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Critical Metrics Related to Quadratic Curvature Functionals over Generalized Symmetric Spaces of Dimension Four

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Abstract

Our examination of quadratic curvature functionals in Generalized Symmetric Spaces has resulted in the comprehensive classification of critical metric sets within diverse categories of these spaces.

Keywords: Generalized symmetric space, Quadratic curvature functional, Critical metric, Homogeneous space.

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1. Introduction

Exploration of critical points (critical metrics) of functionals that are associated with second-order scalar curvature invariants on pseudo-Riemannian manifolds is frequently utilized in theoretical physics, specifically in quantum and relativity theory. A critical metric refers to a metric that produces the minimum or maximum value of a functional and includes factors such as curvature and other metric-associated parameters. While studying functionals in pseudo-Riemannian geometry, it is important to consider two key factors. Firstly, the construction

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of functionals must be carefully considered, in order to reflect the desired physical interpretation. Secondly, the mathematical language used to describe these functionals must also be taken into account.

Given that M is an oriented and closed manifold and M_1 is the space of Riemannian metrics with unit volume on M, it's crucial to note that the total scalar curvature functional $g \to \int \tau \mathrm{d}vol_g$ has been extensively investigated and is well-understood. Critical metrics of this functional are Einstein metrics on M_1. The quadratic scalar curvature invariants are created by the set of $\{\Delta\tau, \tau^2, \|\varrho\|^2, \|R\|^2\}$, where R, ϱ, and τ denote the curvature tensor, the Ricci tensor, and the scalar curvature respectively. For examination of all curvature functionals that relate to quadratic curvature (QC in short) invariants, it is necessary to analyze a family of three-parameter functionals that were initially introduced in [1], for arbitrary real scalars a, b, and c in $\mathbb{R}$,

$$\Phi_{a,b,c} : g \to \int (a\|R\|^2 + b\|\varrho\|^2 + c\|\tau\|^2)\mathrm{d}vol_g.$$

Working in dimension four, according to the Gauss-Bonnet theorem we have

$$\int_M \|R\|^2 \mathrm{d}vol_g = 32\pi^2\chi(M) + \int_M (4\|\varrho\|^2 - \tau^2)\mathrm{d}vol_g,$$

where $\chi(M)$ denotes the Euler characteristic of M. The preceding relation indicates that the functional critical points for the L^2-norm of the curvature tensor and the critical points for the form $4\|\varrho\|^2 - \tau^2$ coincide. Therefore, for all QC-functionals in a four-dimensional space, the metrics that are critical are those which are critical for both the functionals $\mathcal{H}_t$ and $\mathcal{S}$ simultaneously.

$$\mathcal{H}_t : g \mapsto \int_M \left(t\tau^2 + ||\varrho||^2\right) \mathrm{d}vol_g, \quad t \in \mathbb{R}, \qquad \mathcal{S} : g \mapsto \int_M \tau^2 \mathrm{d}vol_g.$$

Evidently, metrics with vanishing scalar curvature are clear examples of critical metrics for the functional $\mathcal{S}$ (as are Ricci-flat metrics for the functional $\mathcal{H}_t$). Curvature functionals linked to quadratic scalar invariants have been thoroughly researched by numerous scholars, as exemplified by the works [2–5]. These functionals yield a vast variety of critical points, as evidenced by sphere $\mathbb{S}^3$ in [6]. The criticality of a m-dimensional Riemannian manifold for the functional $\mathcal{S}$ in M_1 was determined using the Euler-Lagrange Burger equations in [7]. The equation for criticality must satisfy the condition

$$2\mathrm{Hes}_\tau - \frac{2}{m}\Delta\tau g - 2\tau\varrho + \frac{2}{m}\tau^2 g = 0,$$

where Hes_τ represents the Hessian for scalar curvature. Also, $\Delta\tau = tr_g\mathrm{Hes}_\tau$ is it's Laplacian.

As previously noted in [8], Einstein metrics are well-known families of critical metrics. Einstein metrics are critical for $\mathcal{H}_t$, regardless of the real value of t. However, critical metrics for QC-functionals do not have to be Einstein in general.

If the manifold is compact, the critical metrics for the function $\mathcal{S}$ are either Einstein or have a zero scalar curvature [1], thus there is no particular difficulty in evaluating them; nonetheless, we are seeking for non-Einstein critical metrics. It is shown in [1] that there are non-Einstein four-dimensional Riemannian metrics that are critical for QC-functionals $\mathcal{H}_t$ for all values of t. As an example, we may name the non-smooth cones $\mathbb{R}^+ \times_r N$, where N is an Einstein manifold of dimension three with constant sectional curvature of -3.

Generalized symmetric spaces are classes of homogeneous pseudo-Riemannian manifolds with numerous geometric properties. It's worth mentioning that these spaces can be divided into four distinct categories labeled as **A**, **B**, **C**, and **D**.

