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Ricci Bi-Conformal Vector Fields on Siklos Spacetimes

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Abstract

Ricci bi-conformal vector fields have find their place in geometry as well as in physical applications. In this paper, we consider the Siklos spacetimes and we determine all the Ricci bi-conformal vector fields on these spaces.

Keywords: Conformal vector fields, Siklos spacetimes, Riemannian metrics.

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1. Introduction

Siklos [1] found a class of metrics on spacetimes which are solutions of Einstein's field equations with an Einstein-Maxwell source. They are a class of Petrov type N with cosmological constant $\Lambda < 0$. All of them admit a null non-twisting Killing field. In global coordinates (x, y, z, w), Siklos metrics are given by:

$$g = -\frac{3}{\Lambda z^2}\left(2dxdy + Hdy^2 + dz^2 + dw^2\right), \tag{1}$$

where $H = H(y, z, w)$ is an arbitrary smooth function [1, 2]. The class of Siklos spacetimes coincides with the subclass $(IV)_0$ of Kundt spacetimes [3]. Homogeneous Siklos metrics correspond to cases admitting at least four linearly independent Killing vector fields. They form five subclasses $I), \cdots, V)$. For each of

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subclasses $I), \cdots, V)$ we consider the special form of H [4] as follows:

$$\begin{aligned} I) &\quad H = A_{-2}(z,w), \\ II) &\quad H = A(z), \\ III) &\quad H = A(y)z^2, \\ IV) &\quad H = y^{2k-2}A(y^k z), \\ V) &\quad H = \pm z^k, \end{aligned}$$

where $A_{-2}(z,w)$ is a homogeneous function of degree -2 of variables z, w and A is a smooth function.

Let (M, g) be a smooth n-dimensional pseudo-Riemannian manifold. A vector field X on a Riemannian manifold (M, g) is said to be a Killing field if

$$\mathcal{L}_X g = 0,$$

where $\mathcal{L}_X$ is the Lie derivative in the direction of X. Killing vector fields were considered in [5]. Recently, various generalizations of Killing vector fields have been investigated. If there is a smooth function ψ on M that is named a potential function, such that $\mathcal{L}_X g = 2\psi g$ then vector field X on a Riemannian manifold (M, g) is called a conformal vector field. If the potential function $\psi = 0$, X is a Killing vector field. Conformal vector fields have been explained [6, 7]. A vector field X on M is called a Kerr-Schild vector field whenever

$$\mathcal{L}_X g = \alpha l \otimes l, \qquad \mathcal{L}_X l = \beta l,$$

where l is a null 1-form field and α, β are real smooth functions over M. Also, the generalized Kerr-Schild vector field is determined by

$$\mathcal{L}_X g = \alpha g + \beta l \otimes l, \qquad \mathcal{L}_X l = \gamma l,$$

where α, β, γ are smooth functions over M. Coll et al. [8] investigated the generalized Kerr-Schild vector field. A symmetric tensor h on M is said to be a square root of g if $h_{ik}h^k_j = g_{ij}$. Garcia-Parrado and Senovilla [9] by using the square root of g defined bi-conformal vector fields. A vector field X is called a bi-conformal vector field if it satisfies the following equations:

$$\mathcal{L}_X g = \alpha g + \beta h, \qquad \mathcal{L}_X h = \alpha h + \beta g,$$

where h is a symmetric square root of g and α, β are smooth functions. The functions α and β are called gauges [8, 9] of the symmetry and they play a role analogous to the factor ψ appearing in the definition of the conformal vector fields. After that, De et al. in [10] using the metric tensor field g and the Ricci tensor field S defined Ricci bi-conformal vector fields as follows:

Definition 1.1. A vector field X on a Riemannian manifold (M, g) is called Ricci bi-conformal vector field if it satisfies the following equations

$$(\mathcal{L}_X g)(Y, Z) = \alpha g(Y, Z) + \beta S(Y, Z), \tag{2}$$

and

$$(\mathcal{L}_X S)(Y, Z) = \alpha S(Y, Z) + \beta g(Y, Z), \tag{3}$$

for any vector fields Y, Z and some smooth functions α and β, where S is the Ricci tensor of M with respect to metric g.

Also, Ricci soliton is introduced by Hamilton [11] as follows:

$$\mathcal{L}_X g + S = \lambda g, \quad \lambda \in \mathbb{R},$$

which is a natural generalization of Einstein metric. For more details, see [12–18]. Calvaruso in [4, 19–21] investigated the Ricci soliton on homogeneous Siklos spacetimes. Recently, various generalizations of Killing vector fields have been investigated.

Motivated by [4, 10, 19–21], we study the Ricci bi-conformal vector fields on Siklos spacetime. This paper is organized as follows. In Section 2 we recall some necessary concepts on Siklos spacetime which be used throughout this paper. In Section 3 we give the main results and their proofs.

2. Preliminaries

We may refer to [2, 19] for the essential information concerning the Levi-Civita connection and curvature of an arbitrary Siklos metric g. Let

$$\partial_1 = \frac{\partial}{\partial x}, \ \partial_2 = \frac{\partial}{\partial y}, \ \partial_3 = \frac{\partial}{\partial z}, \ \partial_4 = \frac{\partial}{\partial w}.$$

With respect to the global coordinates (x, y, z, w) used in (1), the Levi-Civita connection ∇ of g is determined by the following non-vanishing components:

$$\begin{aligned}
&\nabla_{\partial_1}\partial_2 = \frac{1}{z}\partial_3, \qquad \nabla_{\partial_1}\partial_3 = -\frac{1}{z}\partial_1,\\
&\nabla_{\partial_2}\partial_2 = \frac{1}{2}(\partial_2 H)\partial_a + \frac{1}{2z}(2H - z\partial_3 H)\partial_3 - \frac{1}{2}(\partial_4 H)\partial_4,\\
&\nabla_{\partial_2}\partial_3 = \frac{1}{2}(\partial_3 H)\partial_1 - \frac{1}{z}\partial_2,\\
&\nabla_{\partial_2}\partial_4 = \frac{1}{2}(\partial_4 H)\partial_1, \qquad \nabla_{\partial_3}\partial_3 = -\frac{1}{z}\partial_3,\\
&\nabla_{\partial_3}\partial_4 = -\frac{1}{z}\partial_4, \qquad \nabla_{\partial_4}\partial_4 = \frac{1}{z}\partial_3.
\end{aligned}$$

The non-vanishing components of Riemannian-Christoffel curvature tensor R of g are the following:

$$
\begin{aligned}
&R_{1212} = -\frac{3}{\Lambda z^4}, \qquad R_{1323} = \frac{3}{\Lambda z^4},\\
&R_{1424} = \frac{3}{\Lambda z^4}, \qquad R_{2323} = \frac{3\left(2H - z(\partial_3 H) + z^2(\partial^2_{33} H)\right)}{2\Lambda z^4},\\
&R_{2324} = \frac{3\partial^2_{34} H}{2\Lambda z^2}, \qquad R_{2424} = \frac{3\left(2H - z(\partial_3 H) + z^2(\partial^2_{44} H)\right)}{2\Lambda z^4},\\
&R_{3434} = -\frac{3}{\Lambda z^4},
\end{aligned}
$$

where $R_{ijkl} = g(R(\partial_i, \partial_j)\partial_k, \partial_l)$ and the Ricci tensor S of g is described by

$$
S = \begin{pmatrix} 0 & -\frac{3}{z^2} & 0 & 0\\ -\frac{3}{z^2} & -\frac{6H-2z(\partial_3 H)+z^2(\partial^2_{33}H+\partial^2_{44}H)}{2z^2} & 0 & 0\\ 0 & 0 & -\frac{3}{z^2} & 0\\ 0 & 0 & 0 & -\frac{3}{z^2} \end{pmatrix}, \tag{4}
$$

with respect to $\{\partial_i\}$. Let $X = X_i\partial_i$ be an arbitrary vector field where $X_i = X_i(x, y, z, w)$, $i = 1, 2, 3, 4$ are smooth functions. The Lie derivative $\mathcal{L}_X g$ is given by

$$
\begin{cases}
(\mathcal{L}_X g)_{11} = -\frac{6}{\Lambda z^2}\partial_1 X_2,\\
(\mathcal{L}_X g)_{12} = -\frac{3}{\Lambda z^3}\{z\partial_1 X_1 + zH\partial_1 X_2 + z\partial_2 X_2 - 2X_3\},\\
(\mathcal{L}_X g)_{13} = -\frac{3}{\Lambda z^2}\{\partial_3 X_2 + \partial_1 X_3\},\\
(\mathcal{L}_X g)_{14} = -\frac{3}{\Lambda z^2}\{\partial_4 X_2 + \partial_1 X_4\},\\
(\mathcal{L}_X g)_{22} = -\frac{3}{\Lambda z^3}\{2z\partial_2 X_1 + z\partial_2 H X_2 + 2zH\partial_2 X_2 - 2HX_3\\
\qquad\qquad + z\partial_3 H X_3 + z\partial_4 H X_4\},\\
(\mathcal{L}_X g)_{23} = -\frac{3}{\Lambda z^2}\{\partial_3 X_1 + H\partial_3 X_2 + \partial_2 X_3\},\\
(\mathcal{L}_X g)_{24} = -\frac{3}{\Lambda z^2}\{\partial_4 X_1 + H\partial_4 X_2 + \partial_2 X_4\},\\
(\mathcal{L}_X g)_{33} = -\frac{6}{\Lambda z^3}\{z\partial_3 X_3 - X_3\},\\
(\mathcal{L}_X g)_{34} = -\frac{3}{\Lambda z^2}\{\partial_4 X_3 + \partial_3 X_4\},\\
(\mathcal{L}_X g)_{44} = -\frac{6}{\Lambda z^3}\{z\partial_4 X_4 - X_3\},
\end{cases} \tag{5}
$$

where $(\mathcal{L}_X g)_{ij} = \mathcal{L}_X g(\partial_i, \partial_j),\ i \leq j,\ i, j \in \{1, 2, 3, 4\}$. The Lie derivative $\mathcal{L}_X S$ is determined by

$$\begin{cases} (\mathcal{L}_X S)_{11} = -\frac{6}{z^2}\partial_1 X_2, \\ (\mathcal{L}_X S)_{12} = \frac{6}{z^3}X_3 - \frac{3}{z^2}\partial_1 X_1 - \frac{6H-2z(\partial_3 H)+z^2(\partial^2_{33}H+\partial^2_{44}H)}{2z^2}\partial_1 X_2 \\ \qquad -\frac{3}{z^2}\partial_2 X_2, \\ (\mathcal{L}_X S)_{13} = -\frac{3}{z^2}\{\partial_1 X_3 + \partial_3 X_2\}, \\ (\mathcal{L}_X S)_{14} = -\frac{3}{z^2}\{\partial_1 X_4 + \partial_4 X_2\}, \\ (\mathcal{L}_X S)_{22} = -\sum_{i=2}^4 X_i \partial_i\left(\frac{6H-2z(\partial_3 H)+z^2(\partial^2_{33}H+\partial^2_{44}H)}{2z^2}\right) \\ \qquad -\frac{6}{z^2}\partial_2 X_1 - \frac{6H-2z(\partial_3 H)+z^2(\partial^2_{33}H+\partial^2_{44}H)}{z^2}\partial_2 X_2, \\ (\mathcal{L}_X S)_{23} = -\frac{3}{z^2}\{\partial_2 X_3 + \partial_3 X_1\} - \frac{6H-2z(\partial_3 H)+z^2(\partial^2_{33}H+\partial^2_{44}H)}{2z^2}\partial_3 X_2, \\ (\mathcal{L}_X S)_{24} = -\frac{3}{z^2}\{\partial_2 X_4 + \partial_4 X_1\} - \frac{6H-2z(\partial_3 H)+z^2(\partial^2_{33}H+\partial^2_{44}H)}{2z^2}\partial_4 X_2, \\ (\mathcal{L}_X S)_{33} = \frac{6}{z^3}X_3 - \frac{6}{z^2}\partial_3 X_3, \\ (\mathcal{L}_X S)_{34} = -\frac{3}{z^2}\{\partial_3 X_4 + \partial_4 X_3\}, \\ (\mathcal{L}_X S)_{44} = \frac{6}{z^3}X_3 - \frac{6}{z^2}\partial_4 X_4, \end{cases} \tag{6}$$

where $(\mathcal{L}_X S)_{ij} = \mathcal{L}_X S(\partial_i, \partial_j),\ i \leq j,\ i, j \in \{1, 2, 3, 4\}$.

