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Upgrading Uncapacitated Multiple Allocation P-Hub Median Problem Using Benders Decomposition Algorithm

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Abstract

The Hub Location Problem (HLP) is a significant problem in combinatorial optimization consisting of two main components: location and network design. The HLP aims to develop an optimal strategy for various applications, such as product distribution, urban management, sensor network design, computer network, and communication network design. Additionally, the upgrading location problem arises when modifying specific components at a cost is possible. This paper focuses on upgrading the uncapacitated multiple allocation p-hub median problem (u-UMApHMP), where a predetermined budget and bound of changes are given. The aim is to modify certain network parameters to identify the p-hub median that improves the objective function value concerning the modified parameters. We propose a non-linear mathematical formulation for u-UMApHMP to achieve this goal. Then, we employ the McCormick technique to linearize the model. Subsequently, we solve the linearized model using the CPLEX solver and the Benders decomposition method. Finally, we present experimental results to demonstrate the effectiveness of the proposed approach.

Keywords: Hub location, Multiple allocation p-hub median problem, Benders decomposition, Combinational optimization.

2020 Mathematics Subject Classification: 90C27, 90C11, 90C05.

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1. Introduction

The hub location problem has been widely studied in facility locations for several decades and has gained significant attention [1]. Hubs are specialized facilities that provide services within networks by creating connections between origins and destinations. Such facilities are primarily recognized as sites for collecting, ordering, and distributing goods, services, and information. In network design, a hub network connects origin-destination pairs indirectly through a small set of intermediary nodes called hubs [2]. The utilization of hubs offers several advantages, which can be attributed to the following factors: (a) reduced transportation costs due to consolidated flows that leverage economies of scale, particularly between hubs, (b) decreased costs from establishing a more widely spaced network that connects numerous dispersed origin-destination pairs, and (c) improved service delivery by enabling more frequent connections [3].
The hub location has wide-ranging applications in various industries, including telecommunication and communication [4], air transportation [5, 6], ground transportation [7], postal services [8], etc. This problem was first introduced by Goldman [9] and Hakimi & Maheshwari [10], rooted in location and network analysis. Subsequently, O'Kelly [11, 12] research spurred the emergence of studies and development of hub location as a distinct subfield within the broader domain of facility location analysis. The allocation in HLPs can be classified into two main categories: single and multiple allocation. Each origin and destination can only be connected to one hub in a single allocation, as noted by [13, 14]. In multiple allocation-each non-hub node can be assigned to several hubs [15]. Also, this problem can be further divided into three classes: the hub median problem, the hub covering problem, and the hub center problem.

Over recent years, there has been a significant surge in scholarly research regarding the hub location problem, with a particular focus on the hub median problem (HMP). This has led to several advancements in the theory and methodology of different aspects of the problem [16]. This problem revolves around the selection of optimal hub locations, coupled with the allocation of non-hub nodes to hubs, with the overarching objective of minimizing the total transportation cost, encompassing factors such as time, distance, and related metrics. When the number of hubs is equal to p, the problem transforms into the p-HMP. O'Kelly [13] formulated the p-HMP using quadratic programming. Campbell [17] presented a model for the p-HMP that considered the multiple and single allocation conditions. Skorin-Kapov et al., [18] demonstrated that the single allocation hub location problem is a special case of the multiple allocation problem. Campbell [19] defined a p-HMP, which is analogous to a p-median, and proposed integer programming formulations for both multiple and single p-HMPs. Subsequently, Ernst and Krishnamoorthy [15] developed a more efficient p-hub multiple allocation formulation based on the concepts presented in their previous work in 1996. They [20] also demonstrated that their method outperformed the approach proposed by Skorin-Kapov et al., [18]. The hub location problem has been extensively studied in the literature,

and several methods have been proposed for its solution. Klincewicz [21] applied exchange clustering methods to the problem, while Klincewicz [22] used greedy and tabu search methods and GRAB. O'Kelly [13] employed an enumeration algorithm, Campbell [19] used a greedy algorithm, and Abdinnour-Helm [23] utilized the Simulated Annealing method. Klincewicz [24] and Silva & Cunha [25] applied the tabu search method, while Chen [26] combined the Simulated Annealing algorithm and tabu search. Other approaches include the genetic algorithm and the Lagrange method used by Kratica et al., [27] and Yaman [28], the Benders decomposition method employed by [29–31], and the multi-criteria decision-making method utilized by [32–34].
Moreover, in numerous combinatorial optimization problems, modifications can be implemented on the problem parameters to augment the efficacy of the network. Fulkerson and Harding [35] examined the shortest path upgrading problem, while Hambrusch and Tu [36] explored the longest path problem by improving the length of edges. The minimum spanning tree and Steiner tree upgrading problems were investigated by [37–39], while Sepasian and Monabbati [40] investigated the upgrading problem of the min-max spanning tree, utilizing some combinatorial algorithms to solve it. Some improvement issues have also been reviewed by [41, 42]. Gassner [43–45] conducted separate studies on upgrading problems related to 1-median and 1-center problems.
The uncapacitated multiple allocation p-hub median problem (UMApHMP) refers to a specific variant of the p-HMP. In this problem, there are no capacity constraints imposed on either hubs or edges, and every non-hub node within the network can be connected to all p-hubs. The research on UMApHMP has primarily focused on identifying the optimal hubs and assigning non-hub nodes to these hubs. However, the potential for network enhancement by adjusting its parameters has received insufficient attention.