These spaces are explored from several perspectives following categorization in [9]. In this paper, we investigate generalized symmetric spaces (GS-space in short) and explicitly determine classes of critical metrics for QC-functionals $\mathcal{S}$ and $\mathcal{H}_t$.

Following arguments in [10], a proper, simply connected GS-space (M, g) of dimension $n = 4$ is either of order 3 or infinity. All of these spaces are indecomposable and belong to one of the types **A-D** (up to an isometry).

This study is organized as follows: In the next section, we present known facts as well as information required for the study of critical metrics on GS-spaces. Section 3 is devoted to displaying calculations and geometric findings on the related spaces. Finally, in Section 4, we look at the categorization of critical metrics on four-dimensional GS-spaces.

2. Preliminaries

Quadratic curvature functionals: Riemannian settings are used to compute the well-known Euler-Lagrange equations for a QC-functional [7, 11]. Since the signature of the base metric is not involved in arguments, the conclusions of the Riemannian case may be extended to the pseudo-Riemannian circumstances.

The gradient for the functionals $\mathcal{H}_t : g \mapsto \int_M \left(t\tau^2 + \|\varrho\|^2\right) dvol_g$ and $\mathcal{S} : g \mapsto \int_M \tau^2 dvol_g$ is as follows:

$$\begin{aligned}(\nabla\mathcal{S})_{ij} &= 2\nabla^2_{ij}\tau - 2(\Delta\tau)g_{ij} - 2\tau\varrho_{ij} + \tfrac{1}{2}\tau^2 g_{ij},\\ (\nabla\mathcal{H}_t)_{ij} &= -\Delta\varrho_{ij} + (1+2t)\nabla^2_{ij}\tau - \tfrac{1+4t}{2}(\Delta\tau)g_{ij}\\ &\qquad -2t\tau\varrho_{ij} - 2\varrho_{kl}R_{ikjl} + \tfrac{1}{2}\left(||\varrho||^2 + t\tau^2\right)g_{ij}.\end{aligned}$$

If $(\nabla\mathcal{H}_t) = cg$ for some real constant c, then g is critical for $\mathcal{H}_t$ and vice versa. By tracing the above equation, we have

$$(m-4)\left(t\tau^2 + ||\varrho||^2\right) - (m + 4(m-1)t)\Delta\tau = 2mc.$$

Thus, g is critical for $\mathcal{H}_t$ if and only if

$$-\Delta\varrho_{ij} + (2t+1)\nabla^2_{ij}\tau - \frac{2t}{m}(\Delta\tau)g_{ij} - 2\varrho_{kl}R_{ikjl} - 2t\tau\varrho_{ij} + \frac{2}{m}\left(||\varrho||^2 + t\tau^2\right)g_{ij} = 0, \quad (1)$$

and

$$(m-4)\left(t\tau^2+||\varrho||^2-\lambda\right)=(m+4(m-1)t)\Delta\tau, \tag{2}$$

where $\lambda=\mathcal{H}_t(g)$ (see [2]). As a result, Einstein metrics are crucial for $\mathcal{H}_t$ for any value of t. Since quadratic curvature functionals of dimension four are homothety invariant, one can simplify the exposition by working at the homothetical level. On the other hand, the aforementioned Euler-Lagrange equations can be greatly reduced in to four-dimensional cases with constant scalar curvature (especially in homogeneous spaces). In this situation, $(\nabla\mathcal{S})_{ij}=2\tau\left(\frac{1}{4}\tau g_{ij}-\varrho_{ij}\right)$, Equation (2) is easily met, whereas Equation (1) simplifies to

$$\Delta\varrho+2t\tau\varrho+2R\left[\varrho\right]-\frac{1}{2}\left(||\varrho||^2+t\tau^2\right)g=0, \tag{3}$$

where $\Delta\varrho$ is the Ricci tensor Laplacian and $R[\rho]$ is the Ricci contraction of the curvature tensor deducing from $R[\rho](u,v)=\operatorname{tr}\{w\to R(u,\rho(w))v\}$, noticing that $g(\rho(u),v)=\varrho(u,v)$. Thus, for a metric with constant scalar curvature, the $\mathcal{S}$-criticality condition is equivalent to being either Einstein or with vanishing scalar curvature. In the following sections, we will concentrate on the above Equation (3) and its solutions.