3. The main results and their proofs

Applying (1), (4) and (5) in (2), we get

$$\partial_1 X_2 = 0, \tag{7}$$
$$z\partial_1 X_1 + zH\partial_1 X_2 + z\partial_2 X_2 - 2X_3 = (\alpha + \Lambda\beta)z, \tag{8}$$
$$\partial_3 X_2 + \partial_1 X_3 = 0, \tag{9}$$
$$\partial_4 X_2 + \partial_1 X_4 = 0, \tag{10}$$
$$\begin{aligned} &2z\partial_2 X_1 + z\partial_2 H X_2 + 2zH\partial_2 X_2 - 2HX_3 + z\partial_3 H X_3 + z\partial_4 H X_4 \\ &\qquad = \alpha z H - \left(6H - 2z(\partial_3 H) + z^2(\partial^2_{33}H + \partial^2_{44}H)\right)\frac{z}{6}\Lambda\beta, \end{aligned} \tag{11}$$
$$\partial_3 X_1 + H\partial_3 X_2 + \partial_2 X_3 = 0, \tag{12}$$
$$\partial_4 X_1 + H\partial_4 X_2 + \partial_2 X_4 = 0, \tag{13}$$
$$2\{z\partial_3 X_3 - X_3\} = (\alpha + \Lambda\beta)z, \tag{14}$$
$$\partial_4 X_3 + \partial_3 X_4 = 0, \tag{15}$$
$$2\{z\partial_4 X_4 - X_3\} = (\alpha + \Lambda\beta)z. \tag{16}$$

Inserting (1), (4) and (6) in (3), we have

$$
\begin{aligned}
&\partial_1 X_2 = 0,\\
&-2X_3 + z\partial_1 X_1 + \frac{6H - 2z(\partial_3 H) + z^2(\partial^2_{33}H + \partial^2_{44}H)}{6} z\partial_1 X_2 \qquad (17)\\
&\qquad +z\partial_2 X_2 = (\alpha + \frac{\beta}{\Lambda})z,\\
&\partial_1 X_3 + \partial_3 X_2 = 0,\\
&\partial_1 X_4 + \partial_4 X_2 = 0,\\
&-\sum_{i=2}^{4} X_i \partial_i \left(\frac{6H - 2z(\partial_3 H) + z^2(\partial^2_{33}H + \partial^2_{44}H)}{2z^2} \right) \qquad (18)\\
&\qquad -\frac{6}{z^2}\partial_2 X_1 - \frac{6H - 2z(\partial_3 H) + z^2(\partial^2_{33}H + \partial^2_{44}H)}{z^2} \partial_2 X_2\\
&\qquad = -\frac{6H - 2z(\partial_3 H) + z^2(\partial^2_{33}H + \partial^2_{44}H)}{2z^2}\alpha - \frac{3H}{\Lambda z^2}\beta,\\
&6\{\partial_2 X_3 + \partial_3 X_1\} + \{6H - 2z(\partial_3 H) + z^2(\partial^2_{33}H + \partial^2_{44}H)\}\partial_3 X_2 = 0, \quad (19)\\
&6\{\partial_2 X_4 + \partial_4 X_1\} + \{6H - 2z(\partial_3 H) + z^2(\partial^2_{33}H + \partial^2_{44}H)\}\partial_4 X_2 = 0, \quad (20)\\
&2X_3 - 2z\partial_3 X_3 = -\alpha z - \frac{1}{\Lambda}\beta z,\\
&\partial_3 X_4 + \partial_4 X_3 = 0,\\
&2X_3 - 2z\partial_4 X_4 = -\alpha z - \frac{1}{\Lambda}\beta z.
\end{aligned}
$$

We state integrating the Equation (7), obtaining

$$X_2 = F_1(y, z, w), \tag{21}$$

for some smooth function F_1. Inserting (21) into the Equations (9), (10) and integrating, we respectively get

$$X_3 = -x\partial_3 F_1(y, z, w) + F_2(y, z, w), \tag{22}$$
$$X_4 = -x\partial_4 F_1(y, z, w) + F_3(y, z, w), \tag{23}$$

where F_2, F_3 are smooth functions. Applying (22) and (23) in Equation (15) we have

$$2x\partial^2_{34} F_1(y, z, w) - \partial_4 F_2(y, z, w) - \partial_3 F_3(y, z, w) = 0. \tag{24}$$

Since x is arbitrary, the Equation (24) implies at once

$$\partial^2_{34} F_1(y, z, w) = \partial_4 F_2(y, z, w) + \partial_3 F_3(y, z, w) = 0. \tag{25}$$

Integrating $\partial^2_{34} F_1(y, z, w) = 0$, we infer

$$F_1(y, z, w) = G_1(y, z) + H_1(y, w), \tag{26}$$

for some smooth functions G_1 and H_1. From Equations (14) and (16), we have

$$\partial_3 X_3 - \partial_4 X_4 = 0. \tag{27}$$

Applying (22), (23) and (26) in (27), we obtain

$$-x\partial^2_{33}G_1(y,z) + \partial_3 F_2(y,z,w) + x\partial^2_{44}H_1(y,w) - \partial_4 F_3(y,z,w) = 0,$$

since x is arbitrary, we conclude

$$\partial^2_{44}H_1(y,w) = \partial^2_{33}G_1(y,z) = F_4(y), \tag{28}$$

for some smooth function F_4 and

$$\partial_3 F_2(y,z,w) = \partial_4 F_3(y,z,w). \tag{29}$$

Integrating of (28), we deduce

$$G_1(y,z) = \frac{z^2}{2}F_4(y) + zF_5(y) + G_2(y),$$
$$H_1(y,w) = \frac{w^2}{2}F_4(y) + wF_7(y) + F_6(y),$$

for some smooth functions F_5, F_6, F_7 and G_2. Therefore, we have

$$F_1(y,z,w) = \frac{z^2}{2}F_4(y) + zF_5(y) + \frac{w^2}{2}F_4(y) + wF_7(y) + F_8(y), \tag{30}$$

where $F_8 = G_2 + F_6$. The Equation (14) reduces to

$$\partial_3 F_2(y,z,w) + \frac{x}{z}F_5(y) - \frac{1}{z}F_2(y,z,w) = \frac{\alpha+\Lambda\beta}{2}. \tag{31}$$

Substiuting (7) into (8) and (17), we get

$$z\partial_1 X_1 + z\partial_2 X_2 - 2X_3 = (\alpha+\Lambda\beta)z,$$
$$-2X_3 + z\partial_1 X_1 + z\partial_2 X_2 = (\alpha + \frac{\beta}{\Lambda})z,$$

which imply that

$$(\Lambda+1)\beta = 0.$$

Inserting (21), (22), (31) and (30) in (8), we arrive at

$$\begin{aligned}\partial_1 X_1 \quad = \quad & -\frac{1}{2}(z^2+w^2)F_4'(y) - zF_5'(y) - wF_7'(y) \\ & -F_8'(y) - 2xF_4(y) + 2\partial_3 F_2(y,z,w),\end{aligned}$$

which by integrating provides

$$\begin{aligned} X_1 &= -\frac{1}{2}(z^2+w^2)xF_4'(y) - zxF_5'(y) - wxF_7'(y) \\ &\quad -xF_8'(y) - x^2F_4(y) + 2x\partial_3 F_2(y,z,w) + F_9(y,z,w), \end{aligned} \tag{32}$$

for some smooth function F_9. Substituting (21), (23), (30) and (32) in (13), we find

$$\begin{aligned} &-wxF_4'(y) - xF_7'(y) + 2x\partial^2_{34}F_2(y,z,w) + \partial_4 F_9(y,z,w) + wHF_4(y) \\ &+HF_7(y) - xwF_4'(y) - xF_7'(y) + \partial_2 F_3(y,z,w) = 0. \end{aligned}$$

Since x is arbitrary, we obtain

$$\begin{aligned} &-2wF_4'(y) - 2F_7'(y) + 2\partial^2_{34}F_2(y,z,w) = 0, \\ &\partial_4 F_9(y,z,w) + wHF_4(y) + HF_7(y) + \partial_2 F_3(y,z,w) = 0, \end{aligned}$$

which by integrating it implies that

$$\begin{aligned} F_2(y,z,w) &= \frac{w^2}{2}zF_4'(y) + wzF_7'(y) + F_{10}(y,z), \qquad (33) \\ F_9(y,z,w) &= -\int \left(wHF_4(y) + HF_7(y) + \partial_2 F_3(y,z,w)\right) dw + F_{11}(y,z), \end{aligned}$$

for some smooth functions F_{10} and F_{11}. Therefore, we have

$$\begin{aligned} X_1 &= -\frac{1}{2}(z^2+w^2)xF_4'(y) - zxF_5'(y) - wxF_7'(y) \\ &\quad -xF_8'(y) - x^2F_4(y) + 2x\partial_3 F_2(y,z,w) \\ &\quad -\int \left(wHF_4(y) + HF_7(y) + \partial_2 F_3(y,z,w)\right) dw + F_{11}(y,z), \\ X_2 &= \frac{z^2+w^2}{2}F_4(y) + zF_5(y) + wF_7(y) + F_8(y), \\ X_3 &= -x(zF_4(y) + F_5(y)) + \frac{w^2}{2}zF_4'(y) + wzF_7'(y) + F_{10}(y,z), \\ X_4 &= -x(wF_4(y) + F_5(y)) + F_3(y,z,w). \end{aligned}$$

From (12) and (19), we conclude that

$$\left(2(\partial_3 H) - z(\partial^2_{33}H + \partial^2_{44}H)\right)\partial_3 X_2 = 0. \tag{34}$$

Also, from (13) and (20), we infer

$$\left(2(\partial_3 H) - z(\partial^2_{33}H + \partial^2_{44}H)\right)\partial_4 X_2 = 0. \tag{35}$$

Applying (33) in (25), we deduce

$$\partial_3 F_3(y,z,w) = -\left(wzF_4'(y) + zF_7'(y)\right).$$

Integrating of last equation yields

$$F_3(y,z,w) = -\left(\frac{1}{2}wz^2F_4'(y) + \frac{1}{2}z^2F_7'(y)\right) + F_{12}(y,w), \tag{36}$$

for some smooth function F_{12}. Substituting (33) and (36) in (29), we find

$$\partial_4 F_{12}(y,w) = \frac{1}{2}w^2F_4'(y) + wF_7'(y) + \frac{1}{2}z^2F_4'(y) + \partial_3 F_{10}(y,z).$$

