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# Randić Matrix and Randić Energy of Uniform Hypergraphs

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**Abstract**

The Randić matrix $R = [r_{ij}]$ of a graph $G = (V, E)$ was defined as $r_{ij} = \frac{1}{\sqrt{d_i d_j}}$ if vertices $v_i$ and $v_j$ are adjacent and $r_{ij} = 0$ otherwise, where $d_i$ is the degree of the vertex $v_i \in V$. In this paper, we define the Randić matrix of a uniform hypergraph and study some its spectral properties. We also define the Randić energy of a uniform hypergraph and determine some upper and lower bound for it.

Keywords: Randić matrix, Randić energy, Uniform hypergraph, Eigenvalue.

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## 1. Introduction

Graph theory has a wide application area in many sciences including chemistry. The spectral and topological indices of graphs have long been common in chemical graph theory. One of the most popular graph–spectrum–based quantity is the graph energy concept that helps in approximation of the total $\pi$-electron energy of alkanes (see [1] and the references therein). Let $G$ be a simple graph with $n$ vertices and $m$ edges and $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of its adjacency matrix,

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the energy of $G$ has been defined as:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|,$$

for the first time in 1978 [2]. Its mathematical examinations has led to the publication of many papers in this field (see [3–5]).

Then some researchers introduced other types of energy in graphs. The first candidate for this was the Laplacian energy, that is defined as follows:

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|,$$

where $\mu_1 \geq \cdots \geq \mu_n$ are the eigenvalues of Laplacian matrix of $G$ [6]. Generally, if $M$ is a square symmetric matrix corresponding to graph $G$, the associated energy of $G$ with matrix $M$ is defined as:

$$E_M(G) = \sum_{i=1}^{n} \left| \lambda_i - \frac{tr(M)}{n} \right|,$$

where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of $M$. Some other energies defined on $G$ are the signless Laolacian energy [7], normalized Laplacian energy [8] and Randić energy [9].

Nikiforov in 2007 extended the energy-consept to any matrix [10]. If $M$ is a $n \times m$ matrix then the positive square roots of the eigenvalues of $MM^T$ are called the singular values of $M$. In this way the energy of $M$, $E(M)$, was defined as the sum of its singular values. Note that, if $M$ is a square symmetric real matrix then its singular values, in turn, equal the absolute value of its eigenvalues and therefore the new and old definitions are equivalent.

Nikiforov's definition led to some new energies in the graphs that corresponded to non-square matrices. The first one was the incidence energy [11], followed by normalized incidence energy [12], Laplacian incidence energy [13], Randić incidence energy [14], Randić energy of digraphs [15], etc.

However, unlike the topic of energy of graphs, there is, so far, almost blank for this topic about hypergraphs, a notion that naturally generalizes that of graphs. Few articles have been published in this field. In 2020 and 2022 Cardoso et. al. studied the incidence and the singless Laplacian energies and the adjacency energy of uniform hypergraphs, respectively[16, 17]. They obtained some bounds for these energies as functions of maximum degree, Zagreb index and spectral radius. Then, in 2023, Yalçın introduced the Laplacian energy of uniform hypergraphs and derived bounds. The bounds depond on pair degree, maximum degree, and the first Zagreb index for the greatest Laplacian eigenvalue and this energy of uniform hypergraphs and uniform regular hypergraphs [18]. Recently Sharma et al. extended the concept of distance energy for hypergraphs and obtained some bounds for the distance energy in terms of the determinant of the distance matrix and the number of vertices of uniform hypergraphs [19].

In this paper, we introduce the Randić matrix and then the Randić energy of uniform hypergraphs that are the generalizations of those topics on graphs.

## 2. preliminaries

Here, we present some required concepts of uniform hypergraphs, see [20] for comprehensive references.

An undirected hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$, and edge set $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$, in which $e_s \subset \mathcal{V}$ for $1 \leq s \leq m$, is called a $k$-uniform hypergraph if $|e_s| = k$ for $1 \leq s \leq m$. The order of the hypergraph is $n$ and its size is $m$. Let $v_i, v_j$ be two vertices in $\mathcal{H}$. The degree of $v_i$, which is denoted by $d_i$, is defined by $|E_i|$, where $E_i = \{e \in \mathcal{E} \mid v_i \in e\}$, and the pair-degree of two vertices $v_i, v_j$, which is denoted by $d_{ij}$, is defined by $|E_{ij}|$, where $E_{ij} = \{e \in \mathcal{E} \mid v_i, v_j \in e\}$. The vertex $v_i$ is an isolated vertex if $d_i = 0$. Vertices $v_i, v_j$ are called adjacent and denoted by $v_i \sim v_j$ if there exists an edge that contains both of them. The neighborhood of the vertex $v_i$, denoted by $N(v_i)$, is the set of all adjacent vertices to the vertex $v_i$. A hypergraph is $d$-regular if all its vertices have the same degree $d$. Two different vertices $v_i$ and $v_j$ are connected to each other if there exists a sequence of edges $(e_{l_1}, \ldots, e_{l_p})$ such that $v_i \in e_{l_1}$, $v_j \in e_{l_p}$ and $e_{l_s} \cap e_{l_{s+1}} \neq \emptyset$ for all $s \in \{1, \ldots, p-1\}$. A hypergraph is called connected if every pair of distinct vertices in $\mathcal{H}$ is connected.

