Mathematics Interdisciplinary Research **9** (1) (2024) 111 − 129

Original Scientific Paper

Some Remarks on the Annihilating-Ideal Graph of Commutative Ring with Respect to an Ideal

Zahra Mahmudiankoruie and Mohammad Hasan Naderi ⋆

Abstract

The graph $AG(R)$ of a commutative ring R with identity has an edge linking two unique vertices when the product of the vertices equals the zero ideal and its vertices are the nonzero annihilating ideals of R. The annihilating-ideal graph with respect to an ideal (I), which is denoted by $AG_I(R)$, has distinct vertices K and J that are adjacent if and only if $KJ \subseteq I$. Its vertices are $\{K \mid KJ \subseteq I$ for some ideal J and $K,\ J \nsubseteq I, K$ is a ideal of $R\}$. The study of the two graphs $AG_I(R)$ and $AG(R/I)$ and extending certain prior findings are two main objectives of this research. This studys among other things, the findings of this study reveal that $AG_I(R)$ is bipartite if and only if $AG_I(R)$ is triangle-free.

Keywords: Annihilating-ideal graph, Cut-point, Girth, $r-$Partite graph.

2020 Mathematics Subject Classification: 05C25, 05C38, 05C40.

How to cite this article
Z. Mahmudiankoruie and M. H. Naderi, Some remarks on the annihilating-ideal graph of commutative ring with respect to an ideal, *Math. Interdisc. Res.* **9** (1) (2024) 111-129.

1. Introduction

We fix the premise that all rings are commutative with identity throughout this paper. The main sources for the concepts and notations utilized in this paper are [1, 2]. To keep this note as self-contained as possible, we first establish certain

⋆Corresponding author (E-mail: mh.naderi@qom.ac.ir)
Academic Editor: Ali Reza Ashrafi
Received 28 July 2022, Accepted 2 November 2022
DOI: 10.22052/MIR.2022.246685.1366

definitions and describe the notation we use before offering an outline of our work. If the ring R contains any non-zero nilpotent elements, it is said to be reduced. The length of the shortest cycle in a graph G is known as its girth, and it is shown by the symbol $gr(G)$. If G does not include any cycles, then the girth of G is considered infinite. An r-partite graph is one whose vertex set can be divided into r subsets, none of which includes all of the edge conditions at both ends. A full graph is one that has an edge connecting each pair of vertices. $Z(R)$ and $\min(R)$, respectively, stand for the set of R zero-divisors and minimal prime ideals. A non-zero ideal I of R is referred to be an annihilating-ideal if there exists a non-zero ideal J of R such that $IJ = 0$, with $A(R)$ being the set of annihilating-ideals of R. Anderson and Livingston [3] introduced the graph with zero divisors $\Gamma(R)$ of a ring R with vertices $Z(R)\backslash\{0\}$ for which distinct vertices $x, y \in Z(R)\backslash\{0\}$ are adjacent if and only if $xy = 0$. Redmond [4] extended the zero-divisor graph of a commutative ring to an ideal-based zero-divisor graph of a commutative ring. For a given ideal I of R, he defined a graph $\Gamma_I(R)$ with vertex set $\{x \in R - I \mid xy \in I \text{ for some } y \in R\}$. The vertices x and y of this graph are adjacent if and only if $xy \in I$. According to him there are some relationships between $\Gamma(R/I)$ and $\Gamma_I(R)$. According to Behboodi and Rakeei [5] the annihilating-ideal graph $AG(R)$ is a graph whose vertex set consists of the set of all non-zero annihilating ideals of R and two distinct vertices are connected by an edge when their product is the zero ideal. Aliniaeifard et al. [6] defined the annihilating-ideal graph $AG_I(R)$ with respect to an ideal I. This graph has the vertex set $V(AG_I(R)) = \{K \mid K \trianglelefteq R \text{ and } \exists J \trianglelefteq R \ s.t. \ KJ \subseteq I \ \& \ J \nsubseteq I\}$ and distinct vertices K and L are adjacent if and only if $KL \subseteq I$. They obtained some relationships between $AG(R)$ and $AG_I(R)$.

The most significant research on the annihilating-ideal graph was done by Visweswaran and his co-authors. To determine the necessary and sufficient conditions for the complement of the annihilating ideal graph to be connected in the instance of this graph, Visweswaran and Patel [7] studied commutative rings with identities that admit at least one non-zero annihilating ideal. The diameter was discovered. The complement of the annihilating ideal graph must satisfy both a necessary and sufficient condition, which was also stated in [8]. The same authors demonstrated in [8] that if the set of all zero-divisors of a ring with above conditions is not an ideal, then the complement of the annihilating ideal graph does not contain any infinite clique if and only if its clique number is finite. These authors classified, up to isomorphism, all rings R such that $Z(R)$ is not an ideal and for which the complement of its annihilating ideal graph does not admit any infinite clique. The case that $Z(R)$ is an ideal was also investigated in [8]. Visweswaran and Parmar [9] of commutative rings with identity which is not an integral domain and introduced a graph $H(R)$ with respect to a ring R with given conditions. They studied interplay between the graph structures of $H(R)$ and ring theoretical properties of R. Visweswaran and Lalchandani [8] considered commutative rings with identities that are not integral domains as they investigated the interaction between the graph structures of $H(R)$ and theoretical ring features of R. The authors categorize semi-quasilocal rings with at least two maximal ideals that have planar annihilating-ideal graphs.

This paper tries to investigate some further connections between $AG(R)$ and $AG_I(R)$. It is shown in Section 2 how to draw the graph $AG_I(R)$ in relation to the graph $AG(R/I)$. In Section 3, it is examined how the completeness of $AG_I(R)$, $AG(R/I)$, $\Gamma(R/I)$ and $\Gamma_I(R)$ relate to each other. In particular, it is shown that if I is a radical ideal of a ring R, $AG_I(R)$ cannot be complete. In Section 4, this problem that when $AG_I(R)$ is bipartite and triangle-free is investigated. As a consequence of the results of this section, it is proved that $AG_I(R)$ is triangle-free if and only if $AG_I(R)$ is bipartite. Our study in Section 5, focuses on the situation when $AG_I(R)$ is r-partite and has a cut-point. It is proved among other results that if $\sqrt{I} = I$, then $AG_I(R)$ is not a complete r-partite graph, $r \geq 3$. Theorem 4.4 in [6] is also generalized. To simplify our result, throughout this paper we assume that $V(AG_I(R)) = V$ and $V(AG(R/I)) = V'$.

2. Drawing the $AG_I(R)$ graph based on $AG(R/I)$

In [6, Theorem 2.5], it is only stated that $AG(R/I)$ is isomorphic with a subgraph of $AG_I(R)$. According to the following theorem, we can draw the graph $AG_I(R)$ in relation to the graph $AG(R/I)$. Since now on, we refer to "$AG(R/I)$ as a subgraph of $AG_I(R)$" rather than instead of saying that it is isomorphic with a subgraph of $AG_I(R)$.

Considering $J \in V$, two cases are possible:

1) $I \subset J$, where $J \in V'$.

