, 16]). In the recent decade, the study of Gorenstein modules and their related dimensions on group rings was considered, see [17–25]. Specifically in [24], Asadollahi et al. called the Gorenstein projective dimension of the RΓ-module R, with trivial Γ-action, as the Gorenstein cohomological dimension of Γ over R and denoted it by $\mathrm{Gcd}_{\mathrm{R}}\Gamma$. They investigated the relationship of this dimension with other homological invariants attributed to group Γ and then proved a theorem similar to Serre's Theorem for it. Also, the Gorenstein flat dimension of modules over group rings and especially the Gorenstein flat dimension of the RΓ-module R and its relationship with other homological invariants of Γ were studied in [25].

A class of modules that has received special attention in Gorenstein homological algebra during recent years is the class of modules that are Gorenstein flat and cotorsion at the same time. For example, Gillespie in [26] proved that provided the ring is coherent, the category of modules that are simultaneously Gorenstein flat and cotorsion is Frobenius, while for the category of Gorenstein flat modules, this phenomenon rarely happens, see [27, Theorem 4.5]. Therefore, the Gorenstein flat-cotorsion modules and the dimensions related to them were defined and extensively investigated [27–30].

Our main goal in this paper is to study the class of Gorenstein flat-cotorsion modules for group rings. One of the concerns in the theory of cohomology of groups is related to the transfer of properties of modules along the ring homomorphisms $\mathrm{R}\Gamma' \to \mathrm{R}\Gamma$, $\Gamma' \leqslant \Gamma$, where the main role is played by the restriction and the extension of scalars via the functor $\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} -$. Following this, we examine here the behavior of these functors on the class of Gorenstein flat-cotorsion modules, especially in the case where $\Gamma' \leqslant \Gamma$ is a finite-index subgroup. More precisely, we show that if Γ contains a subgroup Γ' of finite index, then every $\mathrm{R}\Gamma$-module that is flat-cotorsion over $\mathrm{R}\Gamma'$ is Gorenstein flat-cotorsion over $\mathrm{R}\Gamma$ (Proposition 3.4) and that an $\mathrm{R}\Gamma$-module is Gorenstein flat-cotorsion if and only if it is so as an $\mathrm{R}\Gamma'$-module (Corollary 3.8). Also, we define a refinement for the invariant $\mathrm{Ghd}_R\Gamma$, Gorenstein homological dimension Γ over R, which was introduced in [25] and we denote it by $\widetilde{\mathrm{Ghd}}_{\mathrm{R}}\Gamma$. Then we prove a theorem similar to Serre's Theorem and obtain a characterization of finite groups by applying it. At the end of the article, we will focus on the invariant $\mathrm{GFCD}(R)$, which actually stands for "the supremum of Gorenstein flat-cotorsion dimensions of all R-modules". It is shown that if we switch to the group ring $\mathrm{R}\Gamma$, the inequality $\mathrm{GFCD}(\mathrm{R}\Gamma) \leqslant \mathrm{GFCD}(R) + \mathrm{cd}_R\Gamma$ holds, see Theorem 4.9.

2. Notation and terminology

By a complex $\mathbf{A}$ of R-modules, we mean a sequence of homomorphisms of R-modules

$$\mathbf{A} : \cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \longrightarrow \cdots ,$$

such that $d_i d_{i+1} = 0$ for every $i \in \mathbb{Z}$. We write $\mathrm{Z}_i(\mathbf{A})$ for $\mathrm{Ker}(d_i)$ and $\mathrm{C}_i(\mathbf{A})$ for $\mathrm{Coker}(d_{i+1})$. $\mathrm{H}_i(\mathbf{A})$ denotes the i-th homology module of $\mathbf{A}$, and we say that $\mathbf{A}$ is acyclic, when $\mathrm{H}_i(\mathbf{A}) = \mathbf{0}$ for all $i \in \mathbb{Z}$. Following Avramov and Foxby [31], an R-complex $\mathbf{F}$ of flat R-modules is called semi-flat provided $- \otimes_R \mathbf{F}$ preserves acyclicity. Recall that an R-module C is said to be cotorsion if $\mathrm{Ext}^1_R(F, C)$ vanishes for every flat R-module F. When $\mathbf{C}$ is an R-complex of cotorsion R-modules, we say that $\mathbf{C}$ is semi-cotorsion if $\mathrm{Hom}_R(\mathbf{F}, \mathbf{C})$ is acyclic for every semi-flat complex $\mathbf{F}$, see [32]. By definition, an R-complex $\mathbf{K}$ is semi-flat-cotorsion provided that it is simultaneously semi-flat and semi-cotorsion, see [33]. Recall from [28] that a *semi-flat-cotorsion replacement* of an R-complex $\mathbf{A}$ means a semi-flat-cotorsion R-complex that is isomorphic to $\mathbf{A}$ in $\mathcal{D}(R)$, i.e. the derived category of complexes over R.

2.1 Gorenstein flat-cotorsion modules

When it is said that a module is flat-cotorsion, we mean that it is flat and cotorsion at the same time. An acyclic R-complex $\mathbf{L}$ of flat-cotorsion modules is called totally acyclic if for any flat-cotorsion R-module K, the complexes $\mathrm{Hom}_R(K, \mathbf{L})$ and $\mathrm{Hom}_R(\mathbf{L}, K)$ are acyclic. It is noteworthy that if $\mathbf{L}$ is an acyclic complex of flat-cotorsion R-modules, then for every flat-cotorsion R-module K, the complex $\mathrm{Hom}_R(K, \mathbf{L})$ automatically is acyclic, because by [34, Theorem 1.3] the kernels of an acyclic complex of cotorsion modules is cotorsion. We say that an R-module is *Gorenstein flat-cotorsion* if it is isomorphic to $Z_0(\mathbf{L})$ for some totally acyclic complex $\mathbf{L}$ of flat-cotorsion modules, [27]. Clearly, any flat-cotorsion module is Gorenstein flat-cotorsion. The closedness property of the class of Gorenstein flat-cotorsion modules with respect to direct summands and finite direct sums is proved in [28, Proposition 3.3]. If $\mathbf{L}$ is a totally acyclic complex of flat-cotorsion R-modules, then by symmetry, all the kernels, cokernels and the images of the differentials of $\mathbf{L}$ are Gorenstein flat-cotorsion.