The tensor representation of the hypergraph was considered initially in [21]. Since the tensor is a multi-dimensional array, we can specify all the connections and their details in hypergraphs by using it and then it is a more complete representation for hypergraphs. Recently, due to the simplicity of working with matrices, hypergraph matrix representation has attracted the attention of researchers. The adjacency matrix of a $k$-uniform hypergraph, $\mathcal{A} = [a_{ij}]$, was defined in [22] as follows :

$$a_{ij} = \begin{cases} \frac{1}{k-1} d_{ij}, & v_i \sim v_j, \\ 0, & v_i \nsim v_j. \end{cases}$$

In the next section, we will present some information about Randić matrix.

## 3. Randić matrix

Throughout this article, we let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a $k$-uniform hypergraph of order $n$ and size $m$. The Randić matrix of $\mathcal{H}$, denoted by $\mathcal{R} = [r_{ij}]$, is defined to be a matrix with entries

$$r_{ij} = \begin{cases} \frac{a_{ij}}{\sqrt{d_i d_j}}, & v_i \sim v_j, \\ \\ 0, & v_i \nsim v_j. \end{cases}$$

By direct matrix multiplication, if $\mathcal{H}$ has no isolated vertices then $\mathcal{R} = \mathcal{D}^{-\frac{1}{2}}\mathcal{A}\mathcal{D}^{-\frac{1}{2}}$, where $\mathcal{D} = diag(d_1, d_2, \cdots, d_n)$ is the degree matrix of $\mathcal{H}$. This matrix can also be called weighted adjacency matrix in which the connection between the adjacent vertices $v_i, v_j$ is weighted by $\frac{d_{ij}}{(k-1)\sqrt{d_i d_j}}$.

The normalized Laplacian and normalized signless Laplacian matrices of $\mathcal{H}$, denoted by $\mathcal{L}$ and $\mathcal{Q}$, respectively, are defined as follows:

$$\mathcal{L}_{ij} = \begin{cases} 1, & if\ i = j \text{ and } d_i \neq 0, \\ \frac{-a_{ij}}{\sqrt{d_i d_j}}, & v_i \sim v_j, \\ 0, & otherwise. \end{cases}$$

and

$$\mathcal{Q}_{ij} = \begin{cases} 1, & if\ i = j \text{ and } d_i \neq 0, \\ \frac{a_{ij}}{\sqrt{d_i d_j}}, & v_i \sim v_j, \\ 0, & otherwise. \end{cases}$$

If $\mathcal{H}$ has no isolated vertices then $\mathcal{L} = I_n - \mathcal{R}$ and $\mathcal{Q} = I_n + \mathcal{R}$, where $I_n$ is the unit matrix of order $n$.

Let $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ be the eigenvalues of Randić matrix of $\mathcal{H}$. Some of the following theorems are motivated by some classical results for graphs.

**Lemma 3.1.** *With the previous assumptions, we have*

*1)* $\sum_{i=1}^{n} \rho_i = 0,$

*2)* $\sum_{i=1}^{n} \rho_i^2 = \frac{2}{(k-1)^2} \sum_{v_i \sim v_j} \frac{d_{ij}^2}{d_i d_j}.$

*Proof.* 1) $\sum_{i=1}^{n} \rho_i = tr(\mathcal{R}) = 0.$

2)

$$\sum_{i=1}^{n} \rho_i^2 = tr(\mathcal{R}^2) = \sum_{i=1}^{n} [\mathcal{R}^2]_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij} r_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}^2$$

$$= \sum_{i=1}^{n} \sum_{\substack{v_j \\ v_i \sim v_j}} \frac{a_{ij}^2}{d_i d_j} = \frac{1}{(k-1)^2} \sum_{i=1}^{n} \sum_{\substack{v_j \\ v_i \sim v_j}} \frac{d_{ij}^2}{d_i d_j} = \frac{2}{(k-1)^2} \sum_{v_i \sim v_j} \frac{d_{ij}^2}{d_i d_j}.$$

□

**Lemma 3.2.** *Let $\mathcal{H}$ be a $k$-uniform hypergraph of order $n$ and $\mathcal{L}$ be its normalized Laplacian matrix. If the eigenvalues of $\mathcal{L}$ are ordered as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, then $0 \leq \lambda_i \leq 2$ for $i = 2 \cdots, n$ and $\lambda_1 = 0$.*

*Proof.* See Lemma 1.7 in [23]. The proof is also valid for uniform hypergraphs with minor modifications. □

**Theorem 3.3.** *Suppose that the $n$-vertex uniform hypergraph $\mathcal{H}$ has no isolated vertices. If the eigenvalues of Randić matrix of $\mathcal{H}$ are ordered as $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$, then $-1 \leq \rho_i \leq 1$ for $i = 2 \cdots, n$ and $\rho_1 = 1$.*

*Proof.* Since $\mathcal{H}$ has no isolated vertices, then $\mathcal{L} = I_n - \mathcal{R}$. Thus if the eigenvalues of $\mathcal{L}$ are ordered $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, then $\rho_i = 1 - \lambda_i$ for $i = 1 \cdots, n$. Therefore, the result follows from Lemma 3.2. □