2) $I \nsubseteq J$. It can be simply proved that $J \in V$ if and only if $J + I \in V$.

Let $J = K_i + I$, where N is a set, $i \in N$, K_i is an ideal of R, and $I \nsubseteq K_i$ for each $i \in N$. In this instance, a submatrix of the adjacency matrix of J is the adjacency matrix of the ideal K_i. As a result, the $AG_I(R)$ vertices are either an $AG(R/I)$ vertex or are located in one of the columns of an $AG(R/I)$ vertex. As a result, this method is used to extract all $AG_I(R)$ vertices. Suppose $L, J \in V$ and

$$L = L_i + I, \qquad i \in T,$$

$$J = K_j + I, \qquad j \in N,$$

where N and T are sets, for each $i \in T$ and $j \in N$, L_i and K_j are ideals of R, and $I \nsubseteq L_i, K_j$. The following theorem explains the method for drawing the $AG_I(R)$ edges.

Theorem 2.1. With our notations,

a) $JL \subseteq I$ if and only if $K_jL \subseteq I$. This means that if J and L are adjacent, then L is connected to all the members of column J.

b) If $J^2 \subseteq I$, then $JK_j \subseteq I$ and $K_iK_j \subseteq I$ for each $i, j \in N$. Thus, if $J^2 \subseteq I$, then J is adjacent to all members of its column. Moreover, all members of column J are also connected to each other.

c) If $J^2 \nsubseteq I$, then $JK_j \nsubseteq I$ and $K_iK_j \nsubseteq I$ for each $i, j \in N$. Hence, if $J^2 \nsubseteq I$, then J is not adjacent to any member of its column. In addition, none of the members of J column are connected to each other.

Therefore, the edges of $AG_I(R)$ can be obtained based on those of $AG(R/I)$. Suppose K is an ideal of R and define:

$$M = \{J \in V | J = K + I \ s.t \ I \nsubseteq K\}. \tag{1}$$

Remark 1. By Theorem 2.1, it can be seen that if $AG_I(R) \ncong AG(R/I)$, then $M \neq \phi$ and if $AG_I(R) \cong AG(R/I)$, then $M = \phi$.

3. Completeness of $AG_I(R)$

The aim in this section is to look into how the completeness of $AG(R/I)$ and $AG_I(R)$ relate to each other. It is demonstrated that $AG_I(R)$ is not complete if $\frac{R}{I} \cong F_1 \times F_2$, F_1 and F_2 being any fields. Additionary, it will be shown that in [6, Theorem 6.5(c)] (R, m) is a chain ring, $I = m^3$ and $AG_I(R) \cong K_2$. Then [6, Theorem 6.5] is enhanced. Finally, it is investigated how the completeness of $AG_I(R)$ relates to the completeness of $\Gamma_I(R)$ and $\Gamma(R/I)$.

Lemma 3.1. *$AG_I(R)$ is a complete graph if and only if $AG(R/I)$ is complete and for each $J \in M$, we have $J^2 \subseteq I$.*

Proof. First, we assume that $AG(R/I)$ is complete and for every $J \in M$ to have $J^2 \subseteq I$. Now we assume that K and L are two arbitrary vertices of $AG_I(R)$. We show that these two vertices are adjacent. The following three cases are possible:
a) $I \subseteq K, L$. In this cases, we have $K, L \in V'$. Now since $AG(R/I)$ is complete and an inductive subgraph of $AG_(R)$, then L and K are adjacent in $AG_I(R)$.
b) $I \nsubseteq K, L$. If $L + I \neq K + I$, then according to (a), two vertices $(L + I)$ and $(K + I)$ are connected in $AG_I(R)$ and so by part (a) of Theorem 2.1, $KL \subseteq I$. If $L + I = K + I$, hence $(L + I) \in M$ and then according to the assumption, $(L + I)^2 \subseteq I$. On the other hand, since $L + I = K + I$, so K and L are adjacent in $AG_I(R)$.
c) $I \subset L$ and $I \nsubseteq K$. If $L = K + I$, then $L \in M$ and $L^2 \in I$. On the other hand, since $L = K + I$, then K is a member of column L and therefore L and K are adjacent. If $L \neq K + I$, then by part (a), $K + I$ and L are adjacent in $AG_I(R)$. So, according to part (a) of Theorem 2.1, $KL \subseteq I$. Now since we selected optional K and L, then $AG_I(R)$ is complete. $\square$

Theorem 3.2. Suppose $AG(R/I) \ncong K_2$. The following statements are equivalent:

a) $AG(R/I)$ is complete.

b) $AG_I(R)$ is complete.

c) $Z(R/I)$ is an annihilating ideal of $\frac{R}{I}$ with $Z^2(R/I) = I$.

Proof. Let $AG(R/I)$ be complete. By [10, Theorem 3], $Z(R/I)$ is an annihilating ideal of $\frac{R}{I}$ with $Z^2(R/I) = I$. For all $\frac{J}{I} \in, V'$, $(\frac{J}{I})^2 \subseteq Z^2(\frac{R}{I}) = I$ which implies that for all $J \in M$, $J^2 \subseteq I$. We now apply Lemma 3.1 to prove that $AG_I(R)$ is complete. Furthermore, if $AG_I(R)$ is complete, then we may easily see that $AG(R/I)$ is also complete. The equivalence of (b) and (c) is an immediate consequence of [6, Theorem 6.5]. $\square$

Theorem 3.3. Consider I be a non-zero proper ideal of R such that $\frac{R}{I} \cong F_1 \times F_2$, where F_1 and F_2 are fields. Then $AG_I(R)$ is not a complete graph.

Proof. We first note that $\frac{R}{I} = \frac{K}{I} + \frac{J}{I}$, where K and J are the only maximal ideals of R such that $K \cap J = I$, $K^2 + I = K$ and $J^2 + I = J$. Hence, the only ideals of R containing I are I, J and K. Suppose $AG_I(R)$ is complete. Since $K^2 \nsubseteq I$, $J^2 \nsubseteq I$, then according to Lemma 3.1, $K, J \notin M$. So, $V(AG_I(R)) = \{K, J\}$. We prove that K and J are the only maximal ideals of R. Let $L \neq K, J$ be another maximal ideal of R. Then $L + I = R$ which shows that $LJ + I = J$. Since $I \nsubseteq LJ$, $J \in M$ which contradicts this fact that $J \notin M$. Suppose that $a \in J \setminus I$. It is now proved that $J = \langle a \rangle$. If $\langle a \rangle \subset J$ then $\langle a \rangle \nsubseteq I$ and $\langle a \rangle \nsubseteq K$. Since $\langle a \rangle K \subset I$, $K, J \neq \langle a \rangle \in V(AG_I(R))$ which contradicts $V(AG_I(R)) = K, J$. Therefore, for all $a \in J \setminus I$ we have $J = \langle a \rangle$. A similar argument shows that for all $b \in K \setminus I$, we have $K = \langle b \rangle$. Since $J^2 \nsubseteq I$ and $J^2 = \langle a^2 \rangle$, $a^2 \notin I$ and $\langle a \rangle = \langle a^2 \rangle$. This shows that there exists $r \in R$ such that $a(1 - ra) = 0$. If $1 - ra = 0$, then a is an unity, that contradicts the maximality of $\langle a \rangle$. Thus $1 - ra \neq 0$.