Hence, we get

$$F_{12}(y,w) = \frac{1}{6}w^3F_4'(y) + \frac{1}{2}w^2F_7'(y) + w\partial_3 F_{10}(y,z) + \frac{1}{2}z^2wF_4'(y) + F_{13}(y),$$

for some smooth function F_{13}. Deriving with respect to z, we arrive at

$$\partial^2_{33}F_{10}(y,z) + zwF_4'(y) = 0.$$

Since w is arbitrary we have $F_4'(y) = 0$ and $\partial^2_{33}F_{10}(y,z) = 0$ which imply that $F_4(y) = a$ for some constant a and $F_{10}(y,z) = zF_{14}(y) + F_{15}(y)$, for some smooth functions F_{14} and F_{15}, respectively. Therefore,

$$\begin{aligned}
F_3(y,z,w) &= \frac{w^2-z^2}{2}F_7'(y) + wF_{14}(y) + F_{13}(y),\\
X_1 &= -zxF_5'(y) - xF_8'(y) - ax^2 + 2xF_{14}(y) + xwF_7'(y)\\
&\quad -\frac{w^3-3z^2w}{6}F_7''(y) - \frac{1}{2}w^2F_{14}'(y)\\
&\quad -wF_{13}'(y) - \int (awH + HF_7(y))\,dw + F_{11}(y,z), \qquad (37)\\
X_2 &= a\frac{z^2+w^2}{2} + zF_5(y) + wF_7(y) + F_8(y),\\
X_3 &= -x(az + F_5(y)) + wzF_7'(y) + zF_{14}(y) + F_{15}(y),\\
X_4 &= -x(aw + F_5(y)) + \frac{w^2-z^2}{2}F_7'(y) + wF_{14}(y) + F_{13}(y),\\
\alpha + \Lambda\beta &= \frac{2x}{z}F_5(y) - \frac{2}{z}F_{15}(y).
\end{aligned}$$

Substituting X_1, X_2, and X_3 in (12), we get

$$\begin{aligned}
&-xF_5'(y) + zwF_7''(y) - \int (aw + F_7(y))\,\partial_3 H dw + \partial_3 F_{11}(y,z)\\
&+H(az + F_5(y)) + zF_{14}'(y) + F_{15}'(y) = 0.
\end{aligned} \tag{38}$$

Since x is arbitrary we infer $F_5'(y) = 0$, thus, $F_5(y) = b$ for some constant b.

3.1 Siklos metrics with $H = 0$

If $H = 0$ then Equation (11) yields $\partial_2 X_1 = 0$. So, we get

$$F_{11}(y,z) = xF_8'(y) - 2xF_{14}(y) - xwF_7'(y) + \frac{w^3+3z^2w}{6}F_7''(y) + \frac{1}{2}w^2F_{14}'(y) + wF_{13}'(y) + G_3(z), \quad (39)$$

and

$$X_1 = -ax^2 + G_3(z),$$

for some smooth function G_3. Since x is arbitrary we have $F_8'(y) - 2F_{14}(y) - wF_7'(y) = 0$ this implies that $F_7'(y) = 0$ and $F_8'(y) = 2F_{14}(y)$. Therefore $F_7(y) = c$ for some constant c. Also, Since w is arbitrary, from (39) we have $F_{14}'(y) = F_{13}'(y) = 0$. Then $F_{13}(y) = a_{13}, F_{14}(y) = a_{14}, F_8(y) = 2a_{14}y + a_{16}$ for some constants a_{13}, a_{14}, a_{16}. Also, Equation (38) reduces to

$$G_3'(z) + F_{15}'(y) = 0.$$

Therefore $G_3(z) = b_3z + b_4$, $F_{15}(y) = -b_3y + b_5$ and $X_1 = -ax^2 + b_3z + b_4$ for some constants b_3, b_4, b_5. Hence,

$$\begin{aligned} X_1 &= -ax^2 + b_3z + b_4, \\ X_2 &= a\frac{z^2+w^2}{2} + bz + cw + 2a_{14} + a_{16}, \\ X_3 &= -x(az+b) + za_{14} - b_3y + b_5, \\ X_4 &= -x(aw+b) + wa_{14} + a_{13}, \\ \alpha + \Lambda\beta &= \frac{2x}{z}b - \frac{2}{z}(-b_3y + b_5). \end{aligned} \quad (40)$$

Theorem 3.1. *Any homogeneous Siklos spacetime defined by $H = 0$ has Ricci bi-conformal vector field $X = X_i\partial_i$ if and only if α, β and X_i satisfy in (40).*

Now, we consider the vector fields as $X = \nabla f$ for some smooth function f which are Ricci bi-conformal vector fields on homogeneous Siklos spacetime with $H = 0$. On homogeneous Siklos spacetime with $H = 0$, we have

$$\nabla f = -\frac{\Lambda z^2}{3}[(\partial_2 f)\partial_1 + (\partial_1 f)\partial_2 + (\partial_3 f)\partial_3 + (\partial_4 f)\partial_4]. \quad (41)$$

From (41) and (40), we have

$$\begin{aligned} -\frac{\Lambda z^2}{3}\partial_1 f &= a\frac{z^2+w^2}{2} + bz + cw + 2a_{14} + a_{16}, \\ -\frac{\Lambda z^2}{3}\partial_2 f &= -ax^2 + b_3z + b_4, \\ -\frac{\Lambda z^2}{3}\partial_3 f &= -x(az+b) + za_{14} - b_3y + b_5, \\ -\frac{\Lambda z^2}{3}\partial_4 f &= -x(aw+b) + wa_{14} + a_{13}. \end{aligned} \quad (42)$$

Deriving the first Equation (42) with respect to y and deriving the second Equation (42) with respect to x we infer $a = 0$. By similar method we conclude $b = c = a_{14} = a_{13} = a_{16} = b_4 = 0$. Then

$$\partial_1 f = 0, \qquad \partial_2 f = -\frac{3b_3}{\Lambda z},$$
$$\partial_3 f = -\frac{3}{\Lambda}(-b_3 y + b_5)z^{-2}, \qquad \partial_4 f = 0.$$

Therefore we have the following corollary:

Corollary 3.2. *Homogeneous Siklos spacetime defined by $H = 0$ has Ricci bi-conformal vector field as $X = \nabla f$ if and only if $f = \frac{3}{\Lambda z}(-b_3 y + b_5) + k$, where $k \in \mathbb{R}$.*

3.2 Siklos metrics with nonzero constant H

In the following we assume $H \neq 0$ and H be a nonzero constant then (38) yields

$$F_{11}(y,z) = -z^2 w F_7''(y) - H(\frac{a}{2}z^2 + bz) - \frac{1}{2}z^2 F_{14}'(y) - zF_{15}'(y) + F_{16}(y),$$

for some smooth function F_{16}. Thus, we can write (37) as follows:

$$\begin{aligned}
X_1 &= -xF_8'(y) - ax^2 + 2xF_{14}(y) + xwF_7'(y) \\
&\quad -\frac{w^3 - 3z^2 w}{6}F_7''(y) - \frac{1}{2}w^2 F_{14}'(y) \\
&\quad -wF_{13}'(y) - H(\frac{1}{2}aw^2 + wF_7(y)) \\
&\quad -z^2 w F_7''(y) - H(\frac{a}{2}z^2 + bz) - \frac{1}{2}z^2 F_{14}'(y) - zF_{15}'(y) + F_{16}(y), \\
X_2 &= a\frac{z^2 + w^2}{2} + bz + wF_7(y) + F_8(y), \\
X_3 &= -x(az + b) + wzF_7'(y) + zF_{14}(y) + F_{15}(y), \\
X_4 &= -x(aw + b) + \frac{w^2 - z^2}{2}F_7'(y) + wF_{14}(y) + F_{13}(y), \\
\alpha + \Lambda\beta &= \frac{2bx}{z} - \frac{2}{z}F_{15}(y).
\end{aligned}$$

Inserting $\alpha + \Lambda\beta = \frac{2bx}{z} - \frac{2}{z}F_{15}(y)$ and H in (18), we obtain

$$HX_3 - z\partial_2 X_1 - zH\partial_2 X_2 + bHx - HF_{15}(y) = 0. \tag{43}$$

Applying X_1, X_2, X_3, X_4 in (43), we get

$$\begin{aligned}&-xaHz + HwzF_7'(y) + HzF_{14}(y) + zxF_8''(y)\\&-2xzF_{14}'(y) - xzwF_7''(y) + \frac{w^3-3z^2w}{6}zF_7'''(y) + \frac{1}{2}zw^2F_{14}''(y)\\&+zwF_{13}''(y) + z^3wF_7'''(y) + \frac{1}{2}z^3F_{14}''(y) + z^2F_{15}''(y) - zF_{16}'(y)\\&-zHF_8'(y) = 0.\end{aligned} \tag{44}$$

Equation (44) is a polynomial with respect to x. Since x is arbitrary, we infer

$$-Haz + zF_8''(y) - 2zF_{14}'(y) - zwF_7''(y) = 0.$$

Since w is arbitrary we deduce $F_7''(y) = 0$ and $-aH + F_8''(y) - 2F_{14}'(y) = 0$. Also, Equation (44) is a polynomial with respect to z. Since z is arbitrary we get $F_{14}''(y) = F_{15}''(y) = 0$ and

$$HwF_7'(y) + HF_{14}(y) + wF_{13}''(y) - F_{16}'(y) - HF_8'(y) = 0.$$

Thus, $HF_7'(y) + F_{13}''(y) = 0$ and $HF_{14}(y) - F_{16}'(y) - HF_8'(y) = 0$. Therefore,

$$\begin{aligned}&F_7 = b_1y + b_2, \qquad F_8 = \frac{1}{2}b_3y^2 + b_4y + b_5,\\&F_{14} = -\frac{a}{2}Hy + \frac{1}{2}b_3y + b_6, \qquad F_{15} = b_7y + b_8,\\&F_{13} = -\frac{1}{2}b_1Hy^2 + b_9y + b_{10},\\&F_{16} = -\frac{a}{4}H^2y^2 - \frac{1}{4}b_3Hy^2 + (b_6 - b_4)Hy + b_{11},\end{aligned}$$

for some constants $b_1, \cdots, b_{11}$. Then

$$\begin{aligned}X_1 &= -x(b_3y + b_4) - ax^2 + 2x(-\frac{a}{2}Hy + \frac{1}{2}b_3y + b_6) + b_1xw - \frac{b_3 - aH}{4}w^2\\&\quad -w(-2b_1Hy + b_9) - H(\frac{1}{2}aw^2 + w(b_1y + b_2)) - H(\frac{a}{2}z^2 + bz)\\&\quad -\frac{b_3 - aH}{4}z^2 - b_7z - \frac{a}{4}H^2y^2 - \frac{1}{4}b_3Hy^2 + (b_6 - b_4)Hy + b_{11}, \qquad (45)\\X_2 &= a\frac{z^2 + w^2}{2} + bz + w(b_1y + b_2) + \frac{1}{2}b_3y^2 + b_4y + b_5,\\X_3 &= -x(az + b) + b_1wz + z(-\frac{a}{2}Hy + \frac{1}{2}b_3y + b_6) + b_7y + b_8,\\X_4 &= -x(aw + b) + b_1\frac{w^2 - z^2}{2} + w(-\frac{a}{2}Hy + \frac{1}{2}b_3y + b_6) - \frac{1}{2}b_1Hy^2 + b_9y + b_{10}.\end{aligned}$$