**Theorem 3.4.** *Let $\mathcal{A}$ and $\mathcal{R}$ be the adjacency and Randić matrices of hypergraph $\mathcal{H}$, respectively. Then $rank(\mathcal{A}) = rank(\mathcal{R})$.*

*Proof.* Let $\tilde{\mathcal{D}}$ be a diagonal matrix with diagonal entries $\tilde{d}_{ii} = min\{1, d_i\}$ for $i = 1, \cdots, n$. Then $\mathcal{R} = \tilde{\mathcal{D}}^{-\frac{1}{2}} \mathcal{A} \tilde{\mathcal{D}}^{-\frac{1}{2}}$, and since $\tilde{\mathcal{D}}^{-\frac{1}{2}}$ is nonsingular then the result is true. □

In the following, we present the relationship between the $\mathcal{A}$ spectrum and the $\mathcal{R}$ spectrum, in which we use the Sylvester's law. The Sylvester's law of interia is a theorem in matrix algebra about certain properties of the coefficient matrix of a real quadratic form that remain invariant under a chang of basis. Namely, if $A$ is a symmetric matrix that defines the quadratic form, and $S$ is any invertible matrix such that $D = SAS^T$ is diagonal, then the number of negative elements in any diagonalization $D$ is always the same, for all such $S$; and the same goes for the number of positive elements.

**Theorem 3.5.** *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a $k$-uniform hypergraph, and $\mathcal{A}$ and $\mathcal{R}$ be the adjacency and Randić matrices of $\mathcal{H}$, respectively. Then the number of positive eigenvalues of $\mathcal{A}$ is equal to the number of positive eigenvalues of $\mathcal{R}$. The same result is also valid for the number of negative eigenvalues.*

*Proof.* Since $\mathcal{R}$ is symmetric then there exist an invertible matrix $S_1$ and a diagonal matrix $\Sigma$ such that $\Sigma = S_1 \mathcal{R} S_1^T$. Thus by Sylvester's law of inertia the number of positive elements in the diagonal of $\Sigma$ is always the same, for all such $S_1$, and the same goes for the number of negative elements [24]. On the other hand, the number of positive (negative) elements in the diagonal of $\Sigma$ is equal to the number of positive (negative) eigenvalues of $\mathcal{R}$. Now we have:

$$\begin{aligned}\Sigma =& S_1 \tilde{\mathcal{D}}^{-\frac{1}{2}} \mathcal{A} \tilde{\mathcal{D}}^{-\frac{1}{2}} S_1^T \\ =& S_2 \mathcal{A} S_2^T,\end{aligned}$$

where $S_2 = S_1\tilde{\mathcal{D}}^{-\frac{1}{2}}$ is an invertible matrix. Thus the result follows by Sylvester's law of inertia. $\square$

**Theorem 3.6.** *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a $k$-uniform hypergraph, and $\mathcal{A}$ and $\mathcal{R}$ be the adjacency and Randić matrices of $\mathcal{H}$, respectively. If $\mathcal{H}$ has isolated vertices then $det(\mathcal{R}) = det(\mathcal{A}) = 0$, and if $\mathcal{H}$ has no isolated vertices then $det(\mathcal{R}) = \frac{1}{d_1\cdots d_n}det(\mathcal{A})$.*

*Proof.* If $\mathcal{H}$ has isolated vertices then obviously $det(\mathcal{R}) = det(\mathcal{A}) = 0$. Now suppose that $\mathcal{H}$ has no isolated vertices then $\mathcal{RD} = \mathcal{D}^{-\frac{1}{2}}\mathcal{A}\mathcal{D}^{\frac{1}{2}}$. Therefore $\mathcal{RD}$ and $\mathcal{A}$ are similar and then $Spec(\mathcal{RD}) = Spec(\mathcal{A})$, thus

$$det(\mathcal{A}) = det(\mathcal{RD}) = (d_1\cdots d_n)det(\mathcal{R}).$$

$\square$

**Theorem 3.7.** *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a connected $k$-uniform hype rgraph of order $n$ and $1 = \rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ be the eigenvalues of its Randić matrix. Then we have:*

$$\rho_2 - \rho_n \leq 2\sqrt{\Big(\sum_{v_i\sim v_j}\frac{a_{ij}^2}{d_id_j}\Big) - \frac{n}{2(n-1)}}.$$

*Proof.* By Lemma 3.1 we have:

$$\begin{aligned}\sum_{l=3}^{n-1}\Big((\rho_2-\rho_l)^2+(\rho_l-\rho_n)^2\Big)+(\rho_2-\rho_n)^2 &\leq \sum_{2\leq i\leq j\leq n}(\rho_i-\rho_j)^2\\ &= (n-1)\sum_{i=2}^{n}\rho_i^2-\Big(\sum_{i=2}^{n}\rho_i\Big)^2\\ &= 2(n-1)\sum_{v_i\sim v_j}\frac{a_{ij}^2}{d_id_j}-n.\end{aligned}$$

Hence,

$$\sum_{l=3}^{n-1}\Big((\rho_2-\rho_l)^2+(\rho_l-\rho_n)^2\Big)+(\rho_2-\rho_n)^2 \leq 2(n-1)\sum_{v_i\sim v_j}\frac{a_{ij}^2}{d_id_j}-n.$$