Considering our discussion, K and J are the only maximal ideals of R and thus either $1 - ra \in J$ or $1 - ra \in K$. Since $ra \in J$, $1 - ra \in K \setminus I$ and so $K = \langle 1 - ra \rangle$. This proves that $JK = 0$. In contrast, since J and K are both maximal, they are comaximal and hence $JK = J \cap K$. Therefore, $I = 0$ which contradicts our assumption and thus $AG_I(R)$ is incomplete. $\square$

The ring R is a chain ring if and only if its only maximal ideal is a principal ideal.

Theorem 3.4. Assume that $AG_I(R)$ is complete. in the case of $(\frac{R}{I}, \frac{m}{I})$ being local ring with exactly two non-trivial ideals $\frac{m}{I}$ and $(\frac{m}{I})^2$, then (R, m) is a chain ring. In this case, $I = m^3$ and $AG_I(R) \cong K_2$.

Proof. Since $m^2 \nsubseteq I$ then according to Lemma 3.1, $m \notin M$. It is proved that m is the only maximal ideal of R. Assume $J \neq m$ is a maximal ideal of R. Therefore,

$J + I = R$ which implies that $Jm + I = m$. This shows that $m \in M$, which is impossible. Now we claim that m has the form of the principal ideal $\langle x\rangle$.

Consider $y \in m \backslash m^2 + I$. Then $\langle y\rangle \neq m^2 + I$ and $\langle y\rangle \nsubseteq I$. Assume that $\langle y\rangle \subset m$, then $\langle y\rangle(m^2 + I) \subseteq I$. Therefore, $\langle y\rangle \in V$ and by Theorem 2.1, $\langle y\rangle + I = m$. Hence $m \in M$, a contradiction. Therefore, $m = \langle y\rangle$, for all $y \in m \setminus m^2 + I$. Thus (R, m) is a chain ring. Now we prove that except for $\langle y\rangle$ and $\langle y^2\rangle$ all other proper ideals of R are subsets of $\langle y^3\rangle$ and thus $I = \langle y^3\rangle$ and $AG_I(R) \cong K_2$. Consider the proper ideal J of R. Note that all elements of J are of the form of ry $(r \in R)$. If r is a unit in R, then $J = \langle y\rangle$. Otherwise, r is a multiple of y. Hence, if $J \neq \langle y\rangle$ then $J \subseteq \langle y^2\rangle$. If the same trend continues for a finite number of rounds, we have $\langle y^5\rangle \subseteq \langle y^4\rangle \subseteq \langle y^3\rangle \subseteq \langle y^2\rangle \subseteq \langle y\rangle$. □

Corollary 3.5. *$AG_I(R)$ is complete, if and only if either $Z(\frac{R}{I})$ is an annihilating ideal of $\frac{R}{I}$ such that $Z^2(\frac{R}{I}) = I$ or (R, m) is a chain ring, where $I = m^3$ and $AG_I(R) \cong K_2$.*

Proof. Cosider the scenario where $AG_I(R)$ is complete. $AG(R/I)$ is complete according to Lemma 3.1. Its conceivable to encounter one of the following two situations.

1) $AG(R/I) \ncong K_2$. By Theorem 3.2, $AG_I(R)$ is complete if and only if $Z(\frac{R}{I})$ is an annihilating ideal of $\frac{R}{I}$ such that $Z^2(\frac{R}{I}) = I$.

2) $AG(R/I) \cong K_2$. In this case, by [10, Theorem 3] either
   a) $\frac{R}{I} \cong F_1 \times F_2$, where F_1, F_2 are fields. or,
   b) $(\frac{R}{I}, \frac{m}{I})$ is a local ring with exactly two non-trivial ideals $\frac{m}{I}$ and $(\frac{m}{I})^2$.

Since $AG_I(R)$ is complete, Theorem 3.3 implies that a cannot be occurred. We now apply Theorem 3.4 to deduce that (R, m) is a chain ring, where $I = m^3$ and $AG_I(R) \cong K_2$. □

Corollary 3.6. *The followings hold.*

a) *Assume that $AG(R/I) \ncong K_2$. $AG(R/I)$ is complete if and only if $AG_I(R)$ is complete if and only if $Z(R/I)$ is an annihilating ideal of $\frac{R}{I}$ such that $Z^2(R/I) = I$.*

b) *Assume that $AG(R/I) \cong K_2$. $AG_I(R)$ is complete if and only if $AG_I(R) \cong AG(R/I) \cong K_2$ if and only if (R, m) is a chain ring where $I = m^3$.*

Corollary 3.7. *If $\sqrt{I} = I$ then $AG_I(R)$ is not complete.*

Proof. In order for $AG_I(R)$ to be complete, either $Z(\frac{R}{I})$ should be an annihilating ideal of $\frac{R}{I}$ such that $Z^2(\frac{R}{I}) = I$ or (R, m) is a chain ring with $I = m^3$, which both of them contradicts our assumption. □

Theorem 3.8. *The followings hold:*

a) $AG_I(R)$ and $\Gamma(R/I)$ are complete if and only if $Z(R/I)$ is an annihilating ideal of $\frac{R}{I}$ such that $Z^2(R/I) = I$.

b) $AG_I(R)$ is complete and $\Gamma(R/I)$ is not complete, if and only if $(R, \langle a\rangle)$ is a chain ring, where $I = \langle a^3\rangle$.

c) $\Gamma(R/I)$ is complete and $AG_I(R)$ is not complete if and only if $\frac{R}{I} \cong Z_2 \times Z_2$.

Proof. Suppose that $\frac{R}{I} \cong Z_2 \times Z_2$. By Theorem 3.3, $AG_I(R)$ is not complete. If $(R, \langle a\rangle)$ is a chain ring and $I = \langle a^3\rangle$, then it is immediate that $\frac{R}{I} \ncong Z_2 \times Z_2$ and $Z^2(R/I) \neq I$. Therefore, by [3, Theorem 2.8], $\Gamma(R/I)$ is not complete. Our result now follows from Corollary 3.5. □

Corollary 3.9. In Theorem 3.8, $\Gamma(R/I)$ can be replaced with $\Gamma_I(R)$.

Proof. If $AG_I(R)$ is complete, then by Corollary 3.7, we have $\sqrt{I} \neq I$. Hence, $\Gamma(R/I)$ is complete if and only if $\Gamma_I(R)$ is complete. □

Theorem 3.10. Let (R, m) be a local ring such that $m^2 = 0$, K is an ideal of R and r is the number of non-trivial ideals J with the property that $K \subset J \subset m$. Then $r \neq 2$.