Using $\alpha+\Lambda\beta=\frac{2bx}{z}-\frac{2}{z}(b_7y+b_8)$ in (11), we get

$$\begin{aligned}\alpha &= bxz^{-1}-z^{-1}(b_7y+b_8)+\frac{1}{2zH}\{-2b_3zx+4zx(-\frac{a}{2}H+\frac{1}{2}b_3)+4b_1Hzw\\ &\quad -2b_1Hzw-aH^2yz-b_3Hyz+2(b_6-b_4)Hz2b_1Hzw+2b_3Hzy+2b_4Hz,\\ &\quad +2Hx(az+b)-2Hb_1wz-2Hz(-\frac{a}{2}Hy+\frac{1}{2}b_3y+b_6)-2Hb_7y-2Hb_8\}.\end{aligned}$$

Theorem 3.3. *Any homogeneous Siklos spacetime defined by nonzero constant H has Ricci bi-conformal vector field $X=X_i\partial_i$ if and only if α,β and X_i satisfy in (45).*

Now, we consider the vector fields as $X=\nabla f$ for some smooth function f which are Ricci bi-conformal vector fields on homogeneous Siklos spacetime with nonzero constant H. We have

$$\begin{aligned}&-\frac{\Lambda z^2}{3}(-H\partial_1f+\partial_2f)=X_1,\\ &-\frac{\Lambda z^2}{3}\partial_2f=X_2,\\ &-\frac{\Lambda z^2}{3}\partial_3f=X_3,\\ &-\frac{\Lambda z^2}{3}\partial_4f=X_4.\end{aligned}\tag{46}$$

Deriving the third Equation (46) with respect to w and deriving the fourth Equation (46) with respect to z we infer $a=b=b_1=b_3=b_6=b_9=b_{10}=0$. By similar method we conclude $b_2=b_4=b_5=0$. Thus

$$\begin{aligned}&\partial_1f=0, && \partial_2f=0,\\ &\partial_3f=-\frac{3}{\Lambda}b_8z^{-2}, && \partial_4f=0.\end{aligned}$$

Therefore we have the following corollary:

Corollary 3.4. *Homogeneous Siklos spacetime defined by nonzero constant H has Ricci bi-conformal vector field as $X=\nabla f$ if and only if $f=b_8\frac{3}{\Lambda z}+k$, where $k\in\mathbb{R}$.*

3.3 Siklos metrics of case I

We assume that $H=A_{-2}(z,w)$, that is H is a homogeneous function of degree -2 of variable z,w. Similar [4], explicitly we consider

$$H=k_1z^{-2}+k_2z^{-1}w^{-1}+k_3w^{-2},$$

for some real constants k_1, k_2, k_3. We have

$$2(\partial_3 H) - z(\partial_{33}^2 H + \partial_{44}^2 H) = -4k_2 z^{-2}w^{-1} - 10k_1 z^{-3} - 2k_2 w^{-3} - 6k_3 z w^{-4}.$$

Thus Equations (34) and (35) yield $\partial_3 X_2 = \partial_4 X_2 = 0$. Then $a = b = F_7(y) = 0$ and X_2 becomes $X_2 = F_8(y)$. Also, we have

$$\begin{aligned} X_1 &= -xF_8'(y) + 2xF_{14}(y) - \frac{1}{2}w^2 F_{14}'(y) - wF_{13}'(y) + F_{11}(y,z), \\ X_2 &= F_8(y), \\ X_3 &= zF_{14}(y) + F_{15}(y), \\ X_4 &= wF_{14}(y) + F_{13}(y), \\ \alpha + \Lambda\beta &= -\frac{2}{z}F_{15}(y). \end{aligned}$$

From Equation (12), we obtain $\partial_3 F_{11}(y,z) = -zF_{14}'(y) - F_{15}'(y)$. Then

$$F_{11}(y,z) = -\frac{1}{2}z^2 F_{14}'(y) - zF_{15}'(y) + F_{17}(y),$$

for some smooth function F_{17}. Let

$$\begin{aligned} G &:= \frac{6H - 2z(\partial_3 H) + z^2(\partial_{33}^2 H + \partial_{44}^2 H)}{2z^2} \\ &= 8k_1 z^{-4} + 5k_2 z^{-3}w^{-1} + 3k_3 z^{-2}w^{-2} + k_2 z^{-1}w^{-3} + 3k_3 w^{-4}. \end{aligned}$$

Applying $\Lambda\beta = -\frac{2}{z}F_{15}(y) - \alpha$ in Equations (11) and (18), we find

$$2z\partial_2 X_1 + 2zH\partial_2 X_2 - 2HX_3 + z\partial_3 H X_3 + z\partial_4 H X_4 + \frac{2}{3}z^2 G F_{15}(y) = \alpha(zH + \frac{z^3}{3}G), \tag{47}$$

$$-X_3\partial_3 G - X_4\partial_4 G - \frac{6}{z^2}\partial_2 X_1 - 2G\partial_2 X_2 - 6Hz^{-3}F_{15}(y) = (3Hz^{-2} - G)\alpha. \tag{48}$$

Therefore,

$$\begin{aligned} &z^6 w^9 (G - 3Hz^{-2})\,(2z\partial_2 X_1 + 2zH\partial_2 X_2 - 2HX_3 + z\partial_3 H X_3 + z\partial_4 H X_4 \\ &+ \frac{2}{3}z^2 G F_{15}(y)) - z^6 w^9 (zH + \frac{z^3}{3}G)\,(X_3\partial_3 G + X_4 \partial_4 G \\ &+ \frac{6}{z^2}\partial_2 X_1 + 2G\partial_2 X_2 + 6Hz^{-3}F_{15}(y)) = 0. \end{aligned} \tag{49}$$

Differentiate it with respect to x we get $-F_8''(y) + 2F_{14}'(y) = 0$, thus $F_{14}(y) = \frac{1}{2}F_8'(y) + c$ for some constant c. Substituting G, H, X_1, X_2, X_3 and X_4 in (49) we arrive at

$$
\begin{aligned}
&(5k_1z^3w^9 + 2k_2z^4w^8 + k_2z^6w^6 + 3k_3z^7w^5)\{-w^2zF''_{14}(y) - 2zwF''_{13}(y)\\
&-z^3F''_{14}(y) - 2z^2F''_{15}(y) + 2zF'_{17}(y) + (2k_1z^{-1} + 2k_2w^{-1} + 2k_3zw^{-2})F'_8(y)\\
&-2(k_1z^{-2} + k_2z^{-1}w^{-1} + k_3w^{-2})(zF_{14}(y) + F_{15}(y))\\
&-(2k_1z^{-3} + k_2z^{-2}w^{-1})(zF_{14}(y) + F_{15}(y))\\
&-(k_2z^{-1}w^{-2} + 2k_3zw^{-3})(wF_{14}(y) + F_{13}(y))\\
&+\frac{2}{3}(8k_1z^{-2} + 5k_2z^{-1}w^{-1} + 3k_3w^{-2} + k_2zw^{-3} + 3k_3z^2w^{-4})F_{15}(y)\}\\
&+(\frac{11}{3}k_1z^6w^9 + \frac{8}{3}k_2z^7w^8 + 2k_3z^8w^7 + \frac{1}{3}k_2z^9w^6 + k_3z^{10}w^5)\{(32k_1z^{-5}\\
&+15k_2z^{-4}w^{-1} + 6k_3z^{-3}w^{-2} + k_2z^{-2}w^{-3})(zF_{14}(y) + F_{15}(y))\\
&+(5k_2z^{-3}w^{-2} + 6k_3z^{-2}w^{-3} + 3k_2z^{-1}w^{-4} + 12k_3w^{-5})(wF_{14}(y) + F_{13}(y))\\
&+3z^{-2}w^2F''_{14}(y) + 6z^{-2}wF'_{13}(y) + 6F''_{14}(y) + 6z^{-1}F''_{15}(y) - 6z^{-2}F'_{17}(y)\\
&+2(8k_1z^{-4} + 5k_2z^{-3}w^{-1} + 3k_3z^{-2}w^{-2} + k_2z^{-1}w^{-3} + 3k_3w^{-4})F'_8(y)\\
&+6(k_1z^{-2} + k_2z^{-1}w^{-1} + k_3w^{-2})F_{15}(y)\} = 0.
\end{aligned} \tag{50}
$$

Last equation is a polynomial with respect to z and w. Since z and w are arbitrary we conclude that all coefficients of z^rw^s for $0 \leq r, s \leq 11$ are equal to zero. The coefficient of z^4w^{11} in (50) implies that

$$k_1F''_{14}(y) = 0. \tag{51}$$

The coefficients of z^5w^{10} and z^4w^{10} in (50) imply that

$$k_2F''_{14}(y) = 0, \;\; k_1(-10F''_{13}(y) + 22F'_{13}(y)) = 0. \tag{52}$$

The coefficient of z in (50) leads to

$$k_3F_{13}(y) = 0. \tag{53}$$

The coefficients of z^9w^1 and $z^{10}w^1$ in (50) yield

$$k_3F_{15}(y) = 0, \;\; k_3(2F_{14} + F'_8(y)) = 0. \tag{54}$$

The coefficient of z^6w^9 in (50) leads to $k_3F''_{14}(y) = 0$. Since $H \neq 0$, using (51), (52) and (53), we deduce $F''_{14}(y) = 0$. The coefficients of w^9, zw^9, z^2w^9 and z^4w^9 in (50) imply that

$$k_1F_{15}(y) = 0, \;\; k_1F_{14}(y) = 0, \;\; k_1F'_8(y) = 0, \;\; k_1F'_{17}(y) = 0, \tag{55}$$

respectively. The coefficients of z^6w^8, z^5w^8, z^8w^2 and z^7w^3 in (50) yield

$$k_2F''_{15}(y) = 0, \;\; k_2F'_{16}(y) = 0, \;\; k_2F_{13}(y) = 0, \;\; k_2F_{15}(y) = 0, \tag{56}$$

respectively. Since $H \neq 0$, using (54), (55) and (56) we deduce $F_{15}(y) = 0$. The coefficient of $z^8 w^3$ in (50) leads to

$$24k_3^2(F_{14}(y) + F_8'(y)) + \frac{2}{3}k_2^2(2F_{14}(y) + F_8'(y)) = 0. \tag{57}$$

The coefficients of $z^6 w^7$, $z^3 w^7$ and $z^4 w^7$ in (50) imply that

$$k_3 F_{17}'(y) = 0, \;\; k_2 F_{14}(y) = 0, \;\; k_2 F_8'(y) = 0, \tag{58}$$

respectively. Applying (58) in (57) we obtain $k_3(F_{14}(y) + F_8'(y)) = 0$. Inserting it in (54) we conclude $k_3 F_{14}(y) = k_3 F_8'(y) = 0$. Since $H \neq 0$, using (55) and (58) we get $F_{14}(y) = F_8'(y) = 0$, thus $F_8(y) = a_1$ for some constant a_1. Also, using (55), (56), and (58) we get $F_{17}'(y) = 0$, then $F_{17} = a_2$ for some constant a_2. If $k_1 = k_2 = 0$ then $k_3 \neq 0$ and (53) yields $F_{13}(y) = 0$. Thus from (47) and (48) we have $\alpha = 0$. Therefore for case $k_1 = k_2 = 0$ we have,

$$X_1 = a_2, \quad X_2 = a_1, \quad X_3 = 0, \;\; X_4 = 0, \;\; \alpha = \beta = 0.$$