2.2 Group rings

If Γ is a group, then $\mathrm{R}\Gamma$ denotes the group ring formed by Γ over R. By an $\mathrm{R}\Gamma$-module, we mean an R-module A with the action of Γ on it. An important point to note is that over a group ring $\mathrm{R}\Gamma$, every right module A can be converted in a canonical way into a left module by setting $x.a = ax^{-1}$, $\;a \in A, x \in \Gamma$ and vice versa. Therefore, when discussing the category of $\mathrm{R}\Gamma$-modules, we refrain from writing the right and left attributes. When Γ' is a subgroup of Γ, $\mathrm{R}\Gamma'$ is a subring of $\mathrm{R}\Gamma$, and so if A is an $\mathrm{R}\Gamma$-module, then we can restrict it to an $\mathrm{R}\Gamma'$-module. Moreover, if B is an $\mathrm{R}\Gamma'$-module, we have the $\mathrm{R}\Gamma$-modules $\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} B$ and $\mathrm{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, B)$, which are called the induced and coinduced modules, respectively. It is well-known that there is an inclusion $\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} B \hookrightarrow \mathrm{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, B)$, which is an isomorphism whenever the index of Γ' in Γ is finite, see [35]. In the special case where R is viewed as an $\mathrm{R}\Gamma$-module with trivial Γ-action, the projective dimension (the flat dimension, respectively) of R, denoted $\mathrm{cd}_R\Gamma$ ($\mathrm{hd}_R\Gamma$, respectively), is called the cohomological dimension (homological dimension, respectively) of Γ over R. Also note that the exactness of any exact sequence of $\mathrm{R}\Gamma$-modules is preserved by restricting it to $\mathrm{R}\Gamma'$. By definition, an exact sequence of $\mathrm{R}\Gamma$-modules

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0\ ,$$

is called $\mathrm{R}\Gamma'$-split if there exists a homomorphism $\beta' : C \to B$ of $\mathrm{R}\Gamma'$-modules with $\beta\beta' = \mathrm{id}_C$.

The following lemma, which is sometimes called the *Eckmann-Shapiro* lemma, is a direct consequence of [36, Corollary 2.8.4].

Lemma 2.1. *Let Γ' be a subgroup of an arbitrary group Γ. If A is an $\mathrm{R}\Gamma$-module and B is an $\mathrm{R}\Gamma'$-module, then for every non-negative integer n*

- *i)* $\operatorname{Ext}^n_{\mathrm{R}\Gamma}(\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} B, A) \cong \operatorname{Ext}^n_{\mathrm{R}\Gamma'}(B, A)$.
- *ii)* $\operatorname{Ext}^n_{\mathrm{R}\Gamma}(A, \operatorname{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, B)) \cong \operatorname{Ext}^n_{\mathrm{R}\Gamma'}(A, B)$.

2.3 Gorenstein homological dimension

An R-module G is said to be Gorenstein flat if there is a complex $\mathbf{F}$ of flat R-modules with $G \cong \mathrm{C}_0(\mathbf{F})$ such that $\mathrm{H}_i(\mathbf{F}) = 0 = \mathrm{H}_i(\mathbf{F} \otimes_R E)$, for any injective R-module E and for any $i \in \mathbb{Z}$. For an R-module A, the Gorenstein flat dimension of A, $\operatorname{Gfd}_R A$, is defined as

$$\operatorname{Gfd}_R A = \inf \left\{ n \in \mathbb{Z} \;\middle|\; \begin{array}{l} \text{There is an exact sequence of } R\text{-modules} \\ 0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_0 \longrightarrow A \longrightarrow 0 \\ \text{where the } G_i \text{ are Gorenstein flat} \end{array} \right\}.$$

The Gorenstein flat dimension of R as an $\mathrm{R}\Gamma$-module with trivial Γ-action is called the *Gorenstein homological dimension* of Γ over R and is denoted by $\operatorname{Ghd}_R\Gamma$.

3. Gorenstein flat-cotorsion modules over group rings

This section is devoted to the description of Gorenstein flat-cotorsion modules for group rings. The central role of flat-cotorsion modules in the article is our main reason for starting with the following lemmas.

Lemma 3.1. *Suppose Γ is a group and K is an $\mathrm{R}\Gamma$-module. If K is a flat-cotorsion, then K is a flat-cotorsion over $\mathrm{R}\Gamma'$ for every subgroup Γ' of Γ.*

Proof. First, we note that since K is flat as an $\mathrm{R}\Gamma$-module, it is also flat as an $\mathrm{R}\Gamma'$-module. Therefore, it is enough to show that K is a cotorsion $\mathrm{R}\Gamma'$-module. To this end, consider the flat $\mathrm{R}\Gamma'$-module F'. Clearly, the $\mathrm{R}\Gamma$-module $\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} F'$ is flat, so we have $\operatorname{Ext}^1_{\mathrm{R}\Gamma}(\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} F', K) = 0$. But $\operatorname{Ext}^1_{\mathrm{R}\Gamma}(\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} F', K) \cong \operatorname{Ext}^1_{\mathrm{R}\Gamma'}(F', K)$, by [37, Theorem 10.74], hence $\operatorname{Ext}^1_{\mathrm{R}\Gamma'}(F', K) = 0$. Therefore, K as an $\mathrm{R}\Gamma'$-module is cotorsion, hence flat-cotorsion. □

Lemma 3.2. *Let $\Gamma' \leqslant \Gamma$ be a finite-index subgroup of an arbitrary group Γ and let K be an $\mathrm{R}\Gamma'$-module. Then K is flat-cotorsion if and only if $\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} K$ is flat-cotorsion as an $\mathrm{R}\Gamma$-module.*

Proof. Assume that K is a flat-cotorsion $\mathrm{R}\Gamma'$-module. Since $\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} K$ is flat as an $\mathrm{R}\Gamma$-module, to prove the "only if" part, it suffices to show that it is also a cotorsion $\mathrm{R}\Gamma$-module. Therefore, take a flat $\mathrm{R}\Gamma$-module F. We have the following isomorphisms in which the second isomorphism is obtained from Lemma 2.1

$$\begin{aligned} \operatorname{Ext}^1_{\mathrm{R}\Gamma}(F, \mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} K) &\cong \operatorname{Ext}^1_{\mathrm{R}\Gamma}(F, \operatorname{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, K)) \\ &\cong \operatorname{Ext}^1_{\mathrm{R}\Gamma'}(F, K). \end{aligned}$$

The fact that F is flat as an $R\Gamma'$-module implies that $\mathrm{Ext}^1_{R\Gamma}(F, R\Gamma \otimes_{R\Gamma'} K) = 0$, as desired.

For the "if" part, consider the $R\Gamma$-exact sequence

$$0 \longrightarrow E \longrightarrow R\Gamma \otimes_{R\Gamma'} K \xrightarrow{\pi} K \longrightarrow 0,$$

where π is the canonical epimorphism, which is known to be $R\Gamma'$-split see [35, III.3]. The closedness of the class of flat-cotorsion modules with respect to direct summands now gives the result. □

Remark 1. Suppose $\Gamma' \leqslant \Gamma$ is a subgroup of Γ. As stated in Subsection 2.2, if the index of Γ' in Γ is finite, then for every $R\Gamma'$-module K there is a natural $R\Gamma$-isomorphism $\mathrm{Hom}_{R\Gamma'}(R\Gamma, K) \cong R\Gamma \otimes_{R\Gamma'} K$. Therefore, we can immediately conclude from Lemma 3.1 that an $R\Gamma'$-module K is flat-cotorsion if and only if the $R\Gamma$-module $\mathrm{Hom}_{R\Gamma'}(R\Gamma, K)$ is so. Throughout the article, this point will be used without reference.