Proof. Suppose $r = 2$. Since $mJ = 0$, $AG(R/K) \cong K_2$, contradicting [10, Theorem 3]. □

4. Bipartite $AG_I(R)$ graph

This section aims to establish the relationship between the bipartivity of $AG_I(R)$ and $AG(R/I)$. We describe a method for drawing $AG_I(R)$ using the graph structure of $AG(R/I)$ and vice versa, assuming that $AG_I(R)$ and $AG(R/I)$ are bipartite graphs. Additionally, the triangle-free character of $AG_I(R)$ is only proven if and only if $AG_I(R)$ is bipartite. It is also check when $AG_I(R)$ is a star. The elements C, B, and A in Theorem 20 of [11], can now be obtained using an easier technique, which is shown last. our discussion will be divided into two sections.

1) $\sqrt{I} = I$,

2) $\sqrt{I} \neq I$.

Lemma 4.1. Let $V(AG_I(R)) \neq \{J, K\}$, where $J = K + I$ and $I \nsubseteq K$. $AG_I(R)$ is bipartite if and only if $AG(R/I)$ is bipartite and for all $J \in M$, $J^2 \nsubseteq I$.

Proof. Assume that $AG(R/I)$ is bipartite and for all $J \in M$, $J^2 \nsubseteq I$. We also assume that (V_1', V_2') is a bipartition of $AG(R/I)$. Consider $J \in V$. The following two cases are possible:

1) $I \subset J$, and $\frac{J}{I} \in V_i'$ if and only if $J \in V_i$, $i = 1, 2$;

2) $I \nsubseteq J$, and $J \in V_i$ if and only if $J + I \in V_i$.

Apply Theorem 2.1 to check that V_1 and V_2 are independent and $V_1 \cup V_2 = V$. Hence, $AG_I(R)$ is a bipartite graph.

Conversely, we assume that there exists $J \in M$ such that $J^2 \subseteq I$. Therefore, $J = K + I$, where K is an ideal of R and $I \nsubseteq K$. Without loss of generality, we assume that $J \in V_1$. Hence, $K \in V_2$. On the other hand, since $V(AG_I(R)) \neq \{J, K\}$ and $AG_I(R)$ is connected, there exists $L \in V$ such that $JL \subseteq I$ or $LK \subseteq I$. If $JL \subseteq I$, then according to Theorem 2.1, $KL \subseteq I$ which is impossible. The case that $LK \nsubseteq I$ will result in similar contradiction. Therefore, for each $J \in M$ we have $J^2 \nsubseteq I$ and by Theorem 2.1, for each $K \in V$ with $I \nsubseteq K$, $K \in V_i$ if and only if $K + I \in V_i$. Hence, to obtain the $AG(R/I)$ graph it is enough to delete all vertices of $AG_I(R)$ not included in I. Hence, $AG(R/I)$ is a bipartite graph.

Suppose the bipartite graph $AG(R/I)$ is given and $AG_I(R)$ is also bipartite. To draw $AG_I(R)$, we only require to apply the following two conditions.

1) $\frac{J}{I} \in V_i'$ if and only if $J \in V_i$, $i = 1, 2$;

2) If $I \nsubseteq K$, then $J = K + I$ and K will be in the same part.

Hence the result follows. □

Theorem 4.2. $AG_I(R)$ is triangle-free if and only if $AG_I(R)$ is bipartite.

Proof. Suppose $J = K+I$ and $I \nsubseteq K$. If $V = \{J, K\}$, then the proof is immediate. Hence, it can be assumed that $V \neq \{J, K\}$. Assume that $AG_I(R)$ is triangle-free. Since $AG(R/I)$ is a subgraph of $AG_I(R)$, $AG(R/I)$ is also triangle-free. Thus according to Theorem 2 and Note 22 in [11], $AG(R/I)$ is a bipartite graph. We claim that for all $J \in M$, $J^2 \nsubseteq I$. Suppose that $J = K + I$, where $J^2 \subseteq I$, K is an ideal of R, and $I \nsubseteq K$. There exists $J \neq J' \in V$ such that $JJ' \subseteq I$ and $J'K \subseteq I$. On the other hand, since $J^2 \subseteq I$, $JK \subseteq I$. Thus $J - J' - K - J$ is a triangle in $AG_I(R)$, which is impossible. Hence, by Lemma 4.1, $AG_I(R)$ is bipartite. □

The next lemma shows that if in [11, Theorem 20], $C \neq \varnothing$, then $AG_I(R)$ is a complete bipartite.

Lemma 4.3. In [11, Theorem 20], if $C \neq \phi$, then $AG_I(R)$ is a complete bipartite graph, $V_1(AG_I(R)) = V_1(AG(R)) = N(I)$ and $V_2(AG_I(R)) = V_1^c(AG(R)) \setminus \{I\}$, where I is the only minimal of R.

Proof. Since $C \neq \phi$, $B \neq \phi$. Assume that $J \in B$. We claim that $J + I \neq R$. Since $(J + I)I = 0$, $J + I \in A$. Thus $A \neq \phi$.

1. $J + I = R$. In this case, for all $J'' \in C$, $JJ'' + IJ'' = J''$ and so $I = J''$, contradicting definition of C. This shows that for all $J'' \in C$ and $J' \in A$, $J'J''I = 0$. Hence, $J''J' = I$ and thus $V(AG(R)) \setminus \{I\} \subseteq V(AG_I(R))$. We now prove that $V(AG_I(R)) \subseteq V(AG(R)) \setminus \{I\}$. To do this, we choose

$L \in V(AG_I(R))\backslash V(AG(R))$. Then for all $J \in V(AG(R))$, $LJ \neq 0$. We claim that for all $J \in V(AG(R))$, $LJ \neq I$. If this is true, then L is not adjacent to each vertex of the set $V(AG(R)) \setminus \{I\}$. Therefore, $AG_I(R)$ is not connected, which is a contradiction. Hence, $V(AG_I(R)) = V(AG(R)) \setminus \{I\}$.

2. $J \in B$. Since $0 \neq LJ \subseteq J$ and $I \nsubseteq J$, $LJ \neq I$. If $J \in C$, then $LJI = LI = I$ and so $LJ \neq I$. If $J \in A$, then for all $J' \in B$, $J'(JL) = (J'J)L \neq 0$ which implies that $JL \neq I$. Thus, $((J'J)I = J(J'I) = 0$. On the other hand, $J'J \neq 0$. Therefore, $J'J \in B$. Set $V_1(AG_I(R)) = V_1(AG(R))$.

3. $V_1(AG_I(R))$ and $V_2(AG_I(R))$ are independent subsets of the graph. Note that for all $J, J' \in B$, $J'J \subseteq J$. Since $I \nsubseteq J$, $J'J \neq I$. On the other hand, in Theorem 20 in [11], the authors proved that $J'J \neq 0$. Therefore, B is independent. We now prove that A is independent. Suppose $J, J' \in A$. Then $J'J \neq 0$. On the other hand, we assume that $J'' \in C$. Hence, $J''JJ' = (J''J)J' = IJ' = 0$. Therefore, $J'J \neq I$ and A is independent. Note that $(J''J)I = J''(JI) = 0$ and for all $L \in B$, $L(J''J) = (LJ'')J = 0$. So, $J''J = I$. Suppose that $J \in A$ and $J' \in B$. Then, $0 \neq J'J \subseteq J'$. Since $I \nsubseteq J'$, $J'J \neq I$. This proves that $V_1(AG_I(R))$ is independent. By our definition, $V_2(AG_I(R)) = V_2(AG_I(R)) \setminus \{I\}$. To prove that $V_2(AG_I(R))$ is independent, we choose $J, J' \in V_2(AG_I(R))$. Then $(JJ')I = J(J'I) = JI = I$ and hence $J'J \neq I$. So, $V_2(AG_I(R))$ is independent.