Now by Jensen's inequality for the convex function $\phi(x) = x^2$, we have:

$$\begin{aligned}\frac{\sum_{l=3}^{n-1}\Big((\rho_2-\rho_l)^2+(\rho_l-\rho_n)^2\Big)}{n-3} &= \frac{\sum_{l=3}^{n-1}\phi(\rho_2-\rho_l)}{n-3}+\frac{\sum_{l=3}^{n-1}\phi(\rho_l-\rho_n)}{n-3}\\ &\geq \phi\Big(\frac{\sum_{l=3}^{n-1}\rho_2-\rho_l}{n-3}\Big)+\phi\Big(\frac{\sum_{l=3}^{n-1}\rho_l-\rho_n}{n-3}\Big)\\ &\geq 2\phi\Big(\frac{\frac{\sum_{l=3}^{n-1}\rho_2-\rho_l}{n-3}+\frac{\sum_{l=3}^{n-1}\rho_l-\rho_n}{n-3}}{2}\Big) = \frac{(\rho_2-\rho_n)^2}{2}.\end{aligned}$$

Thus,

$$\sum_{l=3}^{n-1}\Big((\rho_2-\rho_l)^2+(\rho_l-\rho_n)^2\Big)\geq\frac{n-3}{2}(\rho_2-\rho_n)^2,$$

and

$$\begin{aligned}2(n-1)\sum_{v_i\sim v_j}\frac{a_{ij}^2}{d_id_j}-n&\geq\frac{n-3}{2}(\rho_2-\rho_n)^2+(\rho_2-\rho_n)^2\\&=\frac{n-1}{2}(\rho_2-\rho_n)^2\geq 0.\end{aligned}$$

Therefore,

$$\rho_2-\rho_n\leq 2\sqrt{\Big(\sum_{v_i\sim v_j}\frac{a_{ij}^2}{d_id_j}\Big)-\frac{n}{2(n-1)}}.$$

□

In the following we study the Randić matrix of two special classes of hypergraphs.

**Definition 3.8.** Let $\mathcal{K}_n^k=(\mathcal{V},\mathcal{E})$ be a $k$-uniform hypergraph with $n$ vertices. We call it complete hypergraph if $\mathcal{E}$ consists of all possible edges; in other words, every $k$ distinct vertices form an edge.

**Lemma 3.9.** *Let $\mathcal{K}_n^k=(\mathcal{V},\mathcal{E})$ be a complete $k$-uniform hypergraph with $n$ vertices and let $\mathcal{R}(\mathcal{K}_n^k)$ be the Randić matrix of $\mathcal{K}_n^k$. Then we have:*

$$[\mathcal{R}(\mathcal{K}_n^k)]_{ij}=\begin{cases}0, & if\ i=j,\\ \frac{1}{n-1}, & otherwise.\end{cases}$$

*and thus $Spec\big(\mathcal{R}(\mathcal{K}_n^k)\big)=\big\{1,\frac{-1}{n-1}^{(n-1)}\big\}$.*

*Proof.* Let $v_i,v_j$ be two arbitrary vertices in $\mathcal{K}_n^k$. It is easy to see that $d_i=\binom{n-1}{k-1}$ and $d_{ij}=\binom{n-2}{k-2}$. Then by straightforward calculations the result follows. □

Lemma 3.9 is interesting from this point of view that the Randić matrix of a complete $k$-uniform hypergraph $\mathcal{K}_n^k$ is the same as the Randić matrix of the complete graph $K_n$ and it does not depend on $k$.

**Definition 3.10.** Let $\mathcal{S}_m^k=(\mathcal{V},\mathcal{E})$ be a $k$-uniform hypergraph and $m$ be a positive integer. We call it a hyper-star of size $m$, if there exists a disjoint partition of the vertex set $\mathcal{V}$ as $\mathcal{V}=\mathcal{V}_1\cup\mathcal{V}_2\cup\cdots\cup\mathcal{V}_m$, such that $|V_1|=1$ and $|V_2|=\cdots=|V_m|=k-1$ and $\mathcal{E}=\{V_1\cup V_i\mid 2\leq i\leq m\}$. The vertex $v\in V_1$ is called the heart of $\mathcal{S}_m^k$.

It is clear that $n := |\mathcal{V}| = m(k-1)+1$. In the following we determine the spectrum of the Randić matrix of $\mathcal{S}_m^k$.

By straightforward calculations, we have the following lemma.

**Lemma 3.11.** *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k-uniform hypergraph of order n with Randić matrix $\mathcal{R}$. If $\mathbf{x} \in \mathbb{R}^n$ then*

$$(\mathcal{R}\mathbf{x})_i = \frac{1}{k-1} \sum_{e \in E_i} \sum_{j \in e \setminus \{v_i\}} \frac{x_j}{\sqrt{d_i d_j}}.$$

**Lemma 3.12.** *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k-uniform hypergraph of order n with Randić matrix $\mathcal{R}$. Suppose that $u, v \in \mathcal{V}$ such that $E_u = E_v$. If $(\lambda, \mathbf{x})$ is an eigenpair of $\mathcal{R}$ and $\lambda \neq \frac{-1}{k-1}$ then $x_u = x_v$.*

*Proof.* By Lemma 3.11 we have:

$$\lambda x_u = \frac{1}{k-1} \sum_{e \in E_u} \sum_{v_j \in e \setminus \{u\}} \frac{x_j}{\sqrt{d_u d_j}}.$$