This paper aims to enhance the UMApHMP by modifying some network components. Specifically, we propose a novel model to improve the total transportation cost in the UMApHMP by adjusting the length of the edges while respecting the constraints of a pre-determined budget and bounded changes. The problem is formulated as a mixed integer non-linear programming, and to linearize the resulting non-linear model, we use the McCormick relaxation technique. We solved this mixed integer linear programming formulation using the CPLEX solver and the Benders decomposition method.
The rest of the paper is organized as follows: Section 2 summarizes the UMApHMP, introduces and linearizes the u-UMApHMP, and presents its corresponding mathematical formulation. Section 3 presents a Benders decomposition method for solving the linear u-UMApHMP. The computational results and concluding remarks are presented in Sections 4 and 5, respectively.

2. Problem statement and modeling

Combinatorial optimization problems may require modifications to their constituent parameters to improve the underlying network's efficiency, commonly known as an upgrading problem. Researchers have investigated various aspects of this topic. For instance, Sepasian and Monabbati [40] conducted a study on upgrading the min-max spanning tree. Sepasian and Rahbarnia [46] focused on upgrading 1-median on the path. In addition, Sepasian [47] attempted to upgrade the 1-center problem on the network.

The p-HMP constitutes a significant variant of the HLP, in which the number of hubs, denoted as 'p', is predetermined and fixed. The objective of the p-HMP is to determine p optimal hubs in the network such that the total cost is minimized. In p-HMP, it is possible to modify certain problem parameters, and by considering the potential for these changes, the optimal location is determined based on the optimal parameters. This study aims to investigate upgrading the UMApHMP through an edge cost modification approach. Specifically, we aimed to improve the p-hub network's overall performance by modifying the length of edges while adhering to a predetermined budget and bounded changes. To achieve this objective, we design a model that considers the conditions of the UMApHMP and incorporates them into the upgrading process.

2.1 A short review of UMApHMP

The hub median location problem is a well-known optimization problem with diverse applications in transportation and industry, including service center location, sensor network design, computer network design, product distribution network design, air transport network design, fuel distribution network design, and postal item distribution network design. The UMApHMP is a specialized version of the HMP in the field of network optimization. In this problem, there are no limitations on the capacity of hubs or edges.

Moreover, each non-hub node in the network has the potential to be connected to all p-hubs without any restrictions. The main objective of this problem is to identify the optimal location of hubs that minimizes the total transportation cost [17, 19, 48]. Scholars in this area, have provided a comprehensive overview of the various models, classifications, solution techniques, and applications of HLPs [49, 50]. Several models have been proposed to address the HMP [15, 17–19, 48]. Mokhtar et al., [31] presented a linear integer programming model to solve the UMApHMP on an undirected network. We present the required variables (see Table 1) and parameters (see Table 2) for this model as follows:

Table 1: Variables of UMApHMP.

Parameter	Type	Value
X_{ijkm}	Non-negative real	Fraction of flow from origin i to destination j that passes through hubs k and m, respectively.
h_k	Binary	If location k is considered as a hub, the value is 1, otherwise, the value is 0.

Table 2: Sets and required parameters of UMApHMP.

Parameter	Type	Value
V		The set of nodes and $\lvert V\rvert = n$.
E		The set of edges.
α		A discount factor to provide reduced unit costs on links between hubs.
W_{ij}	Non-negative integer	Flow (demand) between node $i, j \in V$.
c_{ij}	Non-negative integer	Distance (transmission cost) between node $i, j \in V$.
C_{ijkm}	Non-negative integer	The cost of transferring a unit of flow from origin i to destination j by passing through hubs $k, m \in h$. This cost is equal to $C_{ijkm}{=}c_{ik}{+}\alpha c_{km}{+}c_{mj}$. If $k = m$, origin i sends flow to destination j through hub k.
p	Non-negative integer	The number of hubs.
F_k	Non-negative integer	The cost of opening hub k.

The mathematical model of UMApHMP is expressed as follows:

$$\min \sum_i \sum_j \sum_k \sum_m W_{ij} X_{ijkm} C_{ijkm} + \sum_k F_k h_k,$$

$$s.t$$

$$\sum_k h_k = p, \tag{1}$$

$$\sum_k \sum_m X_{ijkm} = 1, \qquad \forall i, j \in V, \tag{2}$$

$$\sum_m X_{ijkm} \leq h_k, \qquad \forall i, j, k \in V, \tag{3}$$

$$\sum_k X_{ijkm} \leq h_m, \qquad \forall i,j,m \in V, \tag{4}$$

$$0 \leq X_{ijkm} \leq 1, \qquad \forall i,j,k,m \in V, \tag{5}$$

$$h_k \in \{0,1\}, \qquad \forall k \in V. \tag{6}$$

In the above model, the objective function minimizes the total cost, including transportation and establishment costs. Constraint (1) guarantees that the number of hubs exactly equals p. Constraint (2) ensures that the flow for each origin-destination pair is directed through via some hub pair. Constraints (3), (4) guarantee that the flow which goes through a hub only happens if that hub is installed. Constraint (5) is the non-negativity of the continuous variables X_{ijkm}, while constraint (6) restricts the integer variable h_k to be 0 or 1.