For all $J \in V_2(AG_I(R))$ and $J' \in V_1(AG_I(R))$, $JJ' = 0$. If $J' \in B$ and $J'J = I$, then $J' \in A$. Therefore, $AG_I(R)$ is a complete bipartite graph. □

Lemma 4.4. In accordance with Lemma 4.3, we have the following

a) for each $J \in C$, J is idempotent.

b) for each $J' \in A$ we have $(J')^2 \in B$.

Proof. a) Set $P = \mathrm{Ann}(I)$. In [11, Result 24] it is proved that P is a maximal ideal. Since $I^2 = 0$, then $I \subseteq P$. If $I = P$, in [11, Result 24] it is proved that $AG(R)$ is the only point in I. Therefore, by [5, Theorem 1.4], the only non-trivial ideal of R is I, and this result contradicts the non-emptiness of C. Hence, by Lemma 4.4, we have $P \in V_1(AG(R))$. Moreover, since P is maximal we have, for all $J \in V_2(AG(R))$, $P+J = P$ or R. If $P+J = P$, since $PI = 0$, then $JI = 0$ and contradiction is achieved. Therefore, $P + J = R$ which implies that $I + J^2 = J$. Since $J \nsubseteq I$, then $J^2 \nsubseteq I$. On the other hand, for all $J' \in B$ $J'J^2 \subseteq J'J = 0$ which shows that $J'J^2 = 0$. Therefore $J^2 \in C$ and J^2 is concluded I. Then $I + J^2 = J^2 = J$.
b) For all $J' \in A$, $J'' \in C$ $J''J'^2 = (J''J')J' = IJ' = 0$. Now we show that $J'^2 \neq 0$ and thus $J'^2 \in B$. For all $J \in B, JJ' \in B$. Therefore $J'^2J = J'(JJ') \neq 0$. Then $J'^2 \neq 0$. □

Now the following Theorem presents an easy way of obtaining the members of C in [11, Theorem 20].

Theorem 4.5. Considering assumptions of [11, Theorem 20] , if $C \neq \phi$, then:

a) The ring R has only two minimal prime ideals, one of them is ($P_1 = \mathrm{Ann}(I)$)

b) $C = \{0 \neq J \mid I \neq J \subseteq P, J \text{ is an ideal of } R\}$

c) $|A| \neq 1$

Proof. In Lemma 4.3 it was proved that $AG_I(R)$ is a bipartite complete graph. Hence, by Lemma 4.1 $AG(R/I)$ is also a bipartite complete graph. □

Claim: R/I is reduced.
Claim proof: Assume R/I is not reduced. Therefore, by Theorem 20 in [11], R/I has only one non-zero minimal ideal in the form of $\frac{\langle y\rangle+I}{I}$ where $\langle y\rangle^2 \subseteq I$. We have $\frac{\langle y\rangle+I}{I} \in V(AG(R/I))$, then $\langle y\rangle + I \in V(AG_I(R))$ and thus $\langle y\rangle + I \in A \cup C$, which is contradictory to Lemma 4.4. Hence, R/I is reduced and by Result 24 in [11], R/I has only two prime minimal ideals: $(\frac{P_1}{I}, \frac{P_2}{I})$. According to (a), $V_1(AG(R/I)) = \{J/I|I \subset J \subseteq P_1\}$ and $V_2(AG(R/I)) = \{J/I|I \subset J \subseteq P_2\}$. Therefore $C = \{J|I \subset J \subseteq P_2\}$ and $P_1 = \mathrm{Ann}(I)$. We show that $|A| \neq 1$. Suppose $|A| = 1$. By the proof of part (b) , we obtain: $A = \{\mathrm{Ann}(I)\}$. Since $\mathrm{Ann}(I)$ is a maximal ideal of R, then for all $J'' \in C$, we have

$$\mathrm{Ann}(I) + J'' = R, \tag{2}$$

Since the only minimal ideal of R is I and $B \neq \phi$, then

$$|B| = \infty. \tag{3}$$

Consider $J, J' \in B$ with $J \neq J'$. By (2),

$$J + I + J'' = J + J'' = R, \tag{4}$$

and also

$$J' + I + J'' = J' + J'' = R\,. \tag{5}$$

By (4): $JJ' + J''J' = J'$ which shows that $J' \subseteq J$. By (5):

$$JJ' + J''J = J,$$

which shows that $J \subseteq J'$. Therefore $J = J'$. Since $J, J' \in B$ ($J \neq J'$) were arbitrarily selected from B, then $|B| = 1$, which contradicts Lemma 4.4. Therefore, $|A| \neq 1$. □

In the following, we consider two states of $\sqrt{I} = I$ and $\sqrt{I} \neq I$ to investigate the issue of bipartite $AG_I(R)$. First, in Theorem 4.6 to Corollary 4.9, we assume $\sqrt{I} \neq I$.

Theorem 4.6. One of the following situations may occur If $AG_I(R)$ is bipartite, then one of the following cases occurs.

a) R/I consists of exactly two minimal ideals and $R/I \cong F \times S$ where F is a field and S a ring with exactly one non-trivial ideal.

b) The only minimal ideal over I in R is in the form of the principal ideal $\langle x \rangle$ for a $x \in R$ and we have $\langle x \rangle^2 \subseteq I$.

Proof. According to Lemma 4.1, $AG(R/I)$ is bipartite and thus according to Theorem 2 in [11], two cases are possible:

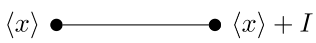
1) R/I admits exactly two minimal ideals, and $R/I \cong F \times S$ where F is a field and S is a ring with exactly one non-trivial ideal.

2) R/I admits only one minimal ideal, in which case the minimal ideal is of the form of $\langle x + I \rangle$ and we have $\langle x + I \rangle^2 = I$.

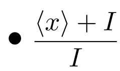
In case (2), we have $\langle x \rangle \nsubseteq I$ and $\langle x \rangle^2 \subseteq I$. Therefore, $\langle x \rangle$ is a vertex in $AG_I(R)$. The following two cases are assumed:

A) $I \subseteq \langle x \rangle$, in which case (b) is proved.

B) $I \nsubseteq \langle x \rangle$, therefore according to Theorem 2.1, $J = \langle x \rangle + I$ is a vertex in $AG_I(R)$. Now since $\langle x \rangle^2 \subseteq I$, then according to Theorem 2.1, $J \in M$ and $J^2 \subseteq I$. On the other hand, $AG_I(R)$ is bipartite, hence $AG_I(R)$ has the following form according to Lemma 4.1.



And $AG(R/I)$ is in the following form



and therefore contradiction is achieved and B does not occur.