Proposition 3.3. *Let $\Gamma' \leqslant \Gamma$ be a finite-index subgroup of an arbitrary group Γ and let G be an $R\Gamma$-module. If G is Gorenstein flat-cotorsion, then it is Gorenstein flat-cotorsion as an $R\Gamma'$-module.*

Proof. G is Gorenstein flat-cotorsion as an $R\Gamma$-module. Therefore, according to [28, Lemma 3.2], G is a cotorsion $R\Gamma$-module. So, as seen in the proof of the Lemma 3.1, it is also cotorsion as an $R\Gamma'$-module. Now, take a flat-cotorsion $R\Gamma'$-module K'. By Lemma 3.2, the $R\Gamma$-module $R\Gamma \otimes_{R\Gamma'} K'$ is flat-cotorsion. Consider the isomorphisms

$$\begin{aligned} \mathrm{Ext}^i_{R\Gamma}(G, R\Gamma \otimes_{R\Gamma'} K') &\cong \mathrm{Ext}^i_{R\Gamma}(G, \mathrm{Hom}_{R\Gamma'}(R\Gamma, K')) \\ &\cong \mathrm{Ext}^i_{R\Gamma'}(G, K'), \end{aligned}$$

where the second isomorphism is obtained from Lemma 2.1. According to [28, Lemma 3.2], we have $\mathrm{Ext}^{i\geqslant 1}_{R\Gamma'}(G, K') = 0$. On the other hand, by [28, Lemma 3.2], there exists an exact sequence

$$\mathbf{L}: \ 0 \longrightarrow G \longrightarrow L_0 \longrightarrow L_{-1} \longrightarrow L_{-2} \longrightarrow \cdots, \qquad (*)$$

where, the L_i are flat-cotorsion $R\Gamma$-modules and $\mathrm{Hom}_{R\Gamma}(\mathbf{L}, K)$ is acyclic for every flat-cotorsion $R\Gamma$-module K. It is clear that, $(*)$ is an exact sequence of $R\Gamma'$-modules and by Lemma 3.1, the L_i are flat-cotorsion $R\Gamma'$-modules. Now, using the isomorphisms

$$\begin{aligned} \mathrm{Hom}_{R\Gamma'}(\mathbf{L}, K') &\cong \mathrm{Hom}_{R\Gamma'}(R\Gamma \otimes_{R\Gamma} \mathbf{L}, K') \\ &\cong \mathrm{Hom}_{R\Gamma}(\mathbf{L}, \mathrm{Hom}_{R\Gamma'}(R\Gamma, K')) \\ &\cong \mathrm{Hom}_{R\Gamma}(\mathbf{L}, R\Gamma \otimes_{R\Gamma'} K'), \end{aligned}$$

we conclude that $\mathrm{Hom}_{R\Gamma'}(\mathbf{L}, K')$ is exact. Therefore, another use of [28, Lemma 3.2] yields that G is a Gorenstein flat-cotorsion $R\Gamma'$-module. □

Proposition 3.4. *Let* $\Gamma' \leqslant \Gamma$ *be a subgroup of finite index. Then an* $\mathrm{R}\Gamma$*-module* G *is Gorenstein flat-cotorsion provided it is flat-cotorsion as an* $\mathrm{R}\Gamma'$*-module.*

Proof. Consider the $\mathrm{R}\Gamma'$-split $\mathrm{R}\Gamma$-exact sequence

$$0 \longrightarrow J_0 \longrightarrow \mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} G \xrightarrow{\pi} G \longrightarrow 0.$$

By assumption and Lemma 3.2 the $\mathrm{R}\Gamma$-module $\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} G$ is flat-cotorsion and therefore by Lemma 3.1, it is flat-cotorsion as an $\mathrm{R}\Gamma'$-module. So, J_0 is a flat-cotorsion $\mathrm{R}\Gamma'$-module. By repeating this process for J_0 instead of G, it is possible to obtain the $\mathrm{R}\Gamma'$-split exact sequence

$$0 \longrightarrow J_1 \longrightarrow L_1 \longrightarrow J_0 \longrightarrow 0,$$

of $\mathrm{R}\Gamma$-modules such that L_1 is a flat-cotorsion $\mathrm{R}\Gamma$-module and J_1 is a flat-cotorsion $\mathrm{R}\Gamma'$-module. In this way, we get an acyclic complex of $\mathrm{R}\Gamma$-modules

$$\cdots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow G \longrightarrow 0, \qquad (\star)$$

in which the L_i are flat-cotorsion and it is split over $\mathrm{R}\Gamma'$. On the other hand, we have the $\mathrm{R}\Gamma'$-split exact sequence of $\mathrm{R}\Gamma$-modules

$$0 \longrightarrow G \xrightarrow{\iota} \mathrm{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, G) \longrightarrow J_{-1} \longrightarrow 0,$$

where ι is the canonical monomorphism. Note that since G is a flat-cotorsion $\mathrm{R}\Gamma'$-module, the $\mathrm{R}\Gamma$-module $\mathrm{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, G)$ is also flat-cotorsion, and so J_{-1} is flat-cotorsion as an $\mathrm{R}\Gamma'$-module. In the same way and by substituting J_{-1} instead of G, one can obtain an $\mathrm{R}\Gamma'$-split exact sequence of $\mathrm{R}\Gamma$-modules

$$0 \longrightarrow J_{-1} \longrightarrow L_{-1} \longrightarrow J_{-2} \longrightarrow 0,$$

where L_{-1} is a flat-cotorsion $\mathrm{R}\Gamma$-module and J_{-2} is a flat-cotorsion $\mathrm{R}\Gamma'$-module. Therefore, the exact sequence of $\mathrm{R}\Gamma$-modules

$$0 \longrightarrow G \longrightarrow L_{-1} \longrightarrow L_{-2} \longrightarrow \cdots, \qquad (\star\star)$$

is obtained, in which, the L_i are flat-cotorsion $\mathrm{R}\Gamma$-modules. Moreover, this sequence is split over $\mathrm{R}\Gamma'$. Splicing together the sequences $(\star)$ and $(\star\star)$, an acyclic complex of flat-cotorsion $\mathrm{R}\Gamma$-modules

$$\mathbf{L} : \cdots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow L_{-1} \longrightarrow L_{-2} \longrightarrow \cdots,$$

can be obtained such that $G = \mathrm{Ker}(L_{-1} \longrightarrow L_{-2})$. Now assume that K is a flat-cotorsion $\mathrm{R}\Gamma$-module. We claim that the functor $\mathrm{Hom}_{\mathrm{R}\Gamma}(-, K)$ preserves the exactness of $\mathbf{L}$. Since K is a flat $\mathrm{R}\Gamma$-module, by Lazard's Theorem, K can be written as a direct limit of a family of finitely generated free $\mathrm{R}\Gamma$-modules. Clearly, for every free $R\Gamma$-module P, there exists an $R\Gamma$-isomorphism $P \cong R\Gamma \otimes_{R\Gamma'} P'$, where P' is a projective $R\Gamma'$-module. Now, since the functor $R\Gamma \otimes_{R\Gamma'} -$ commutes with direct limits, so K is $R\Gamma$-isomorphic to $R\Gamma \otimes_{R\Gamma'} K'$ for some flat $R\Gamma'$-module K'. Hence, there are the isomorphisms of complexes:

$$\begin{aligned} \operatorname{Hom}_{R\Gamma}(\mathbf{L}, K) &\cong \operatorname{Hom}_{R\Gamma}(\mathbf{L}, R\Gamma \otimes_{R\Gamma'} K') \\ &\cong \operatorname{Hom}_{R\Gamma}(\mathbf{L}, \operatorname{Hom}_{R\Gamma'}(R\Gamma, K')) \\ &\cong \operatorname{Hom}_{R\Gamma'}(R\Gamma \otimes_{R\Gamma} \mathbf{L}, K') \\ &\cong \operatorname{Hom}_{R\Gamma'}(\mathbf{L}, K'). \end{aligned}$$