2.2 The problem of u-UMApHM

To state the problem, consider the complete undirected graph $G = (V, E)$, such that V is the set of nodes of this graph, including origin nodes, destination nodes, and V=$\{1, 2, ..., n\}$ $(p \leq n)$. Also, suppose that E is the edge set of the graph assigned to edge $ij \in E$ of length (cost) $c_{ij} \in \mathbb{Z}^+$ and $W_{ij} \in \mathbb{Z}^+$ is the amount of flow (demand) required between node i and node j. Now suppose that in order to improve the service delivery, it is possible to modify the edge lengths of the context graph under the predefined budget B. Due to some real-world restrictions in the existing structures, modifying the edges to any desired size is impossible. Therefore, for each edge $ij \in E$, the amount of edge length modification is displayed with the variable $y_{ij} \in \mathbb{Z}^+$, and the maximum amount of length modification is not less than $l_{ij} \in \mathbb{Z}^+$. Also, the cost paid for the correction of each unit of edge length $ij \in E$ is denoted by u_{ij}. To avoid redundancy, we will avoid repeating similar information presented in Table 1 and Table 2.

The primary objective of this problem is to strategically adjust the edge lengths within graph G under the constraints of the available budget, aiming to improve the UMApHM's objective function value by optimizing the modified edge lengths. The u-UMApHMP is formulated as follows:

$$\min \sum_i \sum_j \sum_k \sum_m W_{ij} X_{ijkm} C_{ijkm} - \sum_i \sum_j \sum_k \sum_m W_{ij} X_{ijkm} (y_{ik} + \alpha y_{km} + y_{mj}) + \sum_k F_k h_k,$$

$s.t$

(1) − (6),

$$\sum_i \sum_j u_{ij} y_{ij} \leq B, \tag{7}$$

$$y_{ij} \leq (c_{ij} - l_{ij}), \qquad \forall i,j \in V, \tag{8}$$

$$y_{ij} \geq 0, \qquad \forall i,j \in V. \tag{9}$$

The objective function of the above model minimizes the total cost, cost improvement due to edge modification with respect to the budget consuming, and the cost of establishing hub nodes. Constraint (7) states that the cost of edge improvements should not exceed the available budget. Constraint (8) bounds the maximum value of edge length improvement from above.

The presented model for the u-UMApHMP contains non-linear terms in the form of the product of two continuous variables. Non-linear models are more complicated than linear models. The McCormick linearization is a suitable method for relaxing a non-linear expression of the product of two continuous variables. To accomplish this, a change in variables is implemented in the problem, accompanied by the corresponding constraints [51].

The changes in the variables are as follows:

$$\begin{aligned} Q_{ijkm} &= X_{ijkm} \times y_{ik}, \\ P_{ijkm} &= X_{ijkm} \times y_{km}, \\ R_{ijkm} &= X_{ijkm} \times y_{mj}. \end{aligned} \tag{10}$$

So, the integer linear model of the u-UMApHMP is obtained as follows:

$$\min \sum_i \sum_j \sum_k \sum_m W_{ij} X_{ijkl} C_{ijkm} - \sum_i \sum_j \sum_k \sum_m W_{ij} \big(Q_{ijkm} + \alpha\ P_{ijkm} + R_{ijkm}\big) + \sum_k F_k h_k,$$

$s.t$

(1) – (9),

$$Q_{ijkm} \geq (l_{ik} - c_{ik}) + (c_{ik} - l_{ik})\ X_{ijkm} + y_{ik}, \qquad \forall i,j,k,m \in V, \tag{11}$$

$$Q_{ijkm} \leq y_{ik}, \qquad \forall i,j,k,m \in V, \tag{12}$$

$$Q_{ijkm} \leq (c_{ik} - l_{ik})\ X_{ijkm}, \qquad \forall i,j,k,m \in V, \tag{13}$$

$$P_{ijkm} \geq (l_{km} - c_{km}) + (c_{km} - l_{km})\ X_{ijkm} + y_{km}, \qquad \forall i,j,k,m \in V, \tag{14}$$

$$P_{ijkm} \leq y_{km}, \qquad \forall i,j,k,m \in V, \tag{15}$$

$$P_{ijkm} \leq (c_{km} - l_{km})\ X_{ijkm}, \qquad \forall i,j,k,m \in V, \tag{16}$$

$$R_{ijkm} \geq (l_{mj} - c_{mj}) + (c_{mj} - l_{mj})\ X_{ijkm} + y_{mi}, \qquad \forall i,j,k,m \in V, \tag{17}$$