So far, we have proved that if $AG_I(R)$ is a bipartite graph, one of the cases of the Theorem 4.6 occurs. Now using the following two theorems, we try to determine when $AG_I(R)$ is a star or bipartite graph. □

Theorem 4.7. Suppose $AG_I(R)$ is a bipartite graph. The two parts of $AG_I(R)$ are named V_1 and V_2. Then one of the following cases occurs.

a) $V_1 = \{J_1, J_2\} \cup \{K \mid K \text{ is an ideal of } R, I \nsubseteq K \nsubseteq I\}$ and $V_2 = \{J_3, J_4\}$, where $\frac{J_1}{I} \cong F \times T$, $\frac{J_2}{I} \cong F \times \langle o \rangle$, $\frac{J_3}{I} \cong \langle o \rangle \times S$ and $\frac{J_4}{I} \cong \langle o \rangle \times T$. Therefore $|V_1| \geq 2$ and $|V_2| = 2$. Moreover, we have for all $J \in V_1$, $J_4 J \subseteq I$ and for all $J_1 \neq J \in V_1$, $J_3 J \subseteq I$ or

b) $V_1 = \{J|J \subseteq P_1, \ \ J \nsubseteq I\}$ and $V_2 = \{J|\langle x\rangle \subseteq J \subseteq P_2\}$ where P_1 and P_2 are the only two minimal ideals R over $\langle x\rangle$ and $\frac{P_2}{I} = \mathrm{Ann}(x+I)$. or

c) $V_1 = \{J|J \subseteq P_1, \ J \neq \langle x\rangle, \ J \subseteq I\}$ and $V_2 = \{\langle x\rangle\}$ where $\frac{P_1}{I} = \mathrm{Ann}(x+I)$.

Proof. If case (a) occurs in the Theorem 4.6, we have $V(AG(\frac{R}{I})) = \{\frac{J_1}{I}, \frac{J_2}{I}, \frac{J_3}{I}, \frac{J_4}{I}\}$. Clearly, $\frac{J_1}{I}$, $\frac{J_2}{I}$, $\frac{J_3}{I}$ and $\frac{J_4}{I}$ from R/I have one-to-one correspondence with the $F \times T$, $F \times \langle o\rangle$, $\langle o\rangle \times S$ and $\langle o\rangle \times T$ that T is non-trivial ideal of S. We prove that $J_4 \notin M$. Since $(\langle o\rangle \times T)^2 = \langle o\rangle \times \langle o\rangle$, then $J_4^2 \subseteq I$ and thus by Lemma 4.1,

$$J_4 \notin M. \tag{6}$$

Assume that $J_3 \in M$, that is to say, $J_3 = K + I$ where K is an ideal of R and $I \nsubseteq K$. J_3 is a maximal ideal of R and on the other hand, $J_1 \nsubseteq J_3$, therefore $J_3 + J_1 = R$ which shows that $K + I + J_1 = R$ which shows that $KJ_4 + I = J_4$, $KJ_4 \nsubseteq I$ and $I \nsubseteq KJ_4$. Hence $J_4 \in M$ which contradicts (6) and therefore $J_3 \notin M$, then $V_2' = \left\{\frac{J_3}{I}, \frac{J_4}{I}\right\}$. On the other hand, $J_3, \ J_4 \notin M$ and thus $V_2 = \left\{J_3, J_4\right\}$ and $|V_2| = 2$. Assume that $J_1 \in M$, thus $J_1 = K + I$ such that $I \nsubseteq K \nsubseteq I$. On the other hand, $J_4 \subset J_1$. Therefore $I \subset J_4 \subset K + I$, then $J_4 = (K \cap J_4) + I$, which is contradictory to $J_4 \notin M$ and thus $J_1 \notin M$.

Now it is shown that all ideals of R that do not contain I, as well as all ideals of R that are not subsets of I, are vertices in $AG_I(R)$. We assume that L is an ideal of R so that $I \nsubseteq L \nsubseteq I$ and $L \notin V$. Therefore, according to Theorem 2.1, $L + I = R$ then $LJ_4 + I = J_4$, $I \nsubseteq LJ_4 \nsubseteq I$ and thus $J_4 \in M$, which contradicts (6).

We have $V_1' = \left\{\frac{J_1}{I}, \frac{J_2}{I}\right\}$. On the other hand, we proved that only J_2 can be in M. Therefore,

$$V_1 = \left\{J_1, J_2\right\} \cup \left\{J|I \nsubseteq J \nsubseteq I \qquad J \text{ is an ideal of} R\right\}.$$

Therefore, $|V_2| \geq 2$.

If case 2 occurs in Theorem 4.7, $\frac{R}{I}$ has only one minimal ideal $(\frac{\langle x\rangle}{I})$. Hence, by [11, Theorem 20], the following cases are possible.

i) $C \neq \phi$ which by Theorem 4.5 we have:

$$V_1' = \left\{\frac{J}{I}|I \subset J \subseteq P_1, \ J \neq \langle x\rangle\right\},$$

and

$$V_2' = \left\{\frac{J}{I}|I \subset J \subseteq P_2\right\},$$

where P_1 and P_2 are the only two minimal ideals of R over $\langle x\rangle$. We claim that all ideals of R, which do not contain I, as well as all ideals of R that are not subsets of I, are placed in V_1 and thus part (b) is proved.

Claim proof: We first show that $M \subseteq B$. Assume that $M \nsubseteq B$ and thus there exists $I \nsubseteq K \nsubseteq I$ such that $(K+I) \in V \setminus B$. Since $\langle x\rangle^2 \subseteq I$, by Lemma 4.1 we have $K+I \neq \langle x\rangle$ and $I \subset \langle x\rangle \subset K+I$ which shows that $\langle x\rangle = (K \cap \langle x\rangle) + I$ which shows that $\langle x\rangle \in M$. This is contradictory to Lemma 4.1. Hence, $M \subseteq B$ and therefore

$$\text{for all } K \in V, \quad s.t \;\; I \nsubseteq K \nsubseteq I.$$

Then,

$$K \in B.$$

Now we should prove that all R ideals, which do not contain I, as well as all ideals of R that are not subsets of I, are vertices in $AG_I(R)$.

We assume that K is an ideal of R so that $I \nsubseteq K \nsubseteq I$ and $K \notin V$. Therefore, by Theorem 2.1, $L = K + I \notin V$ and thus $\frac{L}{I} \notin V'$. Hence,

$$\frac{L}{I} \times \frac{\langle x\rangle}{I} = \frac{\langle x\rangle}{I},$$

which shows that

$$L\langle x\rangle + I = \langle x\rangle,$$

then we have:

$$K\langle x\rangle + I = \langle x\rangle,$$

then $\langle x\rangle \in M$. Therefore, by Lemma 4.1, we get a contradiction.
ii) $C = \phi$, we have two cases:

1) $B = \phi$. Similar to the proof of part (i) we obtain:

$$M \subseteq B.$$

   Now since $B = \phi$, then $AG_I(R) \cong AG(R/I)$.