The construction of $\mathbf{L}$ shows that the last complex is exact, so the first complex is also exact. Finally, by Subsection 2.1 the complex $\operatorname{Hom}_{R\Gamma}(K, \mathbf{L})$ is automatically exact, so the acyclic complex $\mathbf{L}$ is a totally acyclic complex of flat-cotorsion $R\Gamma$-modules. Therefore, G is a Gorenstein flat-cotorsion $R\Gamma$-module. □

Example 3.5. Consider a finite group Γ with trivial subgroup $\Gamma' = \{e\}$. Then $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \frac{\mathbb{Q}}{\mathbb{Z}})$ can be viewed as an $\mathbb{Z}\Gamma$-module where Γ acts trivially on it. Since, by [38, Lemma 3.2.3], $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \frac{\mathbb{Q}}{\mathbb{Z}})$ is a flat-cotorsion $\mathbb{Z}\Gamma'$-module, therefore the previous proposition shows that it is a Gorenstein flat-cotorsion $\mathbb{Z}\Gamma$-module.

Lemma 3.6. *Let $\Gamma' \leqslant \Gamma$ be a subgroup of finite index, and let G be an $R\Gamma'$-module. Then G is Gorenstein flat-cotorsion if and only if $R\Gamma \otimes_{R\Gamma'} G$ is a Gorenstein flat-cotorsion $R\Gamma$-module.*

Proof. If G is a Gorenstein flat-cotorsion $R\Gamma'$-module, then by definition there exists an acyclic complex of flat-cotorsion $R\Gamma'$-modules

$$\mathbf{L} : \cdots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow L_{-1} \longrightarrow L_{-2} \longrightarrow \cdots,$$

with $G = \operatorname{Ker}(L_{-1} \longrightarrow L_{-2})$. Moreover, the complexes $\operatorname{Hom}_{R\Gamma'}(\mathbf{L}, K')$ and $\operatorname{Hom}_{R\Gamma'}(K', \mathbf{L})$ are exact for every flat-cotorsion $R\Gamma'$-module K'. Applying the functor $R\Gamma \otimes_{R\Gamma'} -$ to $\mathbf{L}$ and using Lemma 3.2, the acyclic complex $R\Gamma \otimes_{R\Gamma'} \mathbf{L}$ of flat-cotorsion $R\Gamma$-modules with $R\Gamma \otimes_{R\Gamma'} G = \operatorname{Ker}(R\Gamma \otimes_{R\Gamma'} L_{-1} \longrightarrow R\Gamma \otimes_{R\Gamma'} L_{-2})$ is obtained. Now pick a flat-cotorsion $R\Gamma$-module K. By the hypothesis, the right side of the isomorphism $\operatorname{Hom}_{R\Gamma}(R\Gamma \otimes_{R\Gamma'} \mathbf{L}, K) \cong \operatorname{Hom}_{R\Gamma'}(\mathbf{L}, K)$ is exact, therefore the left side is also exact. In the same way, by using the isomorphisms

$$\operatorname{Hom}_{R\Gamma}(K, R\Gamma \otimes_{R\Gamma'} \mathbf{L}) \cong \operatorname{Hom}_{R\Gamma}(K, \operatorname{Hom}_{R\Gamma'}(R\Gamma, \mathbf{L})) \cong \operatorname{Hom}_{R\Gamma'}(K, \mathbf{L}),$$

we conclude that $\operatorname{Hom}_{R\Gamma}(K, R\Gamma \otimes_{R\Gamma'} \mathbf{L})$ is exact. Thus $R\Gamma \otimes_{R\Gamma'} \mathbf{L}$ is a totally acyclic complex of flat-cotorsion $R\Gamma$-modules and therefore $R\Gamma \otimes_{R\Gamma'} G$ is Gorenstein flat-cotorsion.

The converse follows from Lemma 3.1 and [28, Proposition 3.3] and the fact that G is a direct summand of $R\Gamma \otimes_{R\Gamma'} G$ as an $R\Gamma'$-module. □

Below, we will examine how the property of being Gorenstein flat-cotorsion ascends and descends between the group Γ, and a given finite-index subgroup Γ'. To this end, we need the following theorem.

Theorem 3.7. *Let $\Gamma' \leqslant \Gamma$ be a finite-index subgroup of an arbitrary group Γ. Then every $\mathrm{R}\Gamma$-module G which is Gorenstein flat-cotorsion over $\mathrm{R}\Gamma'$ is Gorenstein flat-cotorsion over $\mathrm{R}\Gamma$.*

Proof. Since G is a Gorenstein flat-cotorsion $\mathrm{R}\Gamma'$-module, there exists an exact sequence of $\mathrm{R}\Gamma'$-modules

$$0 \longrightarrow J_0 \longrightarrow F_0 \longrightarrow G \longrightarrow 0,$$

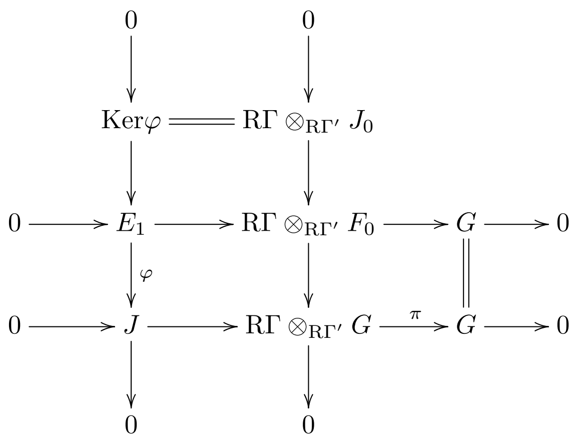
where F_0 is flat-cotorsion and J_0 is Gorenstein flat-cotorsion. Applying the functor $\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} -$ to this exact sequence, we get the exact sequence

$$0 \longrightarrow \mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} J_0 \longrightarrow \mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} F_0 \longrightarrow \mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} G \longrightarrow 0,$$

of $\mathrm{R}\Gamma$-modules. On the other hand, we consider the $\mathrm{R}\Gamma'$-split exact sequence

$$0 \longrightarrow J \longrightarrow \mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} G \xrightarrow{\pi} G \longrightarrow 0,$$

of $\mathrm{R}\Gamma$-modules . So, similar to the proof of [38, Theorem 3.1.9], it can be shown that there is a commutative diagram