$$R_{ijkm} \leq y_{mj}, \qquad \forall i,j,k,m \in V, \tag{18}$$

$$R_{ijkm} \leq (c_{mj} - l_{mj})\ X_{ijkm}, \qquad \forall i,j,k,m \in V, \tag{19}$$

$$Q_{ijkm} \geq 0, \qquad \forall i,j,k,m \in V, \tag{20}$$

$$P_{ijkm} \geq 0, \qquad \forall i,j,k,m \in V, \tag{21}$$

$$R_{ijkm} \geq 0, \qquad \forall i,j,k,m \in V. \tag{22}$$

The above model is a McCormick linearization of the u-UMApHMP, referred to as Lu-UMApHMP. While the multiple allocation hub median problem with fixed hub locations can be solved in polynomial time, the general case of UMApHMP is known to be NP-hard [31]. To address this difficulty, we apply the Benders decomposition algorithm, which is successfully employed to tackle a wide range of optimization problems [52]. To obtain an approximate optimal solution for the original problem, we apply the Benders decomposition method to solve Lu-UMApHMP.

3. The Benders decomposition method

The Benders decomposition method, originally introduced by Benders [53], was initially designed for mixed-integer linear programming problems that involve continuous subproblems. Over time, this method has been extended to tackle a broad range of problems, including nonlinear, integer, stochastic, bilevel, multi-stage, and other optimization problems [52].

The Benders decomposition method is a powerful technique for resource allocation in large-scale problems, and it has been successfully applied to solve numerous practical optimization problems, such as routing, locating, planning, and scheduling. The method involves reformulating the problem using duality and primary duality relations. Specifically, the complex integer problem is divided into a master problem (MP) and a dependent subproblem (SP), which are iteratively solved using each other's solutions [53].

In order to use the Benders decomposition method for a problem, the solution obtained by the MP is used as input for the SP, where integer and complex variables are fixed. The dual of the SP is then solved, and the resulting solution serves as an upper bound for the overall problem. The dual solution of the SP is also used to construct a Benders cut that includes continuous variables and is added to the MP. In the next iteration, this cut is incorporated into the MP, and a new lower bound is obtained by solving the updated problem. This lower bound is guaranteed to be no worse than the current lower bound. The MP and the SP are solved iteratively until a termination condition is met, such as the upper and lower bound difference being less than a small number. Geoffrion and Graves have proven that the Benders decomposition method converges to the optimal solution in a finite number of iterations [54]. Therefore, it is a powerful tool for solving large-scale optimization problems with complex integer variables.

3.1 The Benders decomposition algorithm for the Lu-UMApHMP

In this subsection, we develop a classical Benders decomposition method to solve Lu-UMApHMP. To use this method, we decompose the original problem into a master problem consisting of variables associated with the hub locations and their corresponding constraints and a SP consisting of the remaining variables and constraints. The hub location variable $h = h_k$ is fixed in each iteration, and linear programming is formulated as follows, which is denoted as SP.

SP:

$$\min \sum_i \sum_j \sum_k \sum_m W_{ij} X_{ijkm} C_{ijkm} - \sum_i \sum_j \sum_k \sum_m W_{ij}(Q_{ijkm} + \alpha\ P_{ijkm} + R_{ijkm}),$$

$$
\begin{array}{ll}
s.t & \\
\sum_k \sum_m X_{ijkm} = 1, & \forall i, j \in V, \\
\sum_m X_{ijkm} \leq h_k, & \forall i, j, k \in V, \\
\sum_k X_{ijkm} \leq h_m, & \forall i, j, m \in V, \\
\sum_i \sum_j u_{ij} y_{ij} \leq B & \\
y_{ik} \leq (c_{ik} - l_{ik}), & \forall i, k \in V, \\
y_{km} \leq (c_{km} - l_{km}), & \forall k, m \in V, \\
y_{mj} \leq (c_{mj} - l_{mj}), & \forall m, j \in V, \\
Q_{ijkm} \geq (l_{ik} - c_{ik}) + (c_{ik} - l_{ik})\ X_{ijkm} + y_{ik}, & \forall i, j, k, m \in V, \\
Q_{ijkm} \leq y_{ik}, & \forall i, j, k, m \in V, \\
Q_{ijkm} \leq (c_{ik} - l_{ik})\ X_{ijkm}, & \forall i, j, k, m \in V, \\
P_{ijkm} \geq (l_{km} - c_{km}) + (c_{km} - l_{km})\ X_{ijkm} + y_{km}, & \forall i, j, k, m \in V, \\
P_{ijkm} \leq y_{km}, & \forall i, j, k, m \in V, \\
P_{ijkm} \leq (c_{km} - l_{km})\ X_{ijkm}, & \forall i, j, k, m \in V, \\
R_{ijkm} \geq (l_{mj} - c_{mj}) + (c_{mj} - l_{mj})\ X_{ijkm} + y_{mj}, & \forall i, j, k, m \in V, \\
R_{ijkm} \leq y_{mj}, & \forall i, j, k, m \in V, \\
R_{ijkm} \leq (c_{mj} - l_{mj})\ X_{ijkm}, & \forall i, j, k, m \in V, \\
0 \leq X_{ijkm} \leq 1, & \forall i, j, k, m \in V, \\
Q_{ijkl}, P_{ijkm}, R_{ijkm} \geq 0, & \forall i, j, k, m \in V, \\
y_{ij} \geq 0, & \forall i, j \in V.
\end{array}
$$