2) $B \neq \phi$. Again, with a proof similar to part (i), it is possible to show that all the ideals of R, which do not contain I, as well as all ideals of R that are not subsets of I, are vertices in $AG_I(R)$.

Therefore, part (c) is proved. □

Corollary 4.8. Assume that $AG_I(R)$ is a bipartite graph. Then the following statements are equivalent:

a) $AG_I(R)$ is a complete bipartite graph.

b) $AG_I(R)$ is star.

c) $AG(R/I)$ is star.

d) $Z(R/I) = \mathrm{Ann}(x+I)$, which the only minimal ideal in relation to I in R is of the form of the principal ideal $\langle x\rangle$ for a $x \in R$.

Proof. According to previous Theorem, equality of a, b and c is immediate. Also according to Result 26 in [11], c and d are equivalent. □

Corollary 4.9. If $AG_I(R)$ is bipartite, then girth$(AG_I(R)) = 4$ or girth$(AG_I(R)) = \infty$.

In the continuation of the study of the topic of bipartite $AG_I(R)$ in Theorem 4.10 and Theorem 4.11 we assume $\sqrt{I} = I$.

To prove part two of Theorem 4.1 in [6], it is only proved that $V_i \cup C_I$ is a prime ideal from the $I(R)$ semiring and it is not proved that $C_{P_i} = V_i \cup C_I$. Another method for proving this part of Theorem 4.1 is presented in the following. Moreover, it is proved that P_1 and P_2 introduced in part two of Theorem 4.1 in [6] are only two prime minimal ideals of R in relation to I.

Remark 2. In Remark 22 in [11], we have $V_1 = \{J|0 \neq J \subseteq P_1\}$ and $V_2 = \{J|0 \neq J \subseteq P_2\}$. P_1 and P_2 are the only minimal prime ideals of the ring R.

Proof. Suppose that $J \subseteq P_1$ and $J' \subseteq P_2$. We have $JJ' \subseteq P_1$ and $JJ' \subseteq P_2$. Since $P_1 \cap P_2 = 0$, then $JJ' = 0$. □

Theorem 4.10. Suppose $AG_I(R)$ is a non-empty bipartite graph, the following statements are equivalent:

a) R only has two prime minimal ideals in relation to I: (P_1 and P_2).

b) $V_1(AG_I(R)) = \{J|J \subseteq P_1, J \nsubseteq I\}$ and $V_2(AG_I(R)) = \{J|J \subseteq P_2, J \nsubseteq I\}$.

Proof. a) By Lemma 4.1, $AG(R/I)$ is bipartite. Therefore by [11, Result 24]: $\left|\mathrm{Min}(R/I)\right| = 2$.

b) By above note: $V_1(AG(R/I)) = \left\{\frac{J}{I}\middle| I \subset J \subseteq P_1\right\}$ and $V_2(AG(R/I)) = \left\{\frac{J}{I}\middle| I \subset J \subseteq P_2\right\}$, thus by the proof of Lemma 4.1 and Theorem 2.1, $V_1(AG_I(R)) = \{J|J \subseteq P_1, J \nsubseteq I\}$ and $V_2(AG_I(R)) = \{J|J \subseteq P_2, J \nsubseteq I\}$. □

Let's prove [6, Theorem 4.1] below in a simple way.

Suppose $AG_I(R)$ is a complete bipartite graph. We prove that $I = P_1 \cap P_2$. By Theorem 4.1, R only has two prime minimal ideals in relation to I: (P_1andP_2). Now since $\sqrt{I} = I$ and $\sqrt{I} = P_1 \cap P_2$, therefore $I = P_1 \cap P_2$.
In addition, in part (b) of the previous Theorem we show that $C_{P_i} = V_i \cup C_I$.

Theorem 4.11. The following statements are equivalent:

a) $AG_I(R)$ is bipartite.

c) $AG_I(R)$ is complete bipartite.

d) $AG(R/I)$ is complete bipartite.

e) $AG(R/I)$ is triangle-free.

f) $AG_I(R)$ is triangle - free.

g) $I = P_1 \cap P_2$ where P_1 and P_2 are the only two prime minimal ideals of R in relation to I.

Proof. Since $\sqrt{I} = I$, for each $J \in M$ we have $J^2 \nsubseteq I$ and thus by Lemma 4.1, equivalency of (a) and (b) is proved.
Now since for each $J \in M$ we have $J^2 \nsubseteq I$, then by Theorem 2.1, (e) and (f) are also equivalent.
In Note Remark 2, it is proved that c and g are equivalent.
In [11, Note 22], equivalency of (b), (d), and (e) is proved. Since (a) and (b) are equivalent. We now apply Theorem 4.2 and the proof of Lemma 4.3 to prove that (c) and (d) are equivalent. □

5. Cut point and r-partite $AG_I(R)$ graph

Theorem 5.1. Suppose I is a non-zero proper ideal of the ring R. If $AG_I(R)$ is a complete r-partite graph, $r \geq 3$, then at most one part has more than one vertex and

a) if $V_i = \{A\}$ is a part of $AG_I(R)$, then $A^2 \subseteq I$. Therefore, all vertices of $AG_I(R)$ such as A, where $A^2 \nsubseteq I$, are placed in one part.

b) Consider $T = \{J | J + I \in M,\ J^2 \subseteq I\}$. Each element of T is located in a separate part and so $r \geq |T| + 1$.

Proof. Since the first part of the Theorem is proved in [6, Theorem 4.4], it only requires to prove a and b.

a) If all parts only include one vertex, then $AG_I(R)$ is a complete graph and since $r \geq 3$, then $AG_I(R) \ncong K_2$. By Corollary 3.5 we have $Z^2(R/I) = I$. Therefore, for all vertices of $AG_I(R)$, which include I (such as J), we have $J^2 \subseteq I$ and thus by Theorem 2.1 we have $J^2 \subseteq I$ for all vertices of $AG_I(R)$ such as J. Now without loss of generality, we suppose $|V_1| \geq 2$. Then, we assume that $X \in V_1$, $Y \in V_l$, and $Z \in V_t$ $(t \neq l,\ t, l \neq 1)$. Now we have

$$\{Z\} \cup (I : Z) = C_I \cup (\bigcup_{\substack{i=1 \\ i \neq t}}^{r} V_i) \cup \{Z\},$$

$$\{Y\} \cup (I : Y) = C_I \cup (\bigcup_{\substack{i=1 \\ i \neq l}}^{r} V_i) \cup \{Y\},$$

$$\{X\} \cup (I : X) = C_I \cup (\bigcup_{i=2}^{r} V_i) \cup \{X\}.$$

And therefore we have $(I : Z) \subseteq (I : Y) \cup (I : X)$. On the other hand, $(I : Z) \nsubseteq (I : X)$. Therefore, by [6, Lemma 4.2 and 4.3], we have $(I : Z) \subseteq (I : Y)$. On the other hand, since we know $Y \in (I : Z)$, then $Y \in (I : Y)$, *i.e.* $Y^2 \subseteq I$. In addition, V_l was arbitrarily selected from the $V_2, \cdots$, and V_r sets, hence

$$\text{for all } Y \in V \setminus V_1, \quad Y^2 \subseteq I.$$

b) We claim that for all $J \in T \quad J^2 \notin V_1$ and thus part b is proved.