Generalized symmetric spaces of dimension four: Choose a point p belonging to the pseudo-Riemannian manifold (M,g) which is connected. We refer to an isometry s_p of the manifold M that maintains p as a fixed isolated point as a *symmetry* at p. Each point p on a symmetric space (M,g) allows a symmetry s_p, reversing geodesic to pass through the point. The definition of a *regular s-structure* by A. J. Ledger, who generalized this condition, is a set of $\{s_p : p\in M\}$ symmetries of (M,g) satisfying

$$s_p\circ s_q=s_r\circ s_p,\quad r=s_p(q),$$

for all points p,q of M. A s-structure's *order* is the smallest integer $t\geq 2$ that guarantees that $(s_p)^t=\mathrm{id}_M$ for all $p\in M$.

If (M,g) has a regular s-structure, it is called a *GS-space*. The order of a GS-space is determined by the highest integer $t\geq 2$ that allows for a regular s-structure of order t on M. It's important to note that s-structures on (M,g) are not unique, which is why this definition is necessary.

Four-dimensional GS-spaces were investigated by J. Černy and O. Kowalski, who also categorized these spaces in the local coordinates and algebraically (see [9]). It is an incontrovertible fact that all (pseudo-)Riemannian GS-spaces possess homogeneity. Furthermore, considering the condition of the invariant metric, we can say that there is at least one reductive homogeneous space structure [9].

In the case of four-dimensional instances, such a reductive decomposition corresponds to the realizations as coset spaces G/H throughout of four classes mentioned in [9]. Several investigations have been conducted on GS-spaces of dimension four. Geometric structures, Ricci solitons and conformal geometry, for example, were studied in [12], [10] and [13], respectively. The reference [14] discusses the geometrical properties of GS-Spaces of dimension four. It highlights that the subclass, which has an underlying Lorentzian metric, is locally symmetric and locally isometric to a Cahen-Wallach symmetric space. Therefore, to be specific, we do not consider the type **C** from their original classification of GS-spaces due to this reason.

If a metric is critical for two distinct QC-functionals, then it is unequivocally critical for all quadratic curvature functionals. It is imperative to note that in the homogeneous situation (e.g., GS-spaces), the metric which is critical for all QC-functionals (as that is $\mathcal{S}$-critical) is either Einstein or the scalar curvature vanishes. Apart from the Einstein metrics, it is therefore imperative to understand that only those GS-spaces of dimension four which are conformally Einstein (since the Bach tensor vanishes and thus they are $\mathcal{H}_t$-critical for $t = -1/3$) and with vanishing scalar curvature may be critical for all QC-functionals (see [13]).

3. Geometric computations on GS-spaces

Type A: Riemannian settings. Suppose that $(M = G/H, g)$ is a GS-space of dimension four of type **A**, where the invariant metric signature is $(0, 4)$ or $(4, 0)$. According to [9], the Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ posses a basis $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, e_1\}$, where $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ and $\{e_1\}$ are bases of $\mathfrak{m}$ and $\mathfrak{h}$ respectively, so that (we reverse the metric if necessary) the Lie communicators on $\mathfrak{g}$ and the scalar product on $\mathfrak{m}$ are specified as

$$\begin{array}{lll} [e_1, \epsilon_1] = -\epsilon_2, & [e_1, \epsilon_2] = \epsilon_1, & [e_1, \epsilon_3] = 2\epsilon_4, \\ [e_1, \epsilon_4] = -2\epsilon_3, & [\epsilon_1, \epsilon_3] = -\epsilon_1, & [\epsilon_1, \epsilon_4] = \epsilon_2, \\ [\epsilon_2, \epsilon_3] = \epsilon_2, & [\epsilon_2, \epsilon_4] = \epsilon_1, & [\epsilon_3, \epsilon_4] = -2e_1, \end{array}$$

and

$$g = (\omega^2)^2 + (\omega^1)^2 + \frac{2}{\zeta}\big((\omega^4)^2 + (\omega^3)^2\big), \tag{4}$$

where $\zeta > 0$, is an arbitrary real value. Using direct calculations, one can deduce the Levi-Civita connection according to the basis $\{\epsilon_1, \cdots, \epsilon_4\}$ by the following non-zero covariant derivatives

$$\begin{array}{lll} \nabla_{\epsilon_1}\epsilon_1 = \frac{\zeta}{2}\epsilon_3, & \nabla_{\epsilon_1}\epsilon_2 = -\frac{\zeta}{2}\epsilon_4, & \nabla_{\epsilon_1}\epsilon_3 = -\epsilon_1, \\ \nabla_{\epsilon_1}\epsilon_4 = \epsilon_2, & \nabla_{\epsilon_2}\epsilon_1 = -\frac{\zeta}{2}\epsilon_4, & \nabla_{\epsilon_2}\epsilon_2 = -\frac{\zeta}{2}\epsilon_3, \\ \nabla_{\epsilon_2}\epsilon_3 = \epsilon_2, & \nabla_{\epsilon_2}\epsilon_4 = \epsilon_1. & \end{array} \tag{5}$$