If $k_1 = 0$ and $k_2 \neq 0$ then (56) yields $F_{13}(y) = 0$. Thus from (47) and (48) we have $\alpha = 0$. Therefore for case $k_1 = 0$ and $k_2 \neq 0$, we have

$$X_1 = a_2, \quad X_2 = a_1, \quad X_3 = 0, \;\; X_4 = 0, \;\; \alpha = \beta = 0.$$

If $k_1 \neq 0$ and $k_2 \neq 0$ or $k_3 \neq 0$ then

$$X_1 = a_2, \quad X_2 = a_1, \quad X_3 = 0, \;\; X_4 = 0, \;\; \alpha = \beta = 0.$$

If $k_1 \neq 0$ and $k_2 = k_3 = 0$ then from (52) we have $F_{13}(y) = \frac{10}{22} a_3 e^{\frac{22}{10}y} + a_4$, for some constants a_3 and a_4. Thus from (47) and (48) we have $\alpha = a_3 = 0$. Therefore for case $k_1 = 0$ and $k_2 = k_3 = 0$, we have

$$X_1 = a_2, \quad X_2 = a_1, \quad X_3 = 0, \;\; X_4 = a_4, \;\; \alpha = \beta = 0.$$

Theorem 3.5. *Any homogeneous Siklos spacetime with $H = k_1 z^{-2} + k_2 z^{-1} w^{-1} + k_3 w^{-2}$ has Ricci bi-conformal vector field $X = X_i \partial_i$ if and only if $\alpha = \beta = X_3 = 0$, $X_1 = a_2$, $X_2 = a_1$ and $X_4 = 0$ for $k_1 \neq 0$ and $X_4 = a_4$ for $k_1 \neq 0$.*

Suppose that there exists some smooth function $f = f(x, y, z, w)$ such that $X = \nabla f$. By definition of gradient of f that is $\nabla f = g^{ij}\partial_j f \partial_i$, we have

$$\begin{cases} \frac{1}{3}\Lambda z^2 H \partial_1 f - \frac{1}{3}\Lambda z^2 \partial_2 f = a_2, \\ -\frac{1}{3}\Lambda z^2 \partial_1 f = a_1, \\ -\frac{1}{3}\Lambda z^2 \partial_3 f = 0, \\ -\frac{1}{3}\Lambda z^2 \partial_4 f = a_4. \end{cases}$$

By solving the last system we have $a_1 = a_2 = a_4 = 0$ and $f = k$ for some constant k. Therefore we have the following corollary:

Corollary 3.6. *Homogeneous Siklos spacetime defined by $H = k_1 z^{-2} + k_2 z^{-1} w^{-1} + k_3 w^{-2}$ has Ricci bi-conformal vector field as $X = \nabla f$ if and only if $f = k$ where $k \in \mathbb{R}$.*

3.4 Siklos metrics with $H = A(z)$

Now, assume that $H = A(z)$. We have

$$2(\partial_3 H) - z(\partial_{33}^2 H + \partial_{44}^2 H) = 2A'(z) - zA''(z).$$

If $2A'(z) - zA''(z) = 0$ then $H = \frac{1}{3}a_7 z^3 + a_8$ for some constants a_7, a_8. Assume that $a_7 \neq 0$. In this case we have

$$\frac{6H - 2z(\partial_3 H) + z^2(\partial_{33}^2 H + \partial_{44}^2 H)}{2z^2} = a_7 z + 3a_8 z^{-2}.$$

From Equation (12), we get

$$\begin{aligned}&-a_7 z^2(\frac{1}{2}aw^2 + wF_7(y)) + \partial_3 F_{11}(y,z)\\ &+(\frac{1}{3}a_7 z^3 + a_8)(az + b) + zF'_{14}(y) + F'_{15}(y) = 0.\end{aligned}$$

Since w is arbitrary we deduce $a = 0$,

$$\begin{aligned}&F_7(y) = 0,\\ &\partial_3 F_{11}(y,z) + b(\frac{1}{3}a_7 z^3 + a_8) + zF'_{14}(y) + F'_{15}(y) = 0.\end{aligned}$$

Integrating the last equation, we get

$$F_{11}(y,z) = -\frac{1}{12}ba_7 z^4 - ba_8 z - \frac{1}{2}z^2 F'_{14}(y) - zF'_{15}(y) + F_{18}(y),$$

for some smooth function F_{18}. Therefore,

$$\begin{aligned}X_1 &= -xF'_8(y) + 2xF_{14}(y) - \frac{1}{2}w^2 F'_{14}(y) - wF'_{13}(y)\\ &\quad -\frac{1}{12}ba_7 z^4 - ba_8 z - \frac{1}{2}z^2 F'_{14}(y) - zF'_{15}(y) + F_{18}(y),\\ X_2 &= bz + F_8(y),\\ X_3 &= -bx + zF_{14}(y) + F_{15}(y),\\ X_4 &= -bx + wF_{14}(y) + F_{13}(y),\\ \alpha + \Lambda\beta &= \frac{2x}{z}b - \frac{2}{z}F_{15}(y).\end{aligned}$$

Substituting X_1, X_2, X_3, X_4 and H in (18) and using last equation, we obtain

$$\begin{aligned}&-(a_7 z^3 - 6a_8)\left(-bx + zF_{14}(y) + F_{15}(y)\right) - (2a_7 z^4 + 6a_8 z)F'_8(y)\\ &-6z\left(-xF''_8(y) + 2xF'_{14}(y)\right.\\ &\left.-\frac{1}{2}w^2 F''_{14}(y) - wF''_{13}(y) - \frac{1}{2}z^2 F''_{14}(y) - zF''_{15}(y) + F'_{18}(y)\right)\\ &= -(a_7 z^3 + 3a_8)(2bx - 2F_{15}(y)).\end{aligned}$$

Since x is arbitrary, we have

$$b(a_7z^3 - 6a_8) - 6z(-F_8''(y) + 2F_{14}'(y)) + 2b(a_7z^3 + 3a_8) = 0.$$

This implies that $b = 0$ and $-F_8''(y) + 2F_{14}'(y) = 0$. Also, the above equation is a polynomial with respect to z, w. The coefficient of z^4 implies that $F_{14}(y) + 2F_8'(y) = 0$. Thus $F_{14}'(y) = F_8''(y) = 0$. Then $F_{14}(y) = c_1$ and $F_8(y) = -\frac{1}{2}c_1y + c_2$ for some constants c_1, c_2. The coefficient of z^3 yields $F_{15}(y) = 0$. The coefficient of z leads

$$a_8F_{14}(y) + wF_{13}''(y) - F_{18}'(y) - a_8F_8'(y) = 0.$$

Since w is arbitrary, we infer $F_{13}''(y) = 0$ and $a_8F_{14}(y) - F_{18}'(y) - a_8F_8'(y) = 0$. Hence, $F_{13}(y) = c_3y + c_4$ and $F_{18}(y) = \frac{3}{2}c_1a_8y + c_5$ for some constants c_3, c_4 and c_5. Therefore,

$$\begin{aligned} X_1 &= \frac{5}{2}c_1x - c_3w + \frac{3}{2}c_1a_8y + c_5, \\ X_2 &= -\frac{1}{2}c_1y + c_2, \\ X_3 &= c_1z, \\ X_4 &= c_1w + c_3y + c_4, \\ \alpha + \Lambda\beta &= 0. \end{aligned} \tag{59}$$

From Equation (11) we have $\alpha = \beta = 0$.
Now, we consider $2A'(z) - zA''(z) \neq 0$. Thus Equations (34) and (35) yield $\partial_3X_2 = \partial_4X_2 = 0$. Then $a = b = F_7(y) = 0$ and X_2 becomes $X_2 = F_8(y)$. Also, we have

$$\begin{aligned} X_1 &= -xF_8'(y) + 2xF_{14}(y) - \frac{1}{2}w^2F_{14}'(y) - wF_{13}'(y) + F_{11}(y,z), \\ X_2 &= F_8(y), \\ X_3 &= zF_{14}(y) + F_{15}(y), \\ X_4 &= wF_{14}(y) + F_{13}(y), \\ \alpha + \Lambda\beta &= -\frac{2}{z}F_{15}(y). \end{aligned}$$

From (12) we have $\partial_3F_{11}(y,z) + zF_{14}'(y) + F_{15}'(y) = 0$ then

$$F_{11}(y,z) = -\frac{1}{2}z^2F_{14}'(y) - zF_{15}'(y) + F_{19}(y),$$

for some smooth function F_{19}. Then

$$\begin{aligned} X_1 = &-xF_8'(y) + 2xF_{14}(y) - \frac{1}{2}w^2F_{14}'(y) - wF_{13}'(y) - \frac{1}{2}z^2F_{14}'(y) \\ &-zF_{15}'(y) + F_{19}(y), \end{aligned}$$

and

$$(G-3Hz^{-2})\{2z\partial_2X_1+2zH\partial_2X_2-2HX_3+zH'X_3-\frac{2}{3}z^2GF_{15}(y)\}$$
$$-(zH+\frac{1}{3}z^3G)\{X_3G'+6z^{-2}\partial_2X_1+G\partial_2X_2+6Hz^{-3}F_{15}(y)\}=0.$$

Then

$$\begin{aligned}&(G-3Hz^{-2})\{-2xzF_8''(y)+4xzF_{14}'(y)-zw^2F_{14}''(y)-2zwF_{13}''(y)\\&-z^3F_{14}''(y)-2z^2F_{15}''(y)+2zF_{19}'(y)+2zHF_8'(y)-2H(zF_{14}(y)+F_{15}(y))\\&+zH'(zF_{14}(y)+F_{15}(y))-\frac{2}{3}z^2GF_{15}(y)\}\\&-(zH+\frac{1}{3}z^3G)\{(zF_{14}(y)+F_{15}(y))G'-6xz^{-2}F_8''(y)+12xz^{-2}F_{14}'(y)\\&-3z^{-2}w^2F_{14}''(y)-6z^{-2}wF_{13}''(y)-3F_{14}''(y)\\&-6z^{-1}F_{15}''(y)+6z^{-2}F_{19}'(y)+GF_8'(y)+6Hz^{-3}F_{15}(y)\}=0.\end{aligned}$$

The coefficient of w^2 implies that $F_{14}''(y)=0$ and the coefficient of w leads to $F_{13}''(y)=0$. The coefficient of x yields $F_8''(y)-2F_{14}'(y)=0$. Also, we have

$$\begin{aligned}F_{19}'(y)\;=\;&\frac{z}{12H}(G-3Hz^{-2})\{-2z^2F_{15}''(y)+2zHF_8'(y)-2H(zF_{14}(y)+F_{15}(y))\\&+zH'(zF_{14}(y)+F_{15}(y))-\frac{2}{3}z^2GF_{15}(y)\}-\frac{z}{12H}(zH+\frac{1}{3}z^3G)\{(zF_{14}(y)\\&+F_{15}(y))G'-6z^{-1}F_{15}''(y)+GF_8'(y)+6Hz^{-3}F_{15}(y)\}.\end{aligned}$$