with exact rows and columns. By assumption and Lemma 3.6, $\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} G$ is a Gorenstein flat-cotorsion $\mathrm{R}\Gamma'$-module, and since the bottom row is $\mathrm{R}\Gamma'$-split, we conclude from [28, Proposition 3.3] that J is Gorenstein flat-cotorsion as an $\mathrm{R}\Gamma'$-module. Moreover, $\mathrm{Ker}\varphi = \mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} J_0$ is also Gorenstein flat-cotorsion $\mathrm{R}\Gamma'$-module, so it follows from [28, Proposition 3.4] applied to the exact sequence

$$0 \longrightarrow \mathrm{Ker}\varphi \longrightarrow E_1 \longrightarrow J \longrightarrow 0,$$

that E_1 is a Gorenstein flat-cotorsion $\mathrm{R}\Gamma'$-module. Repeating the argument above for E_1 instead of G, leads to a short exact sequence

$$0 \longrightarrow J_1 \longrightarrow F_1 \longrightarrow E_1 \longrightarrow 0,$$

of $\mathrm{R}\Gamma'$-modules, in which F_1 and J_1 are flat-cotorsion and Gorenstein flat-cotorsion $\mathrm{R}\Gamma'$-modules, respectively. After that, the exact sequence

$$0 \longrightarrow E_2 \longrightarrow \mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} F_1 \longrightarrow E_1 \longrightarrow 0,$$

of $\mathrm{R}\Gamma$-modules is obtained, where E_2 is Gorenstein flat-cotorsion over $\mathrm{R}\Gamma'$. If this process continues, one can provide the exact sequence

$$\cdots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow G \longrightarrow 0, \qquad (\dagger)$$

of $\mathrm{R}\Gamma$-modules such that by Lemma 3.2, the L_i are flat-cotorsion and the syzygies are all Gorenstein flat-cotorsion $\mathrm{R}\Gamma'$-modules.

In order to continue the proof, we need to use the dual of the above argument. To be precise, consider the exact sequence of $\mathrm{R}\Gamma'$-modules

$$0 \longrightarrow G \longrightarrow F_{-1} \longrightarrow J_{-1} \longrightarrow 0,$$

where F_{-1} and J_{-1} are flat-cotorsion and Gorenstein flat-cotorsion, respectively. By applying the functor $\mathrm{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, -)$ to it, the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, G) \longrightarrow \mathrm{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, F_{-1}) \longrightarrow \mathrm{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, J_{-1}) \longrightarrow 0,$$

of $\mathrm{R}\Gamma$-modules is obtained, which together with the canonical exact sequence

$$0 \longrightarrow G \xrightarrow{\iota} \mathrm{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, G) \longrightarrow J \longrightarrow 0,$$

leads to the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G & \xrightarrow{\iota} & \mathrm{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, G) & \longrightarrow & J & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & G & \longrightarrow & \mathrm{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, F_{-1}) & \longrightarrow & E_{-1} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, J_{-1}) & = & \mathrm{Coker}\varphi & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns, see the proof of [38, Proposition 2.2.1]. We know that the first row in the above diagram is $R\Gamma'$-split. Moreover, since the index of Γ' in Γ is finite, $R\Gamma \otimes_{R\Gamma'} G \cong \mathrm{Hom}_{R\Gamma'}(R\Gamma, G)$ and hence by Lemma 3.6, $\mathrm{Hom}_{R\Gamma'}(R\Gamma, G)$ is a Gorenstein flat-cotorsion $R\Gamma$-module. It follows from Proposition 3.3 in view of [28, Proposition 3.3], that the $R\Gamma'$-module J is Gorenstein flat-cotorsion, and so is E_{-1}, by [28, Proposition 3.4]. Repeating the above argument, as we did in the first part of the proof, shows that there exists an exact sequence of $R\Gamma$-modules

$$0 \longrightarrow G \longrightarrow L_{-1} \longrightarrow L_{-2} \longrightarrow \cdots , \qquad (\ddagger)$$

where the L_i are flat-cotorsion and the syzygies are all Gorenstein flat-cotorsion $R\Gamma'$-module. Splicing together the sequences (†) and (‡), we can provide an acyclic complex of flat-cotorsion $R\Gamma$-modules

$$\mathbf{L} : \cdots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow L_{-1} \longrightarrow L_{-2} \longrightarrow \cdots ,$$

in which $G = \mathrm{Ker}(L_{-1} \longrightarrow L_{-2})$. Moreover, $\mathbf{L}$ splits when viewed as an exact sequence of $R\Gamma'$-modules. Thus, $\mathbf{L}$ remains exact when applying the functors $\mathrm{Hom}_{R\Gamma'}(-, K')$ and $\mathrm{Hom}_{R\Gamma'}(K', -)$ for any flat-cotorsion $R\Gamma'$-module K'. Take a flat-cotorsion $R\Gamma$-module K. Since K is a flat $R\Gamma$-module, there exists an $R\Gamma$-isomorphism $K \cong R\Gamma \otimes_{R\Gamma'} K'$, where K' is a flat $R\Gamma'$-module. Now, similar to the proof of Proposition 3.4, it can be proved that the functors $\mathrm{Hom}_{R\Gamma}(-, K)$ and $\mathrm{Hom}_{R\Gamma}(K, -)$ preserve the acyclicity of $\mathbf{L}$. It follows that $\mathbf{L}$ is a totally acyclic complex of flat-cotorsion $R\Gamma$-modules, and so G is a Gorenstein flat-cotorsion $R\Gamma$-module, as desired. □

We end the section with the following result, which is a direct consequence of Proposition 3.3 and Theorem 3.7.

Corollary 3.8. *Let $\Gamma' \leqslant \Gamma$ be a subgroup of finite index, and let G be an $R\Gamma$-module. The necessary and sufficient condition for G to be Gorenstein flat-cotorsion is that it is Gorenstein flat-cotorsion as an $R\Gamma'$-module.*

4. Gorenstein flat-cotorsion dimension with group ring coefficients

The main theme of this section is to investigate the behavior of invariants that are defined based on Gorenstein flat-cotorsion modules over group rings. The Gorenstein flat-cotorsion dimension was originally defined for complexes over associative rings by Christensen et al. in [28]. Their definition was given in terms of replacement of complexes, as followes.

Definition 4.1. The Gorenstein flat-cotorsion dimension of an R-complex $\mathbf{A}$, denoted by $\mathrm{Gfcd}_R\mathbf{A}$, is defined as

$$\mathrm{Gfcd}_R\mathbf{A} = \inf\left\{ n \in \mathbb{Z} \;\middle|\; \begin{array}{c} \mathrm{H}_i(\mathbf{X}) = 0 \text{ for all } i > n \text{ and } \mathrm{C}_n(\mathbf{X}) \\ \text{is Gorenstein flat-cotorsion for some} \\ \text{semi-flat-cotorsion replacement } \mathbf{X} \text{ of } \mathbf{A} \end{array} \right\}.$$

By convention, we set $\inf \varnothing = \infty$.

Remark 2. Supplementary information on Gorenstein flat-cotorsion dimension can be found in [28–30]. However, the following facts are worth mentioning.