Then, we determine by the following non-vanishing tensor values the curvature tensor

$$\begin{array}{lll} R(\epsilon_1,\epsilon_2)\epsilon_1=-\zeta\epsilon_2, & R(\epsilon_1,\epsilon_2)\epsilon_2=\zeta\epsilon_1, & R(\epsilon_1,\epsilon_2)\epsilon_3=-\zeta\epsilon_4,\\ R(\epsilon_1,\epsilon_2)\epsilon_4=\zeta\epsilon_3, & R(\epsilon_1,\epsilon_3)\epsilon_1=\frac{\zeta}{2}\epsilon_3, & R(\epsilon_1,\epsilon_3)\epsilon_2=-\frac{\zeta}{2}\epsilon_4,\\ R(\epsilon_1,\epsilon_3)\epsilon_3=-\epsilon_1, & R(\epsilon_1,\epsilon_3)\epsilon_4=\epsilon_2, & R(\epsilon_1,\epsilon_4)\epsilon_1=\frac{\zeta}{2}\epsilon_4,\\ R(\epsilon_1,\epsilon_4)\epsilon_2=\frac{\zeta}{2}\epsilon_3, & R(\epsilon_1,\epsilon_4)\epsilon_3=-\epsilon_2, & R(\epsilon_1,\epsilon_4)\epsilon_4=-\epsilon_1,\\ R(\epsilon_2,\epsilon_3)\epsilon_1=\frac{\zeta}{2}\epsilon_4, & R(\epsilon_2,\epsilon_3)\epsilon_2=\frac{\zeta}{2}\epsilon_3, & R(\epsilon_2,\epsilon_3)\epsilon_3=-\epsilon_2,\\ R(\epsilon_2,\epsilon_3)\epsilon_4=-\epsilon_1, & R(\epsilon_2,\epsilon_4)\epsilon_1=-\frac{\zeta}{2}\epsilon_3, & R(\epsilon_2,\epsilon_4)\epsilon_2=\frac{\zeta}{2}\epsilon_4,\\ R(\epsilon_2,\epsilon_4)\epsilon_3=\epsilon_1, & R(\epsilon_2,\epsilon_4)\epsilon_4=-\epsilon_2, & R(\epsilon_3,\epsilon_4)\epsilon_1=-2\epsilon_2,\\ R(\epsilon_3,\epsilon_4)\epsilon_2=2\epsilon_1, & R(\epsilon_3,\epsilon_4)\epsilon_3=4\epsilon_4, & R(\epsilon_3,\epsilon_4)\epsilon_4=-4\epsilon_3, \end{array} \tag{6}$$

and the Ricci tensor with respect to the basis $\{\omega^1,\cdots,\omega^4\}$ is calculated immediately as

$$\varrho=-6\big((\omega^3)^2+(\omega^4)^2\big), \tag{7}$$

which indicates that (M,g) is never Einstein. The scalar curvature is then $\tau=-6\zeta$.

Type A: Pseudo-Riemannian settings. Suppose that $(M=G/H,g)$ is a GS-space of dimension four of type **A**, such that the signature of the invariant metric g is $(2,2)$. According to [9], the Lie algebra $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{h}$ accepts a basis $\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,e_1\}$, where $\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\}$ and $\{e_1\}$ are bases of $\mathfrak{m}$ and $\mathfrak{h}$ respectively, so that (if needed we reverse the metric [15]) the Lie communicators on $\mathfrak{g}$ and the scalar product on $\mathfrak{m}$ with respect to the dual basis $\{\omega^1,\cdots,\omega^4\}$ are completely determined by

$$\begin{array}{lll} [e_1,\epsilon_1]=-\epsilon_2, & [e_1,\epsilon_2]=\epsilon_1, & [e_1,\epsilon_3]=2\epsilon_4,\\ [e_1,\epsilon_4]=-2\epsilon_3, & [\epsilon_1,\epsilon_3]=-\eta\epsilon_1, & [\epsilon_1,\epsilon_4]=\eta\epsilon_2,\\ [\epsilon_2,\epsilon_3]=\eta\epsilon_2, & [\epsilon_2,\epsilon_4]=\eta\epsilon_1, & [\epsilon_3,\epsilon_4]=-2\eta^2e_1, \end{array}$$

where $\eta>0$ is a real constant, and

$$g=(\omega^2)^2+(\omega^1)^2-2\big((\omega^4)^2+(\omega^3)^2\big). \tag{8}$$

Applying the basis $\{\epsilon_1,\cdots,\epsilon_4\}$ for $\mathfrak{m}$ and by the well-known Koszul formula, the following non-zero covariant derivatives specify the Levi-Civita connection as follows

$$\begin{array}{lll} \nabla_{\epsilon_1}\epsilon_1=-\frac{\eta}{2}\epsilon_3, & \nabla_{\epsilon_1}\epsilon_2=\frac{\eta}{2}\epsilon_4, & \nabla_{\epsilon_1}\epsilon_3=-\eta\epsilon_1,\\ \nabla_{\epsilon_1}\epsilon_4=\eta\epsilon_2, & \nabla_{\epsilon_2}\epsilon_2=\frac{\eta}{2}\epsilon_3, & \nabla_{\epsilon_2}\epsilon_3=\eta\epsilon_2,\\ \nabla_{\epsilon_2}\epsilon_4=\eta\epsilon_1. & & \end{array} \tag{9}$$