Thus,

$$\begin{aligned}(\lambda + \frac{1}{k-1})x_u &= \frac{1}{k-1} \sum_{e \in E_u} \sum_{v_j \in e} \frac{x_j}{\sqrt{d_u d_j}} \\ =&\frac{1}{k-1} \sum_{e \in E_v} \sum_{v_j \in e} \frac{x_j}{\sqrt{d_v d_j}} = (\lambda + \frac{1}{k-1})x_v.\end{aligned}$$

Since $\lambda \neq \frac{-1}{k-1}$, the result follows. □

**Theorem 3.13.** *Let $\mathcal{S}_m^k = (\mathcal{V}, \mathcal{E})$ be a hyperstar and $\mathcal{R}$ be the Randić matrix of it, then we have:*

$$Spec(\mathcal{R}) = \left\{1, \lambda_0, \left(1 - \frac{1}{k-1}\right)^{n-1}, \left(\frac{-1}{k-1}\right)^{n(k-2)}\right\},$$

*where $\lambda_0 \neq 1$ is the root of $(k-1)\lambda^2 + (2-k)\lambda - 1 = 0$ .*

*Proof.* Let $v_1$ be the heart of $\mathcal{S}_m^k$, therefore $d_{v_1} = m$ and $d_v = 1$ for $v \in \mathcal{V}$ distinct from $v_1$. Suppose that $\mathcal{E} = \{e^j \mid e^j = \{v_1, v_1^j, v_2^j, \cdots, v_{k-1}^j\} \ \ \& \ \ j = 1, \cdots, m\}$. If $(\lambda, \mathbf{x})$ is an eigenpair of $\mathcal{R}$ then by Lemma 3.11, for $j = 1, \cdots, m$ and $i = 1, \cdots, k-1$ we have:

$$\lambda x_{v_i^j} = \frac{1}{k-1}(\frac{x_{v_1}}{\sqrt{m}} + x_{v_1^j} + \cdots + x_{v_{i-1}^j} + x_{v_{i+1}^j} + \cdots + x_{v_{k-1}^j}).$$

Thus,

$$\lambda x_{v_i^j} + \frac{1}{k-1} x_{v_i^j} = \frac{1}{k-1}\left(\frac{x_{v_1}}{\sqrt{m}} + x_{v_1^j} + \cdots + x_{v_{k-1}^j}\right).$$

Therefore,

$$\left(\lambda + \frac{1}{k-1}\right) x_{v_i^j} = \frac{1}{k-1}\left(\frac{x_{v_1}}{\sqrt{m}} + x_{v_1^j} + \cdots + x_{v_{k-1}^j}\right).$$

There are two cases:

I. If $\lambda = \frac{-1}{k-1}$, then there are linearly independent eigenvectors $\mathbf{X}^i$, for $i = 1, \cdots, (k-2)m$, associated with $\lambda = \frac{-1}{k-1}$, such that

$$\mathbf{X}^i = \begin{bmatrix} 0 \\ \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^m \end{bmatrix},$$

where for $j = 1, \cdots, m$,

$$\mathbf{x}^j = \begin{bmatrix} x_1^j \\ \vdots \\ x_{k-1}^j \end{bmatrix},$$

and $x_1^j = 1$ and $x_i^j = -1$ and $x_l^j = 0$ for $l \in e^j \setminus \{v_1^j, v_i^j\}$.

II. If $\lambda \neq \frac{-1}{k-1}$, then by Lemma 3.12, $x_{v_1^j} = \cdots = x_{v_{k-1}^j}$, for $j = 1, \cdots, m$. Thus we have:

$$x_{v_1} = (k-1)\sqrt{m}\left(\lambda + \frac{1}{k-1} - 1\right) x_{v_1^1},$$
$$\vdots$$
$$x_{v_1} = (k-1)\sqrt{m}\left(\lambda + \frac{1}{k-1} - 1\right) x_{v_1^m}$$
$$\lambda x_{v_1} = \frac{1}{(k-1)\sqrt{m}} \sum_{j=1}^{m} (k-1) x_{v_1^j}.$$

Now there are two cases:

IIA. If $\lambda \neq 1 - \frac{1}{k-1}$, then $x_{v_1^1} = \cdots = x_{v_1^m}$, thus:

$$x_{v_1} = (k-1)\sqrt{m}\left(\lambda + \frac{1}{k-1} - 1\right) x_{v_1^1},$$
$$\lambda x_{v_1} = \sqrt{m} x_{v_1^1}.$$

So,

$$\lambda(k-1)\sqrt{m}(\lambda + \frac{1}{k-1} - 1)x_{v_1^1} = \sqrt{m}x_{v_1^1},$$

and since $x_{v_1^1} \neq 0$

$$(k-1)\lambda^2 + (2-k)\lambda - 1 = 0.$$

It is clear that the roots of above equation are $\lambda = 1$ and $\lambda_0 \neq 1$.