- (i) Several characterizations of complexes of finite Gorenstein flat-cotorsion dimension are given in [28]. In particular, according to [28, Theorem 4.5], one can conclude that for every R-module A of finite Gorenstein flat-cotorsion dimension, there is an equality

  $$\mathrm{Gfcd}_R A = \sup\left\{n \mid \mathrm{Ext}^n_R(A, K) \neq 0 \text{ for some flat-cotorsion } R\text{-module } K\right\}.$$

- (ii) A cotorsion R-module C has Gorenstein flat-cotorsion dimension less than or equal to n ($n \in \mathbb{Z},\ \ n \geqslant 0$) if and only if there is an exact sequence

  $$0 \longrightarrow G \longrightarrow K_{n-1} \longrightarrow \cdots \longrightarrow K_0 \longrightarrow C \longrightarrow 0,$$

  with all K_i flat-cotorsion modules and with G a Gorenstein flat-cotorsion module. Therefore, if C is a cotorsion R-module, then the condition $\mathrm{Gfcd}_R C = 0$ is equivalent to C being Gorenstein flat-cotorsion, see [28, Remark 4.6].

- (iii) For any R-module A, there are inequalities $\mathrm{Gfcd}_R A \leqslant \mathrm{Gfd}_R A \leqslant \mathrm{fd}_R A$ and if each of these dimensions is finite, it is equal to the values on its left side, see [28, Theorem B].

Proposition 4.2. *Let $\Gamma' \leqslant \Gamma$ be a subgroup of finite index. For any $\mathrm{R}\Gamma$-module A, $\mathrm{Gfcd}_{\mathrm{R}\Gamma'} A \leqslant \mathrm{Gfcd}_{\mathrm{R}\Gamma} A$, with equality when $\mathrm{Gfcd}_{\mathrm{R}\Gamma} A$ is finite.*

Proof. The inequality $\mathrm{Gfcd}_{\mathrm{R}\Gamma'} A \leqslant \mathrm{Gfcd}_{\mathrm{R}\Gamma} A$ follows from the definition of Gorenstein flat-cotorsion dimension, combined with Lemma 3.1 and Proposition 3.3. Now assume $\mathrm{Gfcd}_{\mathrm{R}\Gamma} A = n < \infty$. Thus, by Remark 2 (i) there exists a flat-cotorsion $\mathrm{R}\Gamma$-module K such that $\mathrm{Ext}^n_{\mathrm{R}\Gamma}(A, K) \neq 0$. The isomorphisms

$$\begin{aligned} \mathrm{Ext}^n_{\mathrm{R}\Gamma'}(A, K) &\cong \mathrm{Ext}^n_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma} A, K) \\ &\cong \mathrm{Ext}^n_{\mathrm{R}\Gamma}(A, \mathrm{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, K)), \end{aligned}$$

and the fact that $\mathrm{Ext}^n_{\mathrm{R}\Gamma}(A, K)$ is a direct summand of $\mathrm{Ext}^n_{\mathrm{R}\Gamma}(A, \mathrm{Hom}_{\mathrm{R}\Gamma'}(\mathrm{R}\Gamma, K))$ show that $\mathrm{Ext}^n_{\mathrm{R}\Gamma'}(A, K) \neq 0$. But since K is flat-cotorsion as an $\mathrm{R}\Gamma'$-module, by Lemma 3.1, again Remark 2 (i) gives that $\mathrm{Gfcd}_{\mathrm{R}\Gamma'} A \geqslant n$, as desired. □

The following Lemma was proved over $\mathbb{Z}$ in Lemma 4.10 of [25], and the same proof works here.

Lemma 4.3. *Let* $\Gamma' \leqslant \Gamma$ *be of finite index. If an* $\mathrm{R}\Gamma$*-module* G *is Gorenstein flat, then it is Gorenstein flat as an* $\mathrm{R}\Gamma'$*-module.*

Corollary 4.4. *Assume that* $\Gamma' \leqslant \Gamma$ *is a subgroup of finite index of an arbitrary group* Γ*. For every* $\mathrm{R}\Gamma$*-module* A *there is an inequality* $\mathrm{Gfd}_{\mathrm{R}\Gamma'} A \leqslant \mathrm{Gfd}_{\mathrm{R}\Gamma} A$*, which is an equality if* $\mathrm{Gfd}_{\mathrm{R}\Gamma} A$ *is finite.*

Proof. The inequality $\mathrm{Gfd}_{\mathrm{R}\Gamma'} A \leqslant \mathrm{Gfd}_{\mathrm{R}\Gamma} A$ is clear, since every Gorenstein flat $\mathrm{R}\Gamma$-module also is Gorenstein flat as an $\mathrm{R}\Gamma'$-module by Lemma 4.3. Since the finiteness of $\mathrm{Gfd}_{\mathrm{R}\Gamma} A$ implies that $\mathrm{Gfd}_{\mathrm{R}\Gamma'} A$ is finite, the equalities $\mathrm{Gfd}_{\mathrm{R}\Gamma'} A = \mathrm{Gfcd}_{\mathrm{R}\Gamma'} A = \mathrm{Gfcd}_{\mathrm{R}\Gamma} A = \mathrm{Gfd}_{\mathrm{R}\Gamma} A$ follow from Remark 2 (iii) and Proposition 4.2. □

Notation. Given a ring R and a group Γ, we denote by $\widetilde{\mathrm{Ghd}}_{\mathrm{R}}\Gamma$ the Gorenstein flat-cotorsion dimension of the $\mathrm{R}\Gamma$-module R, where R is viewed as an $\mathrm{R}\Gamma$-module with trivial Γ-action.

The following assertions are all special cases of some results that have been proved so far in connection with the Gorenstein flat-cotorsion dimension of modules.

Remark 3. For a group Γ, the following statements hold.

- (i) By Remark 2 (iii), we have the inequality $\widetilde{\mathrm{Ghd}}_{\mathrm{R}}\Gamma \leqslant \mathrm{hd}_{\mathrm{R}}\Gamma$ and equality holds whenever the group Γ has finite homological dimension over R.
- (ii) In view of Remark 2 (iii), we can deduce that $\widetilde{\mathrm{Ghd}}_{\mathrm{R}}\Gamma$ refines the notion of Gorenstein homological dimension of Γ over R, and if $\mathrm{Ghd}_R\Gamma$ is finite, these will coincide.
- (iii) An equality, which resembles the well-known Serre's Theorem, is proved for $\widetilde{\mathrm{Ghd}}$. More precisely, if $\Gamma' \leqslant \Gamma$ is a finite-index subgroup, then there is an inequality $\widetilde{\mathrm{Ghd}}_{\mathrm{R}}\Gamma' \leqslant \widetilde{\mathrm{Ghd}}_{\mathrm{R}}\Gamma$, with equality when $\widetilde{\mathrm{Ghd}}_{\mathrm{R}}\Gamma$ is finite, see Corollary 3.8.

By definition, a group Γ is said to be locally finite if every finite subset of elements of Γ is contained in a finite subgroup of Γ. In the sequel, we will examine $\widetilde{\mathrm{Ghd}}_{\mathrm{R}}\Gamma$ when Γ is a finite or locally finite group.