Applying the equation $R_{ij}:=R(v_i,v_j)=[\Lambda_{v_i},\Lambda_{v_j}]-\Lambda_{[v_i,v_j]}$, we can compute the non-vanishing terms of the curvature tensor as following:

$$\begin{array}{lll}
R(\epsilon_1,\epsilon_2)\epsilon_1=\eta^2\epsilon_2, & R(\epsilon_1,\epsilon_2)\epsilon_2=-\eta^2\epsilon_1, & R(\epsilon_1,\epsilon_2)\epsilon_3=\eta^2\epsilon_4,\\
R(\epsilon_1,\epsilon_2)\epsilon_4=-\eta^2\epsilon_3, & R(\epsilon_1,\epsilon_3)\epsilon_1=-\frac{\eta^2}{2}\epsilon_3, & R(\epsilon_1,\epsilon_3)\epsilon_2=\frac{\eta^2}{2}\epsilon_4,\\
R(\epsilon_1,\epsilon_3)\epsilon_3=-\eta^2\epsilon_1, & R(\epsilon_1,\epsilon_3)\epsilon_4=\eta^2\epsilon_2, & R(\epsilon_1,\epsilon_4)\epsilon_1=-\frac{\eta^2}{2}\epsilon_4,\\
R(\epsilon_1,\epsilon_4)\epsilon_2=-\frac{\eta^2}{2}\epsilon_3, & R(\epsilon_1,\epsilon_4)\epsilon_3=-\eta^2\epsilon_2, & R(\epsilon_1,\epsilon_4)\epsilon_4=-\eta^2\epsilon_1,\\
R(\epsilon_2,\epsilon_3)\epsilon_1=-\frac{\eta^2}{2}\epsilon_4, & R(\epsilon_2,\epsilon_3)\epsilon_2=-\frac{\eta^2}{2}\epsilon_3, & R(\epsilon_2,\epsilon_3)\epsilon_4=-\eta^2\epsilon_2,\\
R(\epsilon_2,\epsilon_4)\epsilon_1=\frac{\eta^2}{2}\epsilon_3, & R(\epsilon_2,\epsilon_3)\epsilon_2=-\frac{\eta^2}{2}\epsilon_4, & R(\epsilon_2,\epsilon_4)\epsilon_3=\eta^2\epsilon_1\\
R(\epsilon_2,\epsilon_4)v4=-\eta^2\epsilon_2, & R(\epsilon_3,\epsilon_4)\epsilon_1=-2\eta^2\epsilon_1, & R(\epsilon_3,\epsilon_4)\epsilon_2=2\eta^2\epsilon_1,\\
R(\epsilon_2,\epsilon_4)\epsilon_3=4\eta^2\epsilon_4, & R(\epsilon_3,\epsilon_4)\epsilon_4=-4\eta^2\epsilon_3. &
\end{array} \quad (10)$$

By the relation $\varrho(u,v)=\mathrm{Tr}\{w\to R(w,u)v\}$, we get the Ricci tensor as

$$\varrho=-6\eta^2\big((\omega^3)^2+(\omega^4)^2\big), \quad (11)$$

which shows that (M,g) can never be Einstein. The scalar curvature in then $\tau=6\eta^2$ by contracting the Ricci tensor indices.

It should be noted that the generalized symmetric spaces of type A (in both the Riemannian and neutral signature settings) come with an almost Kähler structure and an opposite Kähler structure. The existence of these structures is the characteristic of generalized symmetric spaces of this type [16, 17]. As a result, these spaces can be locally described as left-invariant metrics on certain Lie groups, which makes analyzing them much simpler [18].

Type B. Assume that (M,g) is a GS-space of dimension four of type **B**. In this type, $M=G/H$. Also, we have $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{h}$ and it is generated by $\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4,e_1\}$, where $\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\}$ and $\{e_1\}$ generate $\mathfrak{m}$ and $\mathfrak{h}$ respectively. The Lie communicators on $\mathfrak{g}$ and the metric product on $\mathfrak{m}$ are specified according to the subsequent relations

$$\begin{array}{lll}
[e_1,\epsilon_3]=-2\epsilon_2, & [e_1,\epsilon_4]=2\epsilon_1, & [\epsilon_1,\epsilon_3]=-\epsilon_1,\\
[\epsilon_1,\epsilon_4]=\varepsilon e_1+\epsilon_2, & [\epsilon_2,\epsilon_3]=-\varepsilon e_1+\epsilon_2, & [\epsilon_2,\epsilon_4]=\epsilon_1,
\end{array}$$

where $\varepsilon=\pm1$, and

$$g=-2\big(\omega^1\omega^3+\omega^2\omega^4\big)+2\lambda\big((\omega^3)^2+(\omega^4)^2\big), \quad (12)$$

such that λ is an undetermined real value (see [9]).