Then

$$\begin{aligned}&F_{14}(y)=c_6y+c_7,\\&F_{13}(y)=c_8y+c_9,\\&F_8(y)=c_6y^2+c_{10}y+c_{11},\end{aligned}$$

and

$$\begin{aligned}F_{19}'(y)\;=\;&\frac{z}{12H}(G-3Hz^{-2})\{-2z^2F_{15}''(y)+2zH(c_6y+c_{10})-2H(c_{7z}+F_{15}(y))\\&+zH'(z(c_6y+c_7)+F_{15}(y))-\frac{2}{3}z^2GF_{15}(y)\}\\&-\frac{z}{12H}(zH+\frac{1}{3}z^3G)\{(z(c_6y+c_7)\\&+F_{15}(y))G'-6z^{-1}F_{15}''(y)+G(2c_6y+c_{10})+6Hz^{-3}F_{15}(y)\},\end{aligned}$$

for some constants $c_7, \cdots, c_{11}$. Also,

$$\begin{aligned} X_1 &= (2c_7 - c_{10})x - \frac{1}{2}c_6 w^2 - c_8 w - \frac{1}{2}c_6 z^2 - zF'_{15}(y) + F_{19}(y), \\ X_2 &= c_6 y^2 + c_{10} y + c_{11}, \\ X_3 &= z(c_6 y + c_7) + F_{15}(y), \\ X_4 &= w(c_6 y + c_7) + c_8 y + c_9, \\ \alpha + \Lambda\beta &= -\frac{2}{z}F_{15}(y), \end{aligned} \tag{60}$$

$$\begin{aligned} \alpha &= -(-G + 3z^{-2}H)^{-1}\{(z(c_6 y + c_7) + F_{15}(y))G' \\ &\quad -6z^{-1}F''_{15}(y) + 6z^{-2}F'_{19}(y) + G(2c_6 y + c_{10}) + 6Hz^{-3}F_{15}(y)\}. \end{aligned}$$

Theorem 3.7. *Any homogeneous Siklos spacetime with $H = A(z)$ has Ricci bi-conformal vector field $X = X_i\partial_i$ if and only if*

i) when $2A'(z) - zA''(z) = 0$ then $X_i, i = 1, 2, 3, 4$ satisfy in (59).

ii) When $2A'(z) - zA''(z) \neq 0$ then $X_i, i = 1, 2, 3, 4$ satisfy in (60).

Suppose that $X = \nabla f$ for some smooth function f which are Ricci bi-conformal vector fields on homogeneous Siklos spacetime with nonzero constant H. We have

$$\begin{aligned} -\frac{\Lambda z^2}{3}(-H\partial_1 f + \partial_2 f) &= X_1, \\ -\frac{\Lambda z^2}{3}\partial_2 f &= X_2, \\ -\frac{\Lambda z^2}{3}\partial_3 f &= X_3, \\ -\frac{\Lambda z^2}{3}\partial_4 f &= X_4. \end{aligned} \tag{61}$$

Deriving the third Equation (61) with respect to w and deriving the fourth Equation (42) with respect to z we infer $a = b = b_1 = b_3 = b_6 = b_9 = b_{10} = 0$. By similar method we conclude $b_2 = b_4 = b_5 = 0$. Thus

$$\begin{aligned} \partial_1 f &= 0, & \partial_2 f &= 0, \\ \partial_3 f &= -\frac{3}{\Lambda}b_8 z^{-2}, & \partial_4 f &= 0. \end{aligned}$$

Therefore we have the following corollary:

Corollary 3.8. *Homogeneous Siklos spacetime defined by nonzero constant H has Ricci bi-conformal vector field as $X = \nabla f$ if and only if $f = b_8\frac{3}{\Lambda z} + k$, where $k \in \mathbb{R}$.*

3.5 Siklos metric with $H = A(y)z^2$

Now, we assume that $H = A(y)z^2$, in this case we have

$$\frac{6H - 2z(\partial_3 H) + z^2(\partial^2_{33}H + \partial^2_{44}H)}{2z^2} = \frac{6A(y)z^2 - 4z^2A(y) + 2z^2A(y)}{2z^2} = 2A(y),$$

$$\begin{aligned}
X_1 &= -zxF_5'(y) - xF_8'(y) - ax^2 + 2xF_{14}(y) + xwF_7'(y)\\
&\quad -\frac{w^3 - 3z^2w}{6}F_7''(y) - \frac{1}{2}w^2F_{14}'(y)\\
&\quad -wF_{13}'(y) - \int (awH + HF_7(y))\,dw + F_{11}(y,z),\\
X_2 &= a\frac{z^2 + w^2}{2} + zF_5(y) + wF_7(y) + F_8(y),\\
X_3 &= -x(az + F_5(y)) + wzF_7'(y) + zF_{14}(y) + F_{15}(y),\\
X_4 &= -x(aw + F_5(y)) + \frac{w^2 - z^2}{2}F_7'(y) + wF_{14}(y) + F_{13}(y),\\
\alpha + \Lambda\beta &= \frac{2x}{z}F_5(y) - \frac{2}{z}F_{15}(y).
\end{aligned}$$

Since

$$2(\partial_3 H) - z(\partial^2_{33}H + \partial^2_{44}H) = 2zA(y).$$

Thus Equations (34) and (35) yield $\partial_3 X_2 = \partial_4 X_2 = 0$. Then $a = b = F_7(y) = 0$ and X_2 becomes $X_2 = F_8(y)$. From Equation (12), we obtain

$$\partial_3 F_{11}(y,z) = -zF_{14}'(y) - F_{15}'(y).$$

Then

$$F_{11}(y,z) = -\frac{1}{2}z^2F_{14}'(y) - zF_{15}'(y) + F_{17}(y),$$

for some smooth function F_{17}. Thus

$$\begin{aligned}
X_1 &= -xF_8'(y) + 2xF_{14}(y) - \frac{1}{2}w^2F_{14}'(y) - wF_{13}'(y) - \frac{1}{2}z^2F_{14}'(y)\\
&\quad -zF_{15}'(y) + F_{17}(y),\\
X_2 &= F_8(y),\\
X_3 &= zF_{14}(y) + F_{15}(y),\\
X_4 &= wF_{14}(y) + F_{13}(y),\\
\alpha + \Lambda\beta &= -\frac{2}{z}F_{15}(y).
\end{aligned}$$

Using (12) and (19), we get

$$\begin{aligned}&-2zxF_8''(y)+4zxF_{14}'(y)-zw^2F_{14}''(y)-2zwF_{13}''(y)-z^3F_{14}''(y)\\&-2z^2F_{15}''(y)+2zF_{17}'(y)+z^3F_8(y)A'(y)+2z^3A(y)F_8'(y)\\&-\frac{4}{3}z^2A(y)F_{15}(y)-\frac{5}{3}z^3\{-2F_8(y)A'(y)+6xz^{-2}F_8''(y)-12xz^{-2}F_{14}'(y)\\&+3z^{-2}w^2F_{14}''(y)+6z^{-2}wF_{13}''(y)+3F_{14}''(y)+6z^{-1}F_{15}''(y)-6z^{-2}F_{17}'(y)\\&-4A(y)F_8'(y)-6z^{-1}A(y)F_{15}(y)\}=0.\end{aligned}$$

Since x is arbitrary, we conclude

$$-F_8''(y)+2F_{14}'(y)=0,$$

and the coefficient of w^2 implies that

$$F_{14}''(y)=0.$$

Hence, $F_{14}(y)=d_1y+d_2$ and $F_8(y)=d_1y^2+d_3y+d_4$. Also the coefficient of w yields $F_{13}''(y)=0$ then $F_{13}(y)=d_5y+d_6$. Hence,

$$\begin{aligned}&-2z^2F_{15}''(y)+2zF_{17}'(y)+z^3F_8(y)A'(y)+2z^3A(y)F_8'(y)-\frac{4}{3}z^2A(y)F_{15}(y)\\&-\frac{5}{3}z^3\{-2F_8(y)A'(y)+6z^{-1}F_{15}''(y)-6z^{-2}F_{17}'(y)\\&-4A(y)F_8'(y)-6z^{-1}A(y)F_{15}(y)\}=0.\end{aligned}$$

The above equation is a polynomial with respect to z. Since z is arbitrary we get $F_8(y)A'(y)+2A(y)F_8'(y)=0$, $F_{17}'(y)=0$, and $-12F_{15}''(y)+13A(y)F_{15}(y)=0$. Thus $F_{17}(y)=d_7$ and $A(y)=\frac{d_8}{(d_1y^2+d_3y+d_4)^2}$. Therefore,

$$\begin{aligned}X_1&=-x(2d_1y+d_3)+2x(d_1y+d_2)-\frac{1}{2}d_1w^2-d_5w-\frac{1}{2}d_1z^2\\&\quad-zF_{15}'(y)+d_7,\\X_2&=d_1y^2+d_3y+d_4,\\X_3&=z(d_1y+d_2)+F_{15}(y),\\X_4&=w(d_1y+d_2)+d_5y+d_6,\\\Lambda\beta&=-\frac{5}{2}z^{-1}F_{15}(y),\\\alpha&=\frac{1}{2}z^{-1}F_{15}(y).\end{aligned}\tag{62}$$

Theorem 3.9. *Any homogeneous Siklos spacetime with $H=A(y)z^2$ has Ricci bi-conformal vector field $X=X_i\partial_i$ if and only if $\alpha,\beta,X_i,i=1,2,3,4$ satisfy in (62).*

Now, we consider the vector fields as $X = \nabla f$ for some smooth function f which are Ricci bi-conformal vector fields on homogeneous Siklos spacetime with $H = A(y)z^2$. We have $d_i = 0$ for $i = 1, \cdots, 7$. Therefore we have the following corollary:

Corollary 3.10. *Homogeneous Siklos spacetime defined by $H = A(y)z^2$ has Ricci bi-conformal vector field as $X = \nabla f$ if and only if $f = F_{15}(y)\frac{3}{\Lambda z} + k$, where $k \in \mathbb{R}$.*

3.6 Siklos metric with $H = y^{2k-2}A(y^k z)$

Now we consider $H = y^{2k-2}A(u)$ where $u = y^k z$. We have

$$2(\partial_3 H) - z(\partial^2_{33}H + \partial^2_{44}H) = y^{3k-2}(2A'(y^k z) - zy^k A''(y^k z)).$$

If $2A'(y^k z) - zy^k A''(y^k z) = 0$ then $H = k_1 y^{5k-2}z^3 + k_2 y^{2k-2}$ for some constants k_1, k_2. In this case, we have

$$\frac{6H - 2z(\partial_3 H) + z^2(\partial^2_{33}H + \partial^2_{44}H)}{2z^2} = 3k_1 y^{5k-2}z + 3k_2 y^{2k-2}z^{-2}.$$

From Equation (38), we get

$$\begin{aligned} &zwF_7''(y) - 3k_1 y^{5k-2}x^2(\frac{1}{2}aw^2 + wF_7(y)) + \partial_3 F_{11}(y, z) \\ &+(k_1 y^{5k-2}z^3 + k_2 y^{2k-2})(az + b) + zF_{14}'(y) + F_{15}'(y) = 0. \end{aligned}$$

Since w is arbitrary we deduce $ak_1 = 0$, $F_7''(y) = 0$,

$$\begin{aligned} &k_1 F_7(y) = 0, \\ &\partial_3 F_{11}(y, z) + ak_2 y^{2k-2}z + b(k_1 y^{5k-2}z^3 + k_2 y^{2k-2}) + zF_{14}'(y) + F_{15}'(y) = 0. \end{aligned}$$