IIB. If $\lambda = 1 - \frac{1}{k-1}$,then there are linearly independent eigenvectors $\mathbf{X}^i$, for $i = 1, \cdots, m-1$, associated with $\lambda = 1 - \frac{1}{k-1}$, such that

$$\mathbf{X}^i = \begin{bmatrix} 0 \\ \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^m \end{bmatrix},$$

where for $j = 1 \cdots, m$,

$$\mathbf{x}^j = \begin{bmatrix} x_1^j \\ \vdots \\ x_{k-1}^j \end{bmatrix},$$

and $\mathbf{x}^1 = \mathbf{1}$ and $\mathbf{x}^i = -\mathbf{1}$ and $\mathbf{x}^j = \mathbf{0}$ for $j \neq 1, i$. This completes the proof. □

# 4. Randić energy

Let $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ be the eigenvalues of $\mathcal{R}$. Randić energy of the hypergraph $\mathcal{H}$ is defined as

$$RE(\mathcal{H}) = \sum_{i=1}^{n} |\rho_i|. \tag{1}$$

**Lemma 4.1.** *Let $\rho_1 \geq \cdots \geq \rho_t$ be the nonnegative eigenvalues of $\mathcal{R}$, then $RE(\mathcal{H}) = 2\sum_{i=1}^{t} \rho_i$.*

*Proof.* We have $\sum_{i=1}^{n} \rho_i = 0$, therefore $\sum_{i=1}^{t} \rho_i = -\sum_{i=t+1}^{n} \rho_i$ and thus $RE(\mathcal{H}) = 2\sum_{i=1}^{t} \rho_i$. □

In [25] the clique multigraph, $C(\mathcal{H}) = (\mathcal{V}, E)$, was associated with a $k$-uniform hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$. The clique multigraph $C(\mathcal{H})$, is obtained by transforming the vertices of $\mathcal{H}$ in its vertices. The number of edges between two vertices of this multigraph is equal the number of hyperedges containing them in $\mathcal{H}$. For more details see [17, 25]. It is easy to see that, $\mathcal{A}(\mathcal{H}) = \frac{1}{k-1}A(C(\mathcal{H}))$ and $d_i = \frac{d_i^{C(\mathcal{H})}}{k-1}$, where $d_i^{C(\mathcal{H})}$ is the degree of vertex $v_i \in \mathcal{V}$ in the multigraph $C(\mathcal{H})$. We also have:

$$\sum_{v_i \sim v_j} \frac{a_{ij}^2}{d_i d_j} = \sum_{v_i \sim v_j} \frac{\frac{d_{ij}^2}{(k-1)^2}}{\frac{d_i^{C(\mathcal{H})} d_j^{C(\mathcal{H})}}{(k-1)^2}} = \sum_{v_i \sim v_j} \frac{d_{ij}^2}{d_i^{C(\mathcal{H})} d_j^{C(\mathcal{H})}} = \sum_{v_i \sim v_j} \frac{(a_{ij}^{C(\mathcal{H})})^2}{d_i^{C(\mathcal{H})} d_j^{C(\mathcal{H})}}, \tag{2}$$

We can refer to $\sum_{v_i \sim v_j} \frac{(a_{ij}^{C(\mathcal{H})})^2}{d_i^{C(\mathcal{H})} d_j^{C(\mathcal{H})}}$ as a special case of general Randić index of the $C(\mathcal{H})$ for $\alpha = -1$, in other words $R_{-1}(C(\mathcal{H})) := \sum_{v_i \sim v_j} \frac{(a_{ij}^{C(\mathcal{H})})^2}{d_i^{C(\mathcal{H})} d_j^{C(\mathcal{H})}}$. Randić index was first defined by Randić in 1975 [26]. Then in 1998, Bollobaś and Erdös introduced the general Randić index for graph $G = (V, E)$ as:

$$R_\alpha(G) = \sum_{u,v \in V} (d_u d_v)^\alpha, \tag{3}$$

where $\alpha$ is an arbitrary real number[27]. In 2023 the general Randić was defined for uniform hypergraph [28].

In the following first we obtain an upper bound for $R_{-1}$ of a multigraph $G$ (similar to Theorem(3.2) in [29]) and then obtain an upper bound for $R_{-1}(C(\mathcal{H}))$, where $C(\mathcal{H})$ is the clique multigraph associated with the hypergraph $\mathcal{H}$.

**Theorem 4.2.** *Let $G = (V, E)$ be a multigraph with a weight-function $w$ that is defined on each edge $v_i v_j \in E$ as $w(v_i v_j) = \frac{a_{ij}^2}{d_i d_j}$ and let $uv$ be an edge with minimum weight in $G$ such that $d_u, d_v > a_{uv}$. Then we have*

$$R_{-1}(G - uv) > R_{-1}(G).$$

*Proof.* Let $W_u = \sum_{\substack{v_i \sim u \\ v_i \neq v}} w(uv_i)$ and $W_v = \sum_{\substack{v_i \sim v \\ v_i \neq u}} w(uv_i)$. It is easy to see that

$$W_u \geq (d_u - a_{uv}) \frac{a_{uv}^2}{d_u d_v},$$
$$W_v \geq (d_v - a_{uv}) \frac{a_{uv}^2}{d_u d_v}.$$