According to the basis $\{\epsilon_1,\cdots,\epsilon_4\}$, the following non-vanishing covariant derivatives are deduced

$$\begin{array}{lll}
\nabla_{\epsilon_3}\epsilon_1=\epsilon_1, & \nabla_{\epsilon_3}\epsilon_2=-\epsilon_2, & \nabla_{\epsilon_3}\epsilon_3=-2\lambda\epsilon_1-\epsilon_3,\\
\nabla_{\epsilon_3}\epsilon_4=2\lambda\epsilon_2+\epsilon_4, & \nabla_{\epsilon_4}\epsilon_1=-\epsilon_2, & \nabla_{\epsilon_4}\epsilon_2=-\epsilon_1,\\
\nabla_{\epsilon_4}\epsilon_3=2\lambda\epsilon_2+\epsilon_4, & \nabla_{\epsilon_4}\epsilon_4=2\lambda\epsilon_1+\epsilon_3. &
\end{array} \quad (13)$$

Now, we specify the curvature tensor by the subsequent non-vanishing tensor values

$$\begin{array}{lll}
R(\epsilon_1,\epsilon_4)\epsilon_3=2\varepsilon\epsilon_2, & R(\epsilon_1,\epsilon_4)\epsilon_4=-2\varepsilon\epsilon_1, & R(\epsilon_2,\epsilon_3)\epsilon_3=-2\varepsilon\epsilon_2,\\
R(\epsilon_2,\epsilon_3)\epsilon_4=2\varepsilon\epsilon_1, & R(\epsilon_3,\epsilon_4)\epsilon_1=2\epsilon_2, & R(\epsilon_3,\epsilon_4)\epsilon_2=-2\epsilon_1,\\
R(\epsilon_3,\epsilon_4)\epsilon_3=2\epsilon_4, & R(\epsilon_3,\epsilon_4)\epsilon_4=-2\epsilon_3, &
\end{array} \quad (14)$$

and immediately the Ricci tensor is

$$\varrho = -2(1+\varepsilon)\big((\omega^3)^2 + (\omega^4)^2\big), \tag{15}$$

which shows that (M, g) is never Einstein for $\varepsilon = 1$ and is trivially Einstein (Ricci-flat) for $\varepsilon = -1$. The scalar curvature vanishes identically.

Type D. Suppose that $(M = G/H, g)$ is a GS-space of type **D**. We apply the results in [9], so for $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ as the Lie algebra of G, a basis $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, e_1\}$ exists such that $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ and $\{e_1\}$ are bases of $\mathfrak{m}$ and $\mathfrak{h}$ respectively. In this case

$$\begin{array}{lll} [e_1, \epsilon_1] = -\epsilon_1, & [e_1, \epsilon_2] = \epsilon_2, & [e_1, \epsilon_3] = -2\epsilon_3, \\ [e_1, \epsilon_4] = 2\epsilon_4, & [\epsilon_1, \epsilon_4] = -\epsilon_2, & [\epsilon_2, \epsilon_3] = -\epsilon_1, \\ [\epsilon_3, \epsilon_4] = -e_1, & & \end{array}$$

and the invariant metric with regard to the basis $\{\omega^2, \cdots, \omega^4\}$ is

$$g = 2\omega^1\omega^2 + 2\lambda\omega^3\omega^4, \tag{16}$$

where λ is a real non-zero scalar. With regard to $\{\epsilon_1, \cdots, \epsilon_4\}$, non-zero covariant derivatives are

$$\nabla_{\epsilon_1}\epsilon_1 = \tfrac{1}{\lambda}\epsilon_3, \quad \nabla_{\epsilon_1}\epsilon_4 = -\epsilon_2, \quad \nabla_{\epsilon_2}\epsilon_2 = \tfrac{1}{\lambda}\epsilon_2, \quad \nabla_{\epsilon_2}\epsilon_3 = -\epsilon_1, \tag{17}$$