Integrating the last equation, we arrive at

$$F_{11}(y, z) = -\frac{1}{2}ak_2 y^{2k-2}z^2 - b(\frac{1}{4}k_1 y^{5k-2}z^4 + k_2 y^{2k-2}z) - \frac{1}{2}z^2 F_{14}'(y) - F_{15}'(y)z + F_{20}(y),$$

for some smooth function F_{20}. Therefore, $F_7(y) = k_3 y + k_4$ for some constants k_3, k_4 and

$$\begin{aligned}
X_1 &= -xF_8'(y) + 2xF_{14}(y) + k_3 xw - \frac{1}{2}w^2 F_{14}'(y) - wF_{13}'(y) \\
&\quad -k_2 y^{2k-2} w(k_3 y + k_4) - \frac{1}{2} a k_2 y^{2k-2} z^2 - b(\frac{1}{4} k_1 y^{5k-2} z^4 + k_2 y^{2k-2} z) \\
&\quad -\frac{1}{2} z^2 F_{14}'(y) - F_{15}'(y) z + F_{20}(y), \\
X_2 &= a\frac{z^2 + w^2}{2} + bz + w(k_3 y + k_4) + F_8(y), \\
X_3 &= -x(az + b) + k_3 wz + zF_{14}(y) + F_{15}(y), \\
X_4 &= -x(aw + b) + k_3 \frac{w^2 - z^2}{2} + wF_{14}(y) + F_{13}(y), \\
\alpha + \Lambda\beta &= \frac{2bx}{z} - \frac{2}{z} F_{15}(y).
\end{aligned}$$

Substituting X_1, X_2, X_3, X_4 and H in (18) and using last equation, we obtain

$$\begin{aligned}
&-3z^3 \left(a\frac{z^2 + w^2}{2} + bz + w(k_3 y + k_4) + F_8(y) \right) ((5k-2) k_1 y^{5k-1} z \\
&+(2k-2) k_2 y^{2k-1} z^{-2}) \\
&-3z^3(-x(az + b) + k_3 wz + zF_{14}(y) + F_{15}(y))(k_1 y^{5k-2} - 2k_2 y^{2k-2} z^{-3}) \\
&-6z\{-xF_8''(y) + 2xF_{14}'(y) - \frac{1}{2} w^2 F_{14}''(y) - wF_{13}''(y) - k_2(2k-2) y^{2k-3} w(k_3 y + k_4) \\
&-k_2 k_3 y^{2k-2} w - a k_2 (k-1) y^{2k-3} z^2 - b(\frac{1}{4} k_1 (5k-2) y^{5k-3} z^4 + k_2 (2k-2) y^{2k-3} z) \\
&-\frac{1}{2} z^2 F_{14}''(y) - F_{15}''(y) z + F_{20}'(y)\} \\
&-6(k_1 y^{5k-2} z^3 + k_2 y^{2k-2})(k_3 w + F_8'(y)) \\
&= -6(k_1 y^{5k-2} z^3 + k_2 y^{2k-2})(bx - F_{15}(y)).
\end{aligned}$$

Since x is arbitrary we have

$$3(az+b)(k_1 y^{5k-2} z^3 - 2k_2 y^{2k-2}) - 6z(-F_8''(y) + 2F_{14}'(y)) = -6b(k_1 y^{5k-2} z^3 + k_2 y^{2k-2}).$$

This implies that $bk_1 = 0$ and $-F_8''(y) + 2F_{14}'(y) = -ak_2 y^{2k-2}$. Also, the above equation is a polynomial with respect to z, w. The coefficient of w^2 implies that $F_{14}''(y) = a(k-1)k_2 y^{2k-1}$. The coefficient of z^4 leads to

$$(w(k_3 y + k_4) + F_8(y))(5k-2) k_1 y^{5k-1} + (k_3 w + F_{14}(y)) k_1 y^{5k-2} = 0.$$

Then $k_1 k_3 = (5k-2) k_1 y F_8(y) + k_1 F_{14}(y) = (5k-2) k_1 k_4 = 0$. The coefficient of z^0 implies that $k_2 k_3 = k_2 F_8'(y) = 0$. Since $k_1 k_3 = k_2 k_3 = 0$ and $H \neq 0$ we infer $k_3 =$

0. The coefficient of z^2 implies that $F''_{15}(y) = b(k-1)k_2y^{2k-1} + 2bk_2(k-1)y^{2k-3}$. The coefficient of z yields

$$(k_4w+F_8(y))(k-1)k_2y^{2k-1}+k_2F_{14}(y)y^{2k-2}-wF''_{13}(y)-k_2(2k-2)wy^{2k-3}+F'_{20}(y)=0.$$

Then

$$F''_{13}(y) = k_4(k-1)y^{2k-1} - k_2(2k-2)y^{2k-3},$$

and

$$F_8(y)(k-1)k_2y^{2k-1} + k_2F_{14}(y)y^{2k-2} + F'_{20}(y) = 0.$$

The coefficient of z^3 implies that

$$\begin{aligned}&-a(k-1)k_2y^{2k-1} - k_1F_{15}(y)y^{5k-1} + 2a(k-1)k_2y^{2k-3}\\&+F''_{14}(y) - 2k_1y^{5k-2}F'_8(y) - 2k_1y^{5k-2}F_{15}(y) = 0.\end{aligned}$$

If $k_1 = 0$ then $k_2 \neq 0$. Equation $k_2F'_8(y) = 0$ implies that $F'_8(y) = 0$ then $F_8(y) = k_5$. The equation $-F''_8(y)+2F'_{14}(y) = -ak_2y^{2k-2}$ yields $F''_{14}(y) = -a(k-1)k_2y^{2k-3}$. Then it together with $F''_{14}(y) = a(k-1)k_2y^{2k-1}$ leads to $a(k-1) = 0$ and $F''_{14}(y) = 0$. Since H is nonconstant then $k \neq 1$ and $a = 0$. Hence, $F_{14} = k_6$. In this case, we have

$$\begin{aligned}X_1 &= 2k_6x - wF'_{13}(y) - k_2y^{2k-2}k_4w - bk_2y^{2k-2}z - F'_{15}(y)z + F_{20}(y),\\X_2 &= bz + k_4w + k_5,\\X_3 &= -bx + k_6z + F_{15}(y),\\X_4 &= -bx + k_6w + F_{13}(y),\\\alpha + \Lambda\beta &= \frac{2bx}{z} - \frac{2}{z}F_{15}(y),\end{aligned} \tag{63}$$

where

$$\begin{aligned}F''_{15}(y) &= b(k-1)k_2(y^{2k-1} + 2y^{2k-3}),\\F''_{13}(y) &= (k-1)(k_4y^{2k-1} - 2k_2y^{2k-3}),\\F'_{20}(y) &= -k_2(k_5(k-1)y^{2k-1} + k_6y^{2k-2}).\end{aligned}$$

If $k_1 \neq 0$ then $a = b = k_4 = 0$. If $k = \frac{2}{5}$ then $F_{14}(y) = 0$ and $F_8(y) = k_{21}y + k_{22}$ where $k_2k_{21} = 0$. Then, $F_{15} = k_{23}y + k_{24}$, $F_{13}(y) = 5k_2y^{-\frac{1}{5}} + k_{25}y + k_{26}$, $F_{20}(y) =$

$\frac{3}{4}k_2k_{22}y^{\frac{4}{5}}$. Thus,

$$\begin{aligned} X_1 &= -k_{21}x - w(-k_2y^{-\frac{6}{5}} + k_{25}) - k_{23}z + \frac{3}{4}k_2k_{22}y^{\frac{4}{5}}, \\ X_2 &= k_{21}y + k_{22}, \\ X_3 &= k_{23}y + k_{24}, \\ X_4 &= 5k_2y^{-\frac{1}{5}} + k_{25}y + k_{26}, \\ \alpha + \Lambda\beta &= -\frac{2}{z}(k_{23}y + k_{24}), \\ \alpha &= \frac{1}{2}(k_1z^4 + k_2y^{-\frac{6}{5}}z)^{-1}\{-\frac{12}{5}zwk_2y^{-\frac{11}{5}} + \frac{6}{5}zk_2k_{22}y^{-\frac{1}{5}} \\ &\quad -\frac{6}{5}k_2k_{22}zy^{-\frac{11}{5}} + 2k_1k_{21}z^4 + (k_1z^3 - 2k_2y^{-\frac{6}{5}})(k_{23}y + k_{24}) \\ &\quad -2z^{-1}(k_{23}y + k_{24})\}. \end{aligned} \tag{64}$$

Now, we assume that $k \neq \frac{2}{5}$. Therefore $F_{14}(y) = F_8(y) = 0$ and

$$\begin{aligned} X_1 &= -wF'_{13}(y) - F'_{15}(y)z + k_{25}, \\ X_2 &= 0, \\ X_3 &= F_{15}(y), \\ X_4 &= F_{13}(y), \\ \alpha + \Lambda\beta &= -\frac{2}{z}F_{15}(y), \\ \alpha &= \frac{1}{2z(k_1y^{5k-2}z^3 + k_2y^{2k-2})}\{-2zwF''_{13}(y) - 2z^2F''_{15}(y) \\ &\quad +(5k-2)k_1y^{5k-3}z^4 + (2k-2)k_2y^{2k-3}z \\ &\quad -4(k_1y^{5k-2}z^3 + k_2y^{2k-2})F_{15}(y) + 3k_1y^{5k-2}z^3F_{15}(y)\}, \end{aligned} \tag{65}$$

where

$$\begin{aligned} F''_{15}(y) &= b(k-1)k_2(y^{2k-1} + 2y^{2k-3}), \\ F''_{13}(y) &= -2(k-1)k_2y^{2k-3}. \end{aligned}$$

Now, we consider $2A'(y^kz) - zy^kA''(y^kz) \neq 0$. Thus Equations (34) and (35) yield $\partial_3X_2 = \partial_4X_2 = 0$. Then $a = b = F_7(y) = 0$ and X_2 becomes $X_2 = F_8(y)$. From (12), we have $\partial_3F_{11}(y,z) + zF'_{14}(y) + F'_{15}(y) = 0$, then

$$F_{11}(y,z) = -\frac{1}{2}z^2F'_{14}(y) - zF'_{15}(y) + F_{21}(y),$$

for some smooth function F_{21}. Then

$$\begin{aligned} X_1 &= -xF'_8(y) + 2xF_{14}(y) - \frac{1}{2}w^2F'_{14}(y) - wF'_{13}(y) - \frac{1}{2}z^2F'_{14}(y) \\ &\quad -zF'_{15}(y) + F_{21}(y), \end{aligned}$$

and from (12) and (19), we have

$$(G-3Hz^{-2})\{2z\partial_2X_1+z\partial_2HX_2+2zH\partial_2X_2-2HX_3+z\partial_3HX_3$$
$$-\frac{2}{3}z^2GF_{15}(y)\}-(zH+\frac{1}{3}z^3G)\{X_2\partial_2G+X_3\partial_3G+6z^{-2}\partial_2X_1+G\partial_2X_2$$
$$+6Hz^{-3}F_{15}(y)\}=0.$$