Now we have:

$$\begin{aligned}
&R_{-1}(G-uv)-R_{-1}(G)\\
&=\sum_{\substack{v_i\sim u\\ v_i\neq v}}\frac{a_{uv_i}^2}{(d_u-a_{uv})d_{v_i}}-\sum_{\substack{v_i\sim u\\ v_i\neq v}}\frac{a_{uv_i}^2}{d_ud_{v_i}}\\
&+\sum_{\substack{v_i\sim v\\ v_i\neq u}}\frac{a_{vv_i}^2}{(d_v-a_{uv})d_{v_i}}-\sum_{\substack{v_i\sim v\\ v_i\neq u}}\frac{a_{vv_i}^2}{d_vd_{v_i}}-\frac{a_{uv}^2}{d_ud_v}\\
&=\frac{d_u}{d_u-a_{uv}}W_u-W_u+\frac{d_v}{d_v-a_{uv}}W_v-W_v-\frac{a_{uv}^2}{d_ud_v}\\
&=W_u(\frac{d_u}{d_u-a_{uv}}-1)+W_v(\frac{d_v}{d_v-a_{uv}}-1)-\frac{a_{uv}^2}{d_ud_v}\\
&\geq(d_u-a_{uv})\frac{a_{uv}^2}{d_ud_v}(\frac{a_{uv}}{d_u-a_{uv}})+(d_v-a_{uv})\frac{a_{uv}^2}{d_ud_v}(\frac{a_{uv}}{d_v-a_{uv}})-\frac{a_{uv}^2}{d_ud_v}\\
&=2\frac{a_{uv}^3}{d_ud_v}-\frac{a_{uv}^2}{d_ud_v}=\frac{a_{uv}^2(2a_{uv}-1)}{d_ud_v}>0.
\end{aligned}$$

□

**Definition 4.3.** Let $S_\Delta^n=(V,E)$ be a multigraph, and $\Delta,n$ be positive integer numbers. We call it a multistar of size $\Delta$ and order $n$, if $V=\{v_1,v_2,\cdots,v_n\}$ and $e=v_1v_i$ , $\forall e\in E$ & $i=2,\cdots,n$ and $d_{v_1}=\Delta$. The vertex $v_1$ is called the heart.

It is clear that in $S_\Delta^n$, $d_{v_i}=a_{v_1v_i}$ for $i=2,\cdots,n$ and $\sum_{i=2}^n a_{v_1v_i}=\Delta$.

**Lemma 4.4.** *Let $S_\Delta^n=(V,E)$ be a multistar of size $\Delta$ and order $n$, then $R_{-1}(S_\Delta)=1$.*

*Proof.* By Definition 4.3 we have:

$$R_{-1}(S_\Delta^n)=\sum_{i=2}^n\frac{a_{v_1v_i}^2}{\Delta a_{v_1v_i}}=\frac{1}{\Delta}\sum_{i=2}^n a_{v_1v_i}=1.$$

□

**Lemma 4.5.** *Let $S_\Delta^n=(V,E)$ be a multistar of size $\Delta$ and order $n$, and let $\mathcal{R}$ be the Randić matrix of it. Then we have:*

$$Spec(\mathcal{R})=\left\{\pm\sqrt{\sum_{i=2}^n\frac{a_{i1}^3}{\Delta}},0^{(n-2)}\right\}.$$

*Proof.* It is determined by simple calculations that there are linearly independent eigenvectors $\mathbf{X}^i \in \mathbb{R}^n$, for $i = 3, \cdots, n$, associated with $\lambda = 0$, such that

$$\mathbf{X}^i = \begin{bmatrix} 0 \\ 1 \\ x_3^i \\ \vdots \\ x_n^i \end{bmatrix},$$

where $x_i^i = \frac{-a_{21}^{\frac{3}{2}}}{a_{i1}^{\frac{3}{2}}}$ and $x_l^i = 0$ for $l \neq 2, i$.

It is also easy to see that $(\sqrt{\sum_{i=2}^n \frac{a_{i1}^3}{\Delta}}, \mathbf{X})$ and $(\sqrt{\sum_{i=2}^n \frac{a_{i1}^3}{\Delta}}, \mathbf{Y})$ are two eigenpair of $\mathcal{R}$, where $x_1 = y_1 = 1$ and $x_i = -y_i = \frac{a_{i1}^{\frac{3}{2}}}{\sqrt{\Delta}} \times \frac{1}{\sqrt{\sum_{i=2}^n \frac{a_{i1}^3}{\Delta}}}$ for $i = 2, \cdots, n$. This completes the proof. □

**Theorem 4.6.** *Let $G = (V, E)$ be a multigraph with $n$ vertices. Then we have*

$$R_{-1}(G) \leq \lfloor \frac{n}{2} \rfloor,$$

*with equality if and only if $G$ is composed of $\frac{n}{2}$ disjoint edges if $n$ is even or is composed of a multistar of order 2 and $\frac{n-3}{2}$ disjoint edges if $n$ is odd.*

*Proof.* The result follows by Theorem 4.2 and Lemma 4.4 and similar to the proof of Theorem 3.2 in [29]. The main idea of the proof is that a multigraph with the maximum $R_{-1}$ must contain the maximum number of multistar. □

**Theorem 4.7.** *Let $\mathcal{H}$ be a $k$ uniform hypergraph with $n$ vertices, and $C(\mathcal{H})$ be the clique multigraph associated with the hypergraph $\mathcal{H}$. Then we have:*

$$\frac{n}{2(k-1)\binom{n-1}{k-1}} \leq R_{-1}(C(\mathcal{H})) \leq \frac{n}{2(k-1)},$$

*with equality on the left if and only if $\mathcal{H}$ is a complete $k$ uniform hypergraph. If $n = qk$, equality in equality on the right holds if and only if $\mathcal{H}$ is composed of $q$ disjoint hyperedges.*