and we get the curvature tensor such that its non-zero components are

$$\begin{array}{lll} R(\epsilon_1, \epsilon_2)\epsilon_1 = \frac{1}{\lambda}\epsilon_1, & R(\epsilon_1, \epsilon_2)\epsilon_2 = -\frac{1}{\lambda}\epsilon_2, & R(\epsilon_1, \epsilon_2)\epsilon_3 = -\frac{1}{\lambda}\epsilon_3, \\ R(\epsilon_1, \epsilon_2)\epsilon_4 = \frac{1}{\lambda}\epsilon_4, & R(\epsilon_1, \epsilon_4)\epsilon_2 = \frac{1}{\lambda}\epsilon_4, & R(\epsilon_1, \epsilon_4)\epsilon_3 = -\epsilon_1, \\ R(\epsilon_2, \epsilon_3)\epsilon_1 = \frac{1}{\lambda}\epsilon_3, & R(\epsilon_2, \epsilon_3)\epsilon_4 = -\epsilon_2, & R(\epsilon_3, \epsilon_4)\epsilon_1 = -\epsilon_1, \\ R(\epsilon_3, \epsilon_4)\epsilon_2 = \epsilon_2, & R(\epsilon_3, \epsilon_4)\epsilon_3 = -2\epsilon_3, & R(\epsilon_3, \epsilon_4)\epsilon_4 = 2\epsilon_4. \end{array} \tag{18}$$

Then, the Ricci tensor will directly be computed as

$$\varrho = -6\omega^3\omega^4, \tag{19}$$

and it follows that (M, g) is never Einstein. The scalar curvature is then $\tau = -\frac{6}{\lambda}$.

We note here that GS-spaces of types **B** and **D** exhibit either conformally symmetric properties (meaning that Weyl tensor is parallel) or they possess almost para-Kähler and opposite para-Kähler structures naturally (refer to [19] for details). In the former case, the metric is a Walker metric with zero scalar curvature and self-dual properties.

Following arguments above, we have the following remark.

Remark 1. The only GS-space of dimension four which is Einstein is the case **B** with $\varepsilon = -1$.

4. Critical metrics for GS-spaces

In this section, we will determine the critical metrics for the functionals $\mathcal{S}$ and $\mathcal{H}_t$ that are inferred from GS-spaces. The results for the critical metrics with respect to the functional $\mathcal{S}$ are resumed in the following proposition.

Proposition 4.1. *If $(M = G/H, g)$ is a GS-space of dimension four belonging to one of the classes* **A-D** *up to an isometry, then the invariant metric g is critical for the functional $\mathcal{S}$ if and only if it belongs to the class* **B**.

Proof. Remind that metrics that are either Einstein or have vanishing scalar curvature are considered critical for the functional $\mathcal{S}$. Based on the computation from the previous section, none of the GS-spaces are Einstein, and the scalar curvature vanishes identically only in case **B**. □

As stated in Section 2, the only GS-spaces of dimension four that can be critical for all QC-functionals are the conformally Einstein spaces of type **B** as proven in [13]. In all other cases, the scalar curvature is non-zero, and therefore, the corresponding metrics are critical at most for a single quadratic curvature functional.

Now, we consider different classes of GS-spaces to analyze critical metrics for the functional $\mathcal{H}_t$ by case-by-case study.

Type A (Riemannian mode). In this case, according to the Equation (4), the invariant metric considering the dual basis $\{\omega^1, \cdots, \omega^4\}$ is

$$g = (\omega^1)^2 + (\omega^2)^2 + \frac{2}{\zeta}\big((\omega^3)^2 + (\omega^4)^2\big),$$

where $\zeta > 0$ is a real scalar. Following the Equation (7), the Ricci tensor is

$$\varrho = -6\big((\omega^3)^2 + (\omega^4)^2\big),$$

and then $\tau = -6\zeta$. In this case, $\|\varrho\|^2 = 18\zeta^2$ and the Laplacian of the Ricci tensor is calculated as:

$$\Delta\varrho = -6\zeta^2\big((\omega^1)^2 + (\omega^2)^2\big) + 12\zeta\big((\omega^3)^2 + (\omega^4)^2\big).$$

Then, the Ricci contraction of the curvature tensor is deduced as:

$$R[\varrho] = 3\zeta^2\big((\omega^1)^2 + (\omega^2)^2\big) + 12\zeta\big((\omega^3)^2 + (\omega^4)^2\big).$$

Finally, by a direct substitution in the Equation (3) we have

$$\mathcal{H}_t = -\left(9\zeta^2 + 18t\zeta^2\right)\big((\omega^1)^2 + (\omega^2)^2\big) + (18\zeta + 36t\zeta)\big((\omega^3)^2 + (\omega^4)^2\big). \tag{20}$$