Hence,

$$(G-3Hz^{-2})\{-2xzF_8''(y)+4xzF_{14}'(y)-zw^2F_{14}''(y)-2zwF_{13}''(y)-z^3F_{14}''(y)$$
$$-2z^2F_{15}''(y)+2zF_{21}'(y)+(2k-2)y^{2k-3}zA(y^kz)F_8(y)+ky^{3k-3}zA'(y^kz)F_8(y)$$
$$+2zy^{2k-2}A(y^kz)F_8'(y)-2y^{2k-2}A(y^kz)(zF_{14}(y)+F_{15}(y))$$
$$+zy^{3k-2}A'(y^kz)(zF_{14}(y)+F_{15}(y))-\frac{2}{3}z^2GF_{15}(y)\}$$
$$-(zH+\frac{1}{3}z^3G)\{F_8(y)\partial_2G+(zF_{14}(y)+F_{15}(y))\partial_3G-6xz^{-2}F_8''(y)$$
$$+12xz^{-2}F_{14}'(y)-3z^{-2}w^2F_{14}''(y)-6z^{-2}wF_{13}''(y)-3F_{14}''(y)$$
$$-6z^{-1}F_{15}''(y)+6z^{-2}F_{19}'(y)+GF_8'(y)+6Hz^{-3}F_{15}(y)\}=0.$$

In the above equation the coefficient of w^2 implies that $F_{14}''(y)=0$ and the coefficient of w yields $F_{13}''(y)=0$. The coefficient of x leads to $F_8''(y)-2F_{14}'(y)=0$. Also, we have

$$\begin{aligned}F_{21}'(y) &= \frac{z}{12H}(G-3Hz^{-2})\{-2z^2F_{15}''(y)+2zHF_8'(y)-2H(zF_{14}(y)+F_{15}(y))\\&\quad+zH'(zF_{14}(y)+F_{15}(y))-\frac{2}{3}z^2GF_{15}(y)\}-\frac{z}{12H}(zH+\frac{1}{3}z^3G)\{(zF_{14}(y)\\&\quad+F_{15}(y))G'-6z^{-1}F_{15}''(y)+GF_8'(y)+6Hz^{-3}F_{15}(y)\}.\end{aligned}$$

Then

$$F_{14}(y)=c_6y+c_7,$$
$$F_{13}(y)=c_8y+c_9,$$
$$F_8(y)=c_6y^2+c_{10}y+c_{11},$$

and

$$\begin{aligned}F_{19}'(y) &= \frac{z}{12H}(G-3Hz^{-2})\{-2z^2F_{15}''(y)+2zH(c_6y+c_{10})-2H(c_{7z}+F_{15}(y))\\&\quad+zH'(z(c_6y+c_7)+F_{15}(y))-\frac{2}{3}z^2GF_{15}(y)\}\\&\quad-\frac{z}{12H}(zH+\frac{1}{3}z^3G)\{(z(c_6y+c_7)+F_{15}(y))G'\\&\quad-6z^{-1}F_{15}''(y)+G(2c_6y+c_{10})+6Hz^{-3}F_{15}(y)\},\end{aligned}$$

for some constants $c_7, \cdots, c_{11}$. Also,

$$\begin{aligned} X_1 &= (2c_7 - c_{10})x - \frac{1}{2}c_6 w^2 - c_8 w - \frac{1}{2}c_6 z^2 - zF'_{15}(y) + F_{19}(y), \\ X_2 &= c_6 y^2 + c_{10} y + c_{11}, \\ X_3 &= z(c_6 y + c_7) + F_{15}(y), \\ X_4 &= w(c_6 y + c_7) + c_8 y + c_9, \\ \alpha + \Lambda\beta &= -\frac{2}{z}F_{15}(y), \end{aligned} \tag{66}$$

$$\begin{aligned} \alpha = & -(-G + 3z^{-2}H)^{-1}\{(z(c_6 y + c_7) + F_{15}(y))G' \\ & -6z^{-1}F''_{15}(y) + 6z^{-2}F'_{19}(y) + G(2c_6 y + c_{10}) + 6Hz^{-3}F_{15}(y)\}. \end{aligned}$$

Theorem 3.11. *Any homogeneous Siklos spacetime with $H = y^{2k-2}A(y^k z)$ has Ricci bi-conformal vector field $X = X_i\partial_i$ if and only if*

i) for $2A'(y^k z) - zy^k A''(y^k z) = 0$ the functions X_i satisfy in (63) or (64), or (65),

ii) for $2A'(y^k z) - zy^k A''(y^k z) \neq 0$ the functions X_i satisfy in (66).

Corollary 3.12. *Homogeneous Siklos spacetime defined by $H = y^{2k-2}A(y^k z)$ has Ricci bi-conformal vector field as $X = \nabla f$ if and only if $f = F_{15}(y)\frac{3}{\Lambda z} + k$, where $k \in \mathbb{R}$.*

3.7 Siklos metrics with $H = \epsilon z^k$

Let $H = \epsilon z^k$ where $\epsilon = \pm 1$ and $k \neq 0$. We have

$$2(\partial_3 H) - z(\partial^2_{33}H + \partial^2_{44}H) = \epsilon(3k - k^2)z^{k-1}.$$

If $k \neq 3$ then Equations (34) and (35) yield $\partial_3 X_2 = \partial_4 X_2 = 0$. Then $a = b = F_7(y) = 0$ and X_2 becomes $X_2 = F_8(y)$. From Equation (12), we obtain

$$\partial_3 F_{11}(y, z) = -zF'_{14}(y) - F'_{15}(y).$$

Then

$$F_{11}(y, z) = -\frac{1}{2}z^2 F'_{14}(y) - zF'_{15}(y) + F_{22}(y).$$

Thus

$$\begin{aligned} X_1 &= -xF'_8(y) + 2xF_{14}(y) - \frac{1}{2}w^2 F'_{14}(y) - wF'_{13}(y) - \frac{1}{2}z^2 F'_{14}(y) \\ & \quad -zF'_{15}(y) + F_{22}(y), \\ X_2 &= F_8(y), \\ X_3 &= zF_{14}(y) + F_{15}(y), \\ X_4 &= wF_{14}(y) + F_{13}(y), \\ \alpha + \Lambda\beta &= -\frac{2}{z}F_{15}(y). \end{aligned}$$

Putting these equations in (12) and (19), we obtain

$$\begin{aligned}&-2zxF_8''(y)+4zxF_{14}'(y)-zw^2F_{14}''(y)-2zwF_{13}''(y)-z^3F_{14}''(y)\\&-2z^2F_{15}''(y)+2zF_{22}'(y)+2\epsilon z^{k+1}F_8'(y)-2\epsilon z^k(zF_{14}(y)+F_{15}(y))\\&+\epsilon kz^k(zF_{14}(y)+F_{15}(y))-\epsilon\frac{1}{3}(k^2-3k+6)z^kF_{15}(y)+\frac{2}{3}z^3(k^2-3k+9)\{\\&-\frac{1}{2}\epsilon(k-2)(k^2-3k+6)z^{k-3}(zF_{14}+F_{15}(y))\\&+6xz^{-2}F_8''(y)-12xz^{-2}F_{14}'(y)+3z^{-2}w^2F_{14}''(y)\\&+6z^{-2}wF_{13}''(y)+3F_{14}''(y)+6z^{-1}F_{15}''(y)-6z^{-2}F_{22}'(y)\\&+\epsilon(k^2-3k+6)z^{k-2}F_8'(y)-6\epsilon z^{k-3}\}.\end{aligned}$$

Since x is arbitrary, we conclude $-F_8''(y)+2F_{14}'(y)=0$, and the coefficient of w^2 implies that $F_{14}''(y)=0$. Then $F_{14}(y)=r_1y+r_2$ and $F_8(y)=r_1y^2+r_3y+r_4$ for some constants $r_1,...,r_4$. Also the coefficient of w yields $F_{13}''(y)=0$ then $F_{13}(y)=r_5y+r_6$ for some constants r_5,r_6. Hence,

$$\begin{aligned}&-2z^2F_{15}''(y)+2zF_{22}'(y)+2\epsilon z^{k+1}(2r_1y+r_3)-2\epsilon z^k(z(r_1y+r_2)+F_{15}(y))\\&+\epsilon kz^k(z(r_1y+r_2)+F_{15}(y))-\epsilon\frac{1}{3}(k^2-3k+6)z^kF_{15}(y)+\frac{2}{3}z^3(k^2-3k+9)\{\\&-\frac{1}{2}\epsilon(k-2)(k^2-3k+6)z^{k-3}(z(r_1y+r_2)+F_{15}(y))\\&++6z^{-1}F_{15}''(y)-6z^{-2}F_{22}'(y)\\&+\epsilon(k^2-3k+6)z^{k-2}(2r_1y+r_3)-6\epsilon z^{k-3}\}.\end{aligned}$$

Therefore,

$$\begin{aligned}X_1&=-x(2r_1y+r_3)+2x(r_1y+r_2)-\frac{1}{2}w^2r_1-wr_5-\frac{1}{2}z^2r_1\\&\quad-zF_{15}'(y)+F_{22}(y),\\X_2&=r_1y^2+r_3y+r_4,\\X_3&=z(r_1y+r_2)+F_{15}(y),\\X_4&=w(r_1y+r_2)+r_5y+r_6,\\\alpha+\Lambda\beta&=-\frac{2}{z}F_{15}(y).\end{aligned}\tag{67}$$

Now we assume that $k=3$ then (38) leads to

$$\begin{aligned}&zwF_7''(y)-3\epsilon z^2(\frac{1}{2}aw^2+wF_7(y))+\partial_3F_{11}(y,z)\\&+\epsilon z^3(az+b)+zF_{14}'(y)+F_{15}'(y)=0.\end{aligned}$$

Since w is arbitrary we conclude $a=F_7(y)=0$ and

$$F_{11}(y,z)=-\frac{1}{4}\epsilon bz^4-\frac{1}{2}z^2F_{14}'(y)-zF_{15}'(y)+F_{23}(y).$$

From (18) we have

$$\begin{aligned}&\epsilon(-bx + zF_{14}(y) + F_{15}(y)) + 2z^{-2}(-xF_8''(y) + 2xF_{14}'(y) - \frac{1}{2}w^2F_{14}''(y)\\&-wF_{13}''(y) - \frac{1}{2}z^2F_{14}''(y) - zF_{15}''(y) + F_{23}'(y)) + 2\epsilon z(bz + F_8'(y))\\&-\epsilon(2bx - 2F_{15}(y)) = 0.\end{aligned}$$

Since x is arbitrary we have $b = -F_8''(y) + 2F_{14}'(y) = 0$. Also, since w is arbitrary we get $F_{14}''(y) = F_{13}''(y) = F_{15}(y) = F_{23}'(y) = F_{14}(y) + 2F_8'(y) = 0$. Then $F_{14} = s_1$, $F_8 = -\frac{1}{2}s_1y + s_2$, $F_{23}(y) = s_3$, $F_{13} = s_4y + s_5$. Therefore

$$\begin{aligned}X_1 &= \frac{1}{2}s_1x + 2s_1x - s_4w + s_3,\\X_2 &= -\frac{1}{2}s_1y + s_2,\\X_3 &= s_1z,\\X_4 &= -s_1w + s_4y + s_5,\\\alpha &= \beta = 0.\end{aligned} \tag{68}$$

Theorem 3.13. *Any homogeneous Siklos spacetime defined by $H = \epsilon z^k$ has Ricci bi-conformal vector field $X = X_i\partial_i$ if and only if α, β and X_i satisfy in (67) for $k \neq 3$ and (68) for $k = 3$.*

Corollary 3.14. *Homogeneous Siklos spacetime defined by nonzero constant H has Ricci bi-conformal vector field as $X = \nabla f$ if and only if $f = F_{15}(y)\frac{3}{\Lambda z} + d_0$ for $k \neq 3$ and $f = d_0$ for $k = 3$ where $d_0 \in \mathbb{R}$.*

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