Corollary 4.5. *Let* Γ *be a group without* R*-torsion. Then* $\widetilde{\mathrm{Ghd}}_{\mathrm{R}}\Gamma = 0$ *provided that* Γ *is locally finite.*

Proof. By [11, Proposition 4.12], a group Γ satisfies $\mathrm{hd}_R\Gamma = 0$ if and only if it is locally finite without R-torsion. Using this fact, in view of Remark 3 (i), we have $\widetilde{\mathrm{Ghd}}_{\mathrm{R}}\Gamma = 0$. □

The next corollary gives another homological characterization of finite groups.

Corollary 4.6. *Suppose* Γ *is a group with* $\mathbb{Z}\Gamma$ *coherent. Then the following statements are equivalent.*

(i) Γ *is finite.*

(ii) $\mathrm{Ghd}_R\Gamma = 0$*, for any commutative ring* R*.*

(iii) $\widetilde{\mathrm{Ghd}}_{\mathrm{R}}\Gamma = 0$*, for any commutative ring* R*.*

(iv) $\widetilde{\mathrm{Ghd}}_{\mathbb{Z}}\Gamma = 0$*.*

(v) $\mathrm{Ghd}_{\mathbb{Z}}\Gamma = 0$*.*

Proof. $(i) \Longrightarrow (ii)$ It is clear that if $\Gamma' = \{\mathrm{e}\}$ is the trivial subgroup of Γ, then R is Gorenstein flat as an $\mathrm{R}\Gamma'$-module, where R is an arbitrary commutative ring. Moreover, since Γ is finite, by [17, Theorem 2.6] (which is done over $\mathbb{Z}$, but the same proof works for R), R is also Gorenstein flat as an $\mathrm{R}\Gamma$-module and therefore $\mathrm{Ghd}_R\Gamma = 0$.

$(ii) \Longrightarrow (iii)$ Follows directly from Remark 3 (ii).

$(iii) \Longrightarrow (iv)$ It is trivial.

$(iv) \Longrightarrow (v)$ Since $\mathbb{Z}\Gamma$ is coherent, [28, Corollary 5.8] indicates that $\mathrm{Ghd}_{\mathbb{Z}}\Gamma$ is equal to $\widetilde{\mathrm{Ghd}}_{\mathbb{Z}}\Gamma$. Thus we have $\mathrm{Ghd}_Z\Gamma = 0$, according to the assumption.

$(v) \Longrightarrow (i)$ It is exactly Proposition 4.12 of [25]. □

For every associative, not necessarily commutative ring R (with unity), we denote the quantity

$$\sup\{\mathrm{Gfcd}_R A \mid A \text{ is a right } R\text{-module}\},$$

by $r.\mathrm{GFCD}(R)$, and refer to it as "the right global Gorenstein flat-cotorsion dimension of R". Similarly, by converting the attribute right to left, the left global Gorenstein flat-cotorsion dimension of R can be defind and is denoted by $l.\mathrm{GFCD}(R)$. In the case of $R \cong R^{\mathrm{op}}$, because the categories of right and left R-modules coincide, we avoid writing the letters r and l, and we use the symbol $\mathrm{GFCD}(R)$. We conclude this section by examining some properties of the invariant

$$\mathrm{GFCD}(\mathrm{R}\Gamma) = \sup\{\mathrm{Gfcd}_{\mathrm{R}\Gamma} A \mid A \text{ is an } \ \mathrm{R}\Gamma\text{-module}\},$$

where R is a ring and Γ is a group.

Proposition 4.7. *Suppose* $\Gamma' \leqslant \Gamma$ *is a finite-index subgroup of an arbitrary group* Γ*. Then* $\mathrm{GFCD}(\mathrm{R}\Gamma) = \mathrm{GFCD}(\mathrm{R}\Gamma')$*.*

Proof. To prove the inequality $\mathrm{GFCD}(\mathrm{R}\Gamma') \leqslant \mathrm{GFCD}(\mathrm{R}\Gamma)$, let us assume that $\mathrm{GFCD}(\mathrm{R}\Gamma)$ is finite, say n, because otherwise there is nothing to prove. Now take an arbitrary $\mathrm{R}\Gamma'$-module A. Since $\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} A$ is an $\mathrm{R}\Gamma$-module, we conclude from the assumption that $\mathrm{Gfcd}_{\mathrm{R}\Gamma}(\mathrm{R}\Gamma \otimes_{\mathrm{R}\Gamma'} A) \leqslant n$. On the other hand, using

Proposition 4.2 together with the fact that the $R\Gamma'$-module A is a direct summand of the $R\Gamma'$-module $R\Gamma \otimes_{R\Gamma'} A$, gives the inequalities

$$\mathrm{Gfcd}_{R\Gamma'} A \leqslant \mathrm{Gfcd}_{R\Gamma'}(R\Gamma \otimes_{R\Gamma'} A) \leqslant \mathrm{Gfcd}_{R\Gamma}(R\Gamma \otimes_{R\Gamma'} A) \leqslant n.$$

Therefore the desired ineuality is proved. For the coverse inequality, it suffices to assume that $\mathrm{GFCD}(R\Gamma')$ is finite and apply Proposition 4.2 again. □

Our next goal is to describe the relationship between $\mathrm{GFCD}(R)$ and $\mathrm{GFCD}(R\Gamma)$ by using $\mathrm{cd}_R\Gamma$. The following lemma, proved by Chouinard in [7], will be needed.

Lemma 4.8. *Let Γ be a group. Given $R\Gamma$-modules A and B with the property $\mathrm{Ext}^i_R(A,B) = 0$ for all $i > 0$, there are isomorphisms*

$$\mathrm{Ext}^n_{R\Gamma}(A,B) \cong \mathrm{Ext}^n_{R\Gamma}(R, \mathrm{Hom}_R(A,B)),$$

for all $n > 0$, where $\mathrm{Hom}_R(A,B)$ carries the diagonal Γ-action defined by

$$(\alpha\beta)(x) = \alpha[\beta(\alpha^{-1}x)], \ \ \alpha \in \Gamma, \ \ x \in A, \ \ \beta \in \mathrm{Hom}_R(A,B).$$

Theorem 4.9. *For an arbitrary group Γ, there is an inequality*

$$\mathrm{GFCD}(R\Gamma) \leqslant \mathrm{GFCD}(R) + \mathrm{cd}_R\Gamma.$$

Proof. Clearly, we may assume that both $\mathrm{GFCD}(R) = r$ and $\mathrm{cd}_R\Gamma = s$ are finite. First, we show that if C is a cotorsion $R\Gamma$-module, then $\mathrm{Gfcd}_{R\Gamma}C \leqslant r + s$. To this end, consider a partial minimal flat resolution (a flat resolution made of flat covers) of the $R\Gamma$-module C as

$$0 \longrightarrow G \longrightarrow F_{r-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow C \longrightarrow 0. \qquad (\dagger)$$