Let $(\Pi^1_{ij}, \Pi^2_{ijk}, \Pi^3_{ijm}, \Pi^4, \Pi^5_{ik}, \Pi^6_{km}, \Pi^7_{mj}, \Pi^8_{ijkm}, ..., \Pi^{16}_{ijkm}, \Pi^{17}_{ijkm})$ be the dual variables associated with constraints SP model, respectively. In the context of mathematical optimization, the dual SP (DSP) can be formulated as follows:

DSP:

$$max \sum_{i,j} \Pi^1_{ij} - \sum_{i,j,k} h_k\ \Pi^2_{ijk} - \sum_{i,j,m} h_m\ \Pi^3_{ijm} - B\ \Pi^4 - \sum_{i,k} (c_{ik} - l_{ik})\ \Pi^5_{ik} - \sum_{k,m} (c_{km} - l_{km})\ \Pi^6_{km} -$$

$$\sum_{m,j}(c_{mj}-l_{mj})\Pi^7_{mj}-\sum_{i,j,k,m}(c_{ik}-l_{ik})\Pi^8_{ijkm}-\sum_{i,j,k,m}(c_{km}-l_{km})\Pi^{11}_{ijkm}-\sum_{i,j,k,m}(c_{mj}-l_{mj})\Pi^{14}_{ijkm}$$
$$-\sum_{i,j,k,m}\Pi^{17}_{ijkm},$$

s.t

$$\Pi^1_{ij}-\Pi^2_{ijk}-\Pi^3_{ijm}-(c_{ik}-l_{ik})\Pi^8_{ijkm}+(c_{ik}-l_{ik})\Pi^{10}_{ijkm}-(c_{km}-l_{km})\Pi^{11}_{ijkm}+(c_{km}-l_{km})\Pi^{13}_{ijkm}-$$
$$(c_{mj}-l_{mj})\ \Pi^{14}_{ijkm}+(c_{mj}-l_{mj})\ \Pi^{16}_{ijkm}+\Pi^{17}_{ijkm}\le W_{ij}\ C_{ijkm},\qquad \forall i,j,k,m\in V, \tag{23}$$

$$-u_{ik}\ \Pi^4-\Pi^5_{ik}-\sum_{m,j}\Pi^8_{ijkm}+\sum_{mj}\Pi^9_{ijkm}\le 0,\qquad \forall i,k\in V, \tag{24}$$

$$-u_{km}\ \Pi^4-\Pi^6_{km}-\sum_{i,j}\Pi^{11}_{ijkm}+\sum_{i,j}\Pi^{12}_{ijkm}\le 0,\qquad \forall k,m\in V, \tag{25}$$

$$-u_{mj}\ \Pi^4-\Pi^7_{mj}-\sum_{ik}\Pi^{14}_{ijkm}+\sum_{ik}\Pi^{15}_{ijkm}\le 0,\qquad \forall m,j\in V, \tag{26}$$

$$\Pi^8_{ijkm}-\Pi^9_{ijkm}-\Pi^{10}_{ijkm}\le -W_{ij},\qquad \forall i,j,k,m\in V, \tag{27}$$

$$\Pi^{11}_{ijkm}-\Pi^{12}_{ijkm}-\Pi^{13}_{ijkm}\le -\alpha\ W_{ij},\qquad \forall i,j,k,m\in V, \tag{28}$$

$$\Pi^{14}_{ijkm}-\Pi^{15}_{ijkm}-\Pi^{16}_{ijkm}\le -W_{ij},\qquad \forall i,j,k,m\in V, \tag{29}$$

$$\Pi^1_{ij}\in\mathbb{R}\qquad \forall i,j\in V, \tag{30}$$

$$\Pi^1_{ij},\Pi^2_{ijk},\Pi^3_{ijm},\Pi^4,\Pi^5_{ik},\Pi^6_{km},\Pi^7_{mj},\Pi^8_{ijkm},...,\Pi^{16}_{ijkm},\Pi^{17}_{ijkm}\ge 0,\qquad \forall i,j,k,m\in V. \tag{31}$$

If the DSP is feasible with an optimal solution, an optimality cut is generated by means of its optimal solution, and then it is added to the MP.