Type A (pseudo-Riemannian mode). In this case according to the Equation (8), the invariant metric regarding the dual basis $\{\omega^1, \cdots, \omega^4\}$ is

$$g = (\omega^1)^2 + (\omega^2)^2 - 2\big((\omega^3)^2 + (\omega^4)^2\big).$$

Following Equation (11), the Ricci tensor is

$$\varrho = -6\eta^2\big((\omega^3)^2 + (\omega^4)^2\big),$$

and then $\tau = 6\eta^2$. Now, by direct calculations $\|\varrho\|^2 = 18\eta^4$ and the Laplacian of the Ricci tensor is deduced as:

$$\Delta\varrho = -6\eta^4\big((\omega^1)^2 + (\omega^2)^2 + 2(\omega^3)^2 + 2(\omega^4)^2\big),$$

and the Ricci contraction of the curvature tensor is

$$R[\varrho] = 3\eta^4\big((\omega^1)^2 + (\omega^2)^2 - 4(\omega^3)^2 - 4(\omega^4)\big).$$

Then, by proper substitution in Equation (3) we have

$$\mathcal{H}_t = -\,(9\eta^4 + 18t\eta^4)\big((\omega^1)^2 + (\omega^2)^2 + 2(\omega^3)^2 + 2(\omega^4)^2\big). \tag{21}$$

Type B. In this case, by using Equation (12), the invariant metric with respect to the basis $\{\omega^1, \cdots, \omega^4\}$ is

$$g = -2\big(\omega^1\omega^3 + \omega^2\omega^4\big) + 2\lambda\big((\omega^3)^2 + (\omega^4)^2\big),$$

where λ is a real constant. According to the Equation (15), the Ricci tensor becomes

$$\varrho = -4\big((\omega^3)^2 + (\omega^4)^2\big),$$

and $\tau = 0$. Now, by direct calculations $\|\varrho\|^2 = 0$ and the Laplacian of the Ricci tensor vanishes identically. Therefore $R[\varrho] = 0$ and so $\mathcal{H}_t = 0$ in this case.

Type D. In this case, by applying the Equation (16), the invariant metric considering the basis $\{\omega^1, \cdots, \omega^4\}$ is $g = 2\omega^1\omega^2 + 2\lambda\omega^3\omega^4$. Applying Equation (19), the Ricci tensor is

$$\varrho = -6\omega^3\omega^4, \quad \tau = -\frac{6}{\lambda}.$$

Now, by direct calculations $\|\varrho\|^2 = \frac{18}{\lambda^2}$ and the Laplacian of the Ricci tensor is deduced as

$$\Delta\varrho = -\frac{12}{\lambda^2}\big(\omega^1\omega^2 - \omega^3\omega^4\big).$$

Then, we calculate the tensor field $R[\varrho]$ as:

$$R[\varrho] = \frac{6}{\lambda^2}\big(\omega^1\omega^2 + 2\omega^3\omega^4\big),$$

and then

$$\mathcal{H}_t = -\frac{18(1+2t)}{\lambda^2}\big(\omega^1\omega^2 - \omega^3\omega^4\big). \tag{22}$$

To sum up, one can deliver the following result from the above arguments.

Theorem 4.2. *Let* $(M = G/H, g)$ *be a four dimension GS-space which belongs to one of the classes* **A-D** *up to an isometry. In this case, the invariant metric* g *is critical for the functional* $\mathcal{H}_t$ *if and only if one of the following cases occurs*

- g *belongs to the class* **B** *for any real value of* t.
- g *belongs to class* **A** *or* **D** *for* $t = -\frac{1}{2}$.

Proof. It is clear that while invariant metrics in the class **B** are critical for the functional $\mathcal{H}_t$ at any value of t, invariant metrics in classes **A** (both Riemannian and pseudo-Riemannian) and **D** are crucial for $\mathcal{H}_t$ when t satisfies $\mathcal{H}_t = 0$. Based on Equations (21), (20), and (22), it is evident that t equals $-\frac{1}{2}$ because η and ζ are not equal to zero. This ends the proof. □

5. Conclusions

We explored critical metrics for certain functionals based on QC-variables. Our study focused on GS-spaces, specifically, we showed that metrics in the class **B** are the only critical metrics for the function $\mathcal{S}$. Also, metrics of type **B** are always critical for the functional $\mathcal{H}_t$, regardless of the value of t. However, metrics in classes **A** (both in Riemannian and pseudo-Riemannian types) and **D**, are only critical for the functional $\mathcal{H}_t$ whenever $t = -\frac{1}{2}$.

We remind here that the Riemannian type **A** of four-dimensional GS-space are homothetic to the left-invariant metric on a semi-direct extension of the Heisenberg group determined by

$$[e_1, e_2] = e_3, \ [e_1, e_4] = \frac{1}{2}e_1, \ [e_2, e_4] = -e_2, \ [e_3, e_4] = -\frac{1}{2}e_3,$$

where $\{e_i\}_{i=1}^4$ is an orthonormal basis. It is also showed in [20] that these metrics are $\mathcal{H}_t$-critical for $t = -\frac{1}{2}$.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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