From the proof of Lemma 3.1, C is cotorsion as an R-module and also by assumption we have $\mathrm{Gfcd}_R C \leqslant r$. Therefore, by Remark 2 (ii), there is an exact sequence of R-modules

$$0 \longrightarrow G' \longrightarrow J_{r-1} \longrightarrow \cdots \longrightarrow J_0 \longrightarrow C \longrightarrow 0, \qquad (\dagger\dagger)$$

in which the J_i are flat-cotorsion and G' is Gorenstein flat-cotorsion. Moreover, when ($\dagger$) is viewed as an exact sequence of R-modules, in light of [39, Corollary 5.3.26], it follows that the F_i are all flat-cotorsion. If we apply [39, Corollary 8.6.4] to ($\dagger$) and ($\dagger\dagger$), we get an isomorphism of R-modules

$$G \oplus J_{r-1} \oplus F_{r-2} \oplus \cdots \cong G' \oplus F_{r-1} \oplus J_{r-2} \oplus \cdots,$$

and hence G' is a Gorenstein flat-cotorsion R-module, thanks to [28, Proposition 3.3]. Take a flat-cotorsion $R\Gamma$-module L, and view it as an R-module. By Lemma 3.1, L is a flat-cotorsion R-module, and so $\mathrm{Ext}^i_R(G,L) = 0$ for all $i > 0$.

Moreover, since $\mathrm{cd}_R\Gamma = s$ we have $\mathrm{Ext}^j_{\mathrm{R}\Gamma}(R, \mathrm{Hom}_R(G, L)) = 0$ for all $j > s$. Thus, Lemma 4.8 implies that $\mathrm{Ext}^j_{\mathrm{R}\Gamma}(G, L) = 0$ for all $j > s$, which in turn implies that $\mathrm{Gfcd}_{\mathrm{R}\Gamma}G \leqslant s$, by [28, Theorem A]. On the other hand, since G is clearly cotorsion as an $\mathrm{R}\Gamma$-module, so by Remark 2 (ii), there exists an exact sequence of $\mathrm{R}\Gamma$-modules

$$0 \longrightarrow H \longrightarrow L_{s-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow G \longrightarrow 0, \qquad (\ddagger)$$

where the L_i are flat-cotorsion and H is Gorenstein flat-cotorsion. By splicing it to $(\dagger)$, we obtain an exact sequence of $\mathrm{R}\Gamma$-modules

$$0 \longrightarrow H \longrightarrow F_{r+s-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow C \longrightarrow 0,$$

with the F_i flat-cotorsion and H a Gorenstein flat-cotorsion module. Therefore, if we use Remark 2 (ii) once again, the inequality $\mathrm{Gfcd}_{\mathrm{R}\Gamma}C \leqslant r + s$ is obtained.

Now suppose A is an arbitrary $\mathrm{R}\Gamma$-module. We claim that $\mathrm{Gfcd}_{\mathrm{R}\Gamma}A \leqslant r + s$. To prove this, we apply the same technique used in the last part of the proof of Theorem 3.3 of [29]. More precisely, let

$$\mathbf{F} : \cdots \longrightarrow F_{s+1} \longrightarrow F_s \longrightarrow \cdots \longrightarrow F_0 \longrightarrow A \longrightarrow 0,$$

be a minimal flat resolution of the $\mathrm{R}\Gamma$-module A. Since $\mathrm{C}_1(\mathbf{F})$ is cotorsion, hence the $\mathrm{R}\Gamma$-module $\mathrm{C}_{r+s+1}(\mathbf{F})$ must be Gorenstein flat-cotorsion. By [40, Corollary 4.10], there exists an exact sequence

$$0 \longrightarrow \mathbf{F} \longrightarrow \mathbf{C} \longrightarrow \mathbf{G} \longrightarrow 0\ ,$$

where $\mathbf{C}$ is semi-cotorsion and $\mathbf{G}$ is acyclic and semi-flat, and in view of [28, Fact 1.4] $\mathbf{C}$ is a semi-flat-cotorsion replacement of A. Also, considering the exact sequence of $\mathrm{R}\Gamma$-modules

$$0 \longrightarrow F_i \longrightarrow C_i \longrightarrow G_i \longrightarrow 0,$$

it is easy to see that, for all $i \geqslant 1$, G_i is cotorsion. So by applying [28, Lemma 5.6] to the exact sequence

$$\mathbf{G}_{\geqslant \mathbf{1}} : \cdots \longrightarrow G_3 \longrightarrow G_2 \longrightarrow G_1 \longrightarrow 0,$$

we infer that $\mathrm{C}_{r+s+1}(\mathbf{G})$ is flat-cotorsion. It follows that the short exact sequence

$$0 \longrightarrow \mathrm{C}_{r+s+1}(\mathbf{F}) \longrightarrow \mathrm{C}_{r+s+1}(\mathbf{C}) \longrightarrow \mathrm{C}_{r+s+1}(\mathbf{G}) \longrightarrow 0,$$

is split and so $\mathrm{C}_{r+s+1}(\mathbf{C})$ is Gorenstein flat-cotorsion. Hence $\mathrm{Gfcd}_{\mathrm{R}\Gamma}A$ is finite. Finally, we must show that $\mathrm{Gfcd}_{\mathrm{R}\Gamma}A \leqslant r + s$. But this holds from [28, Theorem 4.5] in view of the fact that A can be fitted into a short exact sequence

$$0 \longrightarrow A \longrightarrow C' \longrightarrow F' \longrightarrow 0,$$

with C' cotorsion and F' flat.

□

If R is a ring, its global Gorenstein dimension, $\text{Ggldim}(R)$, is defined as

$$\text{Ggldim}(R) = \sup\{\text{Gpd}_R A \mid A \text{ is an } R\text{-module}\},$$

where $\text{Gpd}_R A$ denotes the Gorenstein projective dimension of the R-module A, see [16]. Althought this definition is based on Gorenstein projective dimension of modules, it has been proven that it can be defined using the Gorenstein injective dimension of modules, see [15]. As a direct consequence of the above theorem and [29, Theorem 3.3], one has the following corollary.

Corollary 4.10. *If* Γ *is a group, then* $\text{GFCD}(\text{R}\Gamma) \leqslant \text{Ggldim}(R) + \text{cd}_R\Gamma$.

Example 4.11. Assume that R is an n-Gorenstein ring (meaning that R is Noetherian and has self-injective dimension at most n). According to [39, Theorem 12.3.1], $\text{Ggldim}(R) \leqslant n$. Therefore, by Corollary 4.10, for a given group Γ, the group ring $\text{R}\Gamma$ satisfies the inequality $\text{GFCD}(\text{R}\Gamma) \leqslant n + \text{cd}_R\Gamma$. In the special case that R is the ring of integers $\mathbb{Z}$, we have $\text{GFCD}(\mathbb{Z}\Gamma) \leqslant 1 + \text{cd}_{\mathbb{Z}}\Gamma$, because $\mathbb{Z}$ is 1-Gorenstein. In particular, by [35, Examples VIII 2], we have

- (i) If Γ is the trivial group, then $\text{GFCD}(\mathbb{Z}\Gamma) \leqslant 1$.
- (ii) If Γ is free and non-trivial, then $\text{GFCD}(\mathbb{Z}\Gamma) \leqslant 2$.

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