Lemma 3.1. *The dual of SP is guaranteed to be both feasible and bounded for any valid location of hubs. As a result, only optimality cuts are incorporated into the MP.*

By an optimal solution $(\hat{\Pi}^1_{ij},\hat{\Pi}^2_{ijk},\hat{\Pi}^3_{ijm},\hat{\Pi}^4,\hat{\Pi}^5_{ik},\hat{\Pi}^6_{km},\hat{\Pi}^7_{mj},\hat{\Pi}^8_{ijkm},...,\hat{\Pi}^{16}_{ijkm},\hat{\Pi}^{17}_{ijkm})$ of DSP for a given iteration, we can construct the optimality cut as follows:

$$\varphi\ge\sum_{i,j}\hat{\Pi}^1_{ij}-\sum_{i,j,k,m}h_k\ \hat{\Pi}^2_{ijk}-\sum_{i,j,k,m}h_m\hat{\Pi}^3_{ijm}-B\hat{\Pi}^4-\sum_{i,k}(c_{ik}-l_{ik})\hat{\Pi}^5_{ik}-\sum_{k,m}(c_{km}-l_{km})\hat{\Pi}^6_{km}-$$
$$\sum_{m,j}(c_{mj}-l_{mj})\hat{\Pi}^7_{mj}-\sum_{i,j,k,m}(c_{ik}-l_{ik})\hat{\Pi}^8_{ijkm}-\sum_{i,j,k,m}(c_{km}-l_{km})\hat{\Pi}^{11}_{ijkm}-\sum_{i,j,k,m}(c_{mj}-l_{mj})\hat{\Pi}^{14}_{ijkm}$$
$$-\sum_{i,j,k,m}\hat{\Pi}^{17}_{ijkm}, \tag{32}$$

where φ is a real non-negative variable. Thus, the MP can be formulated as follows:

MP:

$$min\sum_k F_k h_k+\varphi,$$

$$
\begin{aligned}
& s.t \\
& (32), \\
& \sum_k h_k = p, && && (33) \\
& h_k \in \{0,1\}, && \forall k \in V, && (34) \\
& \varphi \geq 0. && && (35)
\end{aligned}
$$

The classical Benders decomposition algorithm (CBDA) is formally stated below in which UB and LB are the upper bound and lower bound, respectively. We denote the optimal solutions obtained by solving the current MP and DSP as Z^*_{MP} and Z^*_{DSP}, respectively.

Algorithm 1 : classical Benders decomposition for Lu-UMApHMP

Start
Set initial feasible h_k and $UB = +\infty$, $LB = -\infty$, $\varepsilon = 10^{-3}$;
  While $(UB - LB) \geq \varepsilon$ **do**
    Solve the dual problem DSP;
    Generate optimality cut (32);
    Set $UB = Z^*_{DSP} + \sum_k F_k h_k$;
    Add Benders cut to MP;
    Solve the MP, obtaining objective value(Z^*_{MP}) and the optimal values for h_k;
    Set $LB = Z^*_{MP}$;
  End While

4. Computational results

This section presents the results of the numerical computations for the proposed model and algorithms. The Benders algorithm and the original model were implemented in the GAMS software environment (version 24.1.2), and the CPLEX optimization solver (version 12.8) was used to solve the SP, DSP, and original models. The computations were performed on a computer with an Intel Core i5 processor, operating at 4200 M processor speed, with 6 Gigabytes of internal memory, and running Windows 11 operating system. To evaluate the proposed model, we utilized the CAB dataset provided by the US Civil Aeronautics Board [13, 55] with some minor modifications. These modifications involved adjustments to the establishment costs, budget allocations for changes, lower bounds for edges, and penalties incurred for corrections.

Our study involved experiments on three different datasets, categorized based on the number of nodes in each instance. Dataset 1 consisted of instances with 10 and 15 nodes (see Table 3), Dataset 2 included instances with 20 and 25 nodes (see Table 4), and Dataset 3 encompassed instances with 30, 35, 40, and 45 nodes (see Table 6).

Table 3: Computational results for Dataset 1.

Parameters N, p, α	Opt. Sol. B=0 ($\times 10^{+7}$)	Opt. Sol. B=20000($\times 10^{+7}$)	CPLEX *Time CPU(s)*	CBDA *Time CPU(s)*	Hub B=20000
10, 2, 0.2	1.624950	1.600549	8	12	{5, 9}
10, 2, 0.4	1.868851	1.844399	8	15	{5, 9}
10, 2, 0.6	2.102095	2.073367	7	12	{5, 9}
10, 2, 0.8	2.304412	2.275910	7	9	{5, 9}
15, 2, 0.2	5.658423	5.578907	90	98	{5, 15}
15, 2, 0.4	6.240293	6.191206	77	97	{5, 14}
15, 2, 0.6	6.727997	6.678911	76	104	{5, 14}
15, 2, 0.8	7.172010	7.125825	76	91	{5, 14}
15, 4, 0.2	2.917392	2.884262	71	78	{3, 5, 9, 14}
15, 4, 0.4	4.058826	4.024079	78	102	{3, 5, 9, 14}
15, 4, 0.6	5.122051	5.084017	78	90	{3, 5, 9, 14}
15, 4, 0.8	6.150809	6.112404	76	103	{3, 5, 9, 14}

In Table 3, the first column represents the input parameters of the problem. The second and third columns display the optimal solution values for the problem under two different scenarios: when the budget is set to 0 (no upgrading can be done) and when a specific budget is allocated for upgrading (the Lu-UMApHMP), respectively. The fourth and fifth columns of the table display the elapsed time (measured in seconds) required by the CPLEX solver and CBDA to solve the given model successfully. Finally, the last column displays the optimal hubs obtained through the utilization of these solvers.