□

Calim proof: Assume that $J \in M \cap T$. Now we have

$$J = K_i + I, \quad i \in N,$$

where N is a set, and for each $i \in N$, K_i is an ideal of R and $I \nsubseteq K_i$. By Theorem 2.1 we have

$$(I : K_i) = (I : J).$$

On the other hand, since $J^2 \subseteq I$, by Theorem 2.1 we have $J K_i \subseteq I$ and thus $J, K_i \notin V_1$.

Corollary 5.2. If $\sqrt{I} = I$, then $AG_I(R)$ cannot be a complete r-partite graph $(r \geq 3)$.

Part (c) of [6, Theorem 3.5] can be developed as follows. Parts (d) and (e) could also be included in this Theorem.

Theorem 5.3. Let I be a non-zero proper ideal of the ring R. In these case we have:

a) Suppose $X = (x)$ is a principal ideal including I, where $X^2 \nsubseteq I$. Moreover, assume that $(x) \neq (x^2)$, in which case X in $AG_I(R)$ is not a cut-point.

b) If X has a column, *i.e.* $X \in M$, then X cannot be a cut-point in $AG_I(R)$.

c) If X in $AG(R/I)$ is not a cut-point, then it is not a cut-point in $AG_I(R)$ either.

Proof. a) Based on the proof by contradiction approach, X is assumed to be situated along all paths from U to W. Since $X^2 \nsubseteq I$, then X^2 is a vertex.The variable X is replaced with the variable X^2 In very path from U to W. Now since $X^2 \neq X$, using the Resulting permutation it is possible to identify a path from U to W, which lacks X, and this Result contradicts the assumption.

b) Based on the proof by contradiction approach, we assume that X represents a cut-point. Therefore, it is possible to assume that U and W are vertices of $AG_I(R)$ such that J is situated along each path from U to W. Set $X = K+I$ where K is an ideal of R and $I \nsubseteq K$. Consider the following path

$$UT_1 \ldots T_n X T_{n+1} \cdots T_r W.$$

By Theorem 2.1, the following permutation exists from U to W.

$$UT_1 \ldots T_n K T_{n+1} \cdots T_r W.$$

Therefore, we found a path from U to W, which does not cross the X vertex, and this result contradicts X being a cut-point.

c) Based on the proof by contradiction approach, it is assumed that in $AG_I(R)$, X is positioned in every path from U to W, where $U, W \in V$. Consider the following three cases:

1) $I \subset U, W$, in which case by Theorem 2.1, X is in every path from U to W in $AG(R/I)$, and this result is contradictory to the assumption.
2) $I \subset W$ and $I \nsubseteq U$. Two cases are possible.
   A) $U + I \neq W$, therefore by Theorem 2.1, X is located in every path from $U + I$ to W, and thus X is situated in every path from $U + I$ to W in $AG(R/I)$, which contradicts the assumption. Note that according to part (d), $X \neq U + I$.
   B) $U + I = W$. Since X is situated in every path from U to W, W is only adjacent to X. Therefore, there exists $J' \in V$ such that $J'X \subseteq I$. (by Theorem 4.7, since $AG_I(R) \ncong K_2$, then $AG(R/I) \ncong K_2$ and J' exists). Thus X is situated in every path from W to J', and thus by case (1), X is one cut-point in $AG(R/I)$ and we achieved a contradiction.
3) $I \nsubseteq W$ And $I \nsubseteq U$. It is assumed that X is in every path from U to W. Two cases are possible.
   A) $U + I \neq W + I$ Therefore, by Theorem 2.1, X it is in every path from $U + I$ to $W + I$, and thus in $AG(R/I)$, X is in every path from $U + I$ to $W + I$, and thus contradiction is obtained.
   B) $U + I = W + I$ Therefore, by Theorem 2.1, X is in every path from U to $W + I$. Now we act similar to case (2) and thus the proposition is proved.

□

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

References

[1] R. Y. Sharp, *Step in Commutative Algebra*, London Mathematical Society Student Texts (51), Cambridge University Press, Cambridge, 2000.

[2] B. R. McDonald, *Finite Rings with Identity*, Pure and Applied Mathematics, **28**, M. Dekker, Inc., New York, 1974.

[3] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217** (1999) 434−447, https://doi.org/10.1006/jabr.1998.7840.

[4] S. P. Redmond, An ideal-based zero-divisor graph of a commutative ring, *Commun. Algebra* **31** (2003) 4425 − 4443, https://doi.org/10.1081/AGB-120022801.

[5] M. Behboodi and Z. Rakeei, The annihilating - ideal graph of a commutative rings I, *J. Algebra Appl.* **10** (2011) 727 − 739, https://doi.org/10.1142/S0219498811004896.

[6] F. Aliniaeifard, M. Behboodi, E. Mehdi-Nezhad and A. M. Rahimi, The annihilating-ideal graph of a commutative ring with respect to an ideal, *Comm. Algebra* **42** (2014) 2269 − 2284, https://doi.org/10.1080/00927872.2012.753606.

[7] S. Visweswaran and H. D. Patel, Some results on the complement of the annihilating ideal graph of a commutative ring, *J. Algebra Appl.* **14** (2015) p. 1550099, https://doi.org/10.1142/S0219498815500991.

[8] S. Visweswaran and H. D. Patel, On the clique number of the complement of the annihilating ideal graph of a commutative ring, *Beitr. Algebra Geom.* **57** (2016) 307 − 320, https://doi.org/10.1007/s13366-015-0247-5.

[9] S. Visweswaran and A. Parmar, Some results on a spanning subgraph of the complement of the annihilating-ideal graph of a commutative reduced ring, *Discrete Math. Algorithms Appl.* **11** (2019) p. 1950012, https://doi.org/10.1142/S1793830919500125.

[10] G. Aalipour, S. Akbari, M. Behboodi, R. Nikandish, M. J. Nikmehr and F. Shaveisi, The classification of the annihilating - ideal graph of a commutative ring, *Algebra Colloq.* **21** (2014) 249 − 256, https://doi.org/10.1142/S1005386714000200.

[11] G. Aalipour, S. Akbari, R. Nikandish, M. J. Nikmehr and F. Shaveisi, On the coloring of the annihilating - ideal graph of a commutative ring, *Discrete Math.* **312** (2012) 2620 − 2626, https://doi.org/10.1016/j.disc.2011.10.020.

[12] S. Visweswaran, P. T. Lalchandani, When is the annihilating ideal graph of a zero-dimensional semiquasilocal commutative ring planar? Nonquasilocal case, *Boll. Unione Mat. Ital.* **9** (2016) 453 – 468, https://doi.org/10.1007/s40574-016-0061-5.

Zahra Mahmudiankoruie
Department of mathematics,
Faculty of science,
University of Qom,
I. R. Iran
e-mail: mahmudianz65@gmail.com

Mohammad Hasan Naderi
Department of mathematics,
Faculty of science,
University of Qom,
I. R. Iran
e-mail: mhnaderi50@gmail.com