According to Table 3, the optimal solution of Lu-UMApHMP with an allocated budget has shown improvement, and the CPLEX solver demonstrates faster solving times compared to CBDA when applied to small-sized datasets. For the test involving the instances with 10 nodes, the CPLEX solver exhibits a time difference rate of 45% compared to the CBDA. The time difference rates for the instances with 15 nodes with 2 and 4 hubs are 18% and 19%, respectively.

The present study conducted a series of tests on Dataset 2 to evaluate the performance of the CPLEX solver and the CBDA. The experiments encompassed instances with different node sizes. The CPLEX solver exhibited remarkable efficiency in achieving the optimal solution for instances consisting of 20 nodes. Specifically, the CPLEX solver demonstrated a 2%-time difference compared to the CBDA, although the disparity between the two was negligible. Notably, for instance, with 25 nodes, the CPLEX solver encountered an out-of-memory error (indicated by ‘M error’), while the CBDA solved the problem (see Table 4).

Table 4: Computational results for Dataset 2.

Parameters N, p, α	Opt. Sol. B=0 ($\times 10^{+7}$)	Opt. Sol. B=20000($\times 10^{+7}$)	CPLEX *Time CPU(s)*	CBDA *Time CPU(s)*	Hub B=20000
20, 4, 0.2	6.737578	6.686855	668	689	{5, 9, 14, 16}
20, 4, 0.4	8.260774	8.219935	451	469	{5, 9, 14, 16}
20, 4, 0.6	9.522764	9.475700	665	678	{5, 9, 14, 16}
20, 4, 0.8	10.69626	10.64740	998	1020	{5, 9, 14, 16}
25, 8, 0.2	13.97660	13.84931	M error	2382	{3, 5, 9, 14, 19, 21, 22, 23}
25, 8, 0.4	21.19367	21.10392	M error	3661	{5, 9, 14, 16, 21, 22, 23, 24}
25, 8, 0.6	27.72824	27.61434	M error	6193	{5, 9, 14, 16, 21, 22, 23, 24}
25, 8, 0.8	33.98784	33.82976	M error	6217	{5, 9, 14, 16, 21, 22, 23, 24}

In this paper, we employed the CAB dataset as a case study. For data with 25 nodes, p=8, and $\alpha = 0.9$, we changed the budget and compared the results. When the budget is set to zero, the optimal total cost aligns precisely with the value reported in reference [55]. The disparity between the two values lies solely in the setup costs for opening the hubs. However, for example, by considering a budget of the amount of 40,000 units for improving the performance of the network, the optimal total cost is decreased 21,417 units. This alteration underscores the pivotal role of budget allocation in optimizing the total cost within the context of our study. Our research findings illustrate that augmenting the budget results in either an improvement in the total cost or its stabilization at a consistent level (see Table 5). Also, with the change in the amount of the budget, the hubs can be changed. For example, by increasing the budget from 25000 to 30000, we observe that the hubs are changed.

Table 5: Computational results related to the changes of the budget.

Budget	Opt. Sol. ($\times 10^{+6}$)	Hub
0	369.0851	{3, 5, 9, 14, 16, 21, 22, 23}
5000	368.4236	{3, 5, 9, 14, 16, 21, 22, 23}
10000	367.8755	{3, 5, 9, 14, 16, 21, 22, 23}
15000	367.6661	{3, 5, 9, 14, 16, 21, 22, 23}
20000	367.4542	{3, 5, 9, 14, 16, 21, 22, 23}
25000	367.3022	{3, 5, 9, 14, 16, 21, 22, 23}
30000	367.1617	{3, 5, 9, 14, 16, 19, 21, 23}
35000	367.0474	{3, 5, 9, 14, 16, 19, 21, 23}
40000	366.9434	{3, 5, 9, 14, 16, 19, 21, 23}
45000	366.8644	{3, 5, 9, 14, 16, 19, 21, 23}

Figure 1 illustrates the process of budget alteration and its consequential impact on the optimal total cost value.

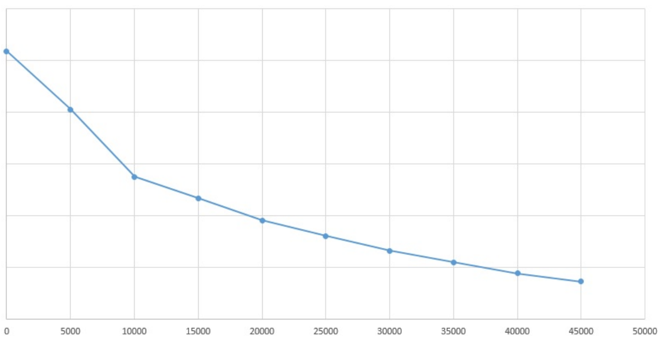

Figure 1: Optimal total cost values with respect to budget change.

It is noteworthy that when working with Dataset 3, which comprises instances of larger sizes, the calculations performed using the CPLEX solver resulted in a memory shortage error and hence could not be included in the analysis. The CBDA is a suitable approach in such scenarios as it decomposes the problem into smaller subproblems. Subsequently, the tests were exclusively conducted on Dataset 3, utilizing the CBDA, as indicated in Table 6. According to the results presented in Table 6, it is evident that the CPLEX solver faced memory shortage errors while attempting to solve instances with 30, 35, 40, and 45 nodes. In contrast, the CBDA demonstrated the ability to solve these instances by decomposing the problem into smaller subproblems.

Table 6: Computational results for Dataset 3.

Parameters N, p, α	Opt. Sol. B=0 ($\times 10^{+8}$)	Opt. Sol. B=20000($\times 10^{+8}$)	CPLEX *Time CPU(s)*	CBDA *Time CPU(s)*	Hub B=20000
30, 8, 0.2	2.574522	2.562840	M error	15112	{5, 9, 14, 21, 23, 28, 29, 30}
30, 8, 0.4	3.902305	3.887137	M error	16282	{5, 9, 14, 21, 23, 28, 29, 30}
30, 8, 0.6	5.122472	5.102279	M error	13982	{5, 9, 14, 21, 22, 23, 28, 30}
30, 8, 0.8	6.267675	6.245660	M error	11982	{5, 9, 14, 21, 22, 23, 28, 30}
35, 8, 0.2	3.746114	3.727827	M error	12524	{5, 9, 14, 21, 28, 30, 31, 32}
35, 8, 0.4	5.332018	5.317167	M error	13926	{5, 9, 14, 21, 28, 29, 30, 32}
35, 8, 0.6	6.791775	6.774643	M error	14975	{5, 9, 14, 21, 28, 29, 30, 32}
35, 8, 0.8	8.112937	8.089146	M error	18251	{5, 9, 14, 21, 22, 23, 28, 30}
40, 4, 0.2	7.732198	7.697829	M error	14450	{5, 21, 23, 28}
40, 4, 0.4	8.764980	8.726314	M error	15103	{5, 21, 23, 28}
40, 4, 0.6	9.544658	9.510205	M error	16111	{5, 21, 23, 28}
40, 4, 0.8	10.15572	10.11534	M error	19224	{5, 21, 23, 28}
45, 4, 0.2	9.666607	9.627035	M error	15519	{5, 28, 38, 44}
45, 4, 0.4	11.21598	11.15735	M error	16725	{5, 21, 23, 28}
45, 4, 0.6	12.24736	12.18669	M error	18758	{5, 21, 23, 28}
45, 4, 0.8	13.05107	12.97818	M error	23253	{5, 21, 23, 28}

In the subsequent analysis, Figure 2 shows the convergence process to the optimal solution using CBDA for instances containing 45 nodes. In this figure, it is clear that the upper and lower bounds are close to each other in some steps, and finally

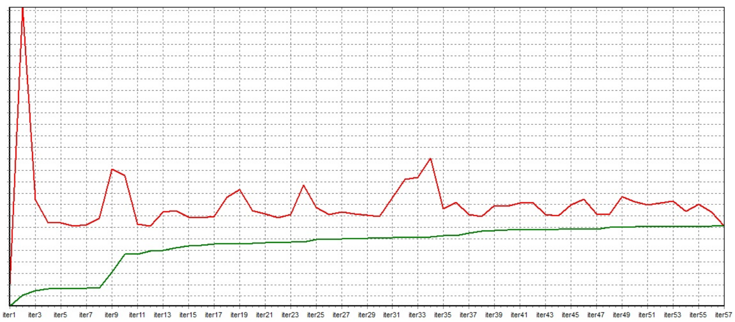

Figure 2: The convergence process to the optimal solution using CBDA for $n = 45$.

they meet.

In summary, the computational findings indicate that the optimal solution of Lu-UMApHMP with a specifically allocated budget consistently demonstrates improvement across all instances. Furthermore, the CPLEX solver effectively solves small problem instances, whereas the CBDA demonstrates greater applicability across a more comprehensive range of instances. As the problem size increases, the CBDA proves advantageous as it can decompose the problem into smaller subproblems, enabling efficient and faster execution. Therefore, utilizing the CBDA for improved efficiency and quicker solution generation is recommended for larger instances.

5. Conclusions

In this paper, we investigated the upgrading of the uncapacitated multiple allocation p-hub median problem (u-UMApHMP). This problem aims to improve the value of the optimal objective function of UMApHMP by adjusting edge lengths while adhering to a pre-defined budget and bound constraints. We developed a nonlinear mixed integer programming model for the u-UMApHMP and then proceeded to linearize the model. We utilized the CPLEX solver and the Benders decomposition method to solve the linearized version. The proposed approach was implemented using the GAMS software, and the resulting outcomes were subsequently compared across multiple datasets for analysis and evaluation.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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