*Proof.* First we prove the right inequality. According to the structure of $C(\mathcal{H})$, this multigraph is composed of a number of cliques (of $k$ vertices) that may connect to each other. Now by Theorem 4.2, $R_{-1}$ is increased by deletion of an edge $uv$ with minimum weight in $C(\mathcal{H})$ such that $d_u^{C(\mathcal{H})}, d_v^{C(\mathcal{H})} > a_{uv}^{C(\mathcal{H})}$. Therefore if $C(\mathcal{H})$ is composed of disjoint cliques then it has the maximum $R_{-1}$. Note that in such a multigraph it is not possible to remove the edge with the mentioned characteristics, because it is in contradiction with the definition of $C(\mathcal{H})$. Then we have:

$$R_{-1}(C(\mathcal{H})) \leq \binom{k}{2}\frac{1}{(k-1)^2} \times \frac{n}{k} = \frac{n}{2(k-1)}.$$

Now we prove the left inequality. Let $W_u$ be the sum of weights of the edges incident with vertex $u$ in $C(\mathcal{H})$, then we have:

$$W_u \geq \frac{d_u}{d_u(k-1)\binom{n-1}{k-1}} = \frac{1}{(k-1)\binom{n-1}{k-1}}.$$

Thus,

$$\begin{aligned} R_{-1}(C(\mathcal{H})) &= \sum_{v_i \sim v_j} \frac{(a_{ij}^{C(\mathcal{H})})^2}{d_i^{C(\mathcal{H})} d_j^{C(\mathcal{H})}} = \frac{1}{2}\sum_{u \in \mathcal{V}} W_u \\ &\geq \frac{1}{2}\sum_{u \in \mathcal{V}} \frac{1}{(k-1)\binom{n-1}{k-1}} = \frac{n}{2(k-1)\binom{n-1}{k-1}}. \end{aligned}$$

It is clear that equality holds if and only if $d_v^{C(\mathcal{H})} = (k-1)\binom{n-1}{k-1}$ for any $v \in \mathcal{V}$, or in other words $\mathcal{H}$ is a complete hypergraph. □

**Theorem 4.8.** *Let $\mathcal{H}$ be a $k$-uniform hypergraph without isolated vertices, then we have:*

$$\frac{n}{(k-1)\binom{k-1}{n-1}} \leq RE(\mathcal{H}) \leq n\sqrt{\frac{1}{k-1}}, \tag{4}$$

*with equality on the left if and only if $\mathcal{H} = \mathcal{K}_2^2$, and if $n = qk$, equality holds on the right if and only if $\mathcal{H}$ is composed of $q$ disjoint hyperedges.*

*Proof.* In the proof of right inequality, Using Cauchy-Schwarz inequality and by Lemma 3.1 we have:

$$\begin{aligned} RE(\mathcal{H}) &= \sum_{i=1}^{n} |\rho_i| \leq \sqrt{n\sum_{i=1}^{n}\rho_i^2} = \sqrt{2n\sum_{v_i \sim v_j}\frac{a_{ij}^2}{d_i d_j}} \\ &\leq \sqrt{2n\frac{n}{2(k-1)}} = n\sqrt{\frac{1}{k-1}}. \end{aligned} \tag{5}$$

The last inequality in (5) follows by Equation(2) and Theorem 4.7. Right equality holds in (4) if and only if $\rho_1 = \cdots = \rho_n$ and $\mathcal{H}$ is composed of disjoint hyperedges in case $n = qk$ (by Theorem 4.7). So the result is valid.

Now we prove the left inequlity. By Theorem 3.3 we have $-1 \leq \rho_i \leq 1$, for $i = 1, \cdots, n$, then $\rho_i^2 \leq |\rho_i|$, for $i = 1, \cdots, n$. Thus

$$RE(\mathcal{H}) = \sum_{i=1}^{n} |\rho_i| \geq \sum_{i=1}^{n} \rho_i^2 = 2 \sum_{v_i \sim v_j} \frac{a_{ij}^2}{d_i d_j}.$$

Now by Theorem 4.7, we have $RE(\mathcal{H}) \geq \frac{n}{(k-1)\binom{n-1}{k-1}}$. The left equality holds in (4) if and only if $|\rho_i| = 1$, for $i = 1, \cdots, n$ and $\mathcal{H}$ is a complete hypergraph, and by Lemma 3.9, it is equivalent to $\mathcal{H} = \mathcal{K}_2^2$. □

**Lemma 4.9.** *([30]). Let $M$ and $N$ be two square matrices, then we have:*

$$E(M + N) \leq E(M) + E(N).$$

**Theorem 4.10.** *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a $k$-uniform hypergraph of order $n$ and $\mathcal{A}$, $\mathcal{R}$ be the adjacency and Randić matrices of $\mathcal{H}$, respectively. Suppose that $C = \{v_1, \cdots, v_t\} \subseteq \mathcal{V}$, such that*

$$a_{ij} \neq 0 \ \rightarrow v_i \in C \text{ or } v_j \in C,$$

*and let $N_i' = N(v_i) \setminus \{v_1, \cdots, v_{i-1}\}$. Then we have :*

$$ER(\mathcal{H}) \leq 2 \min_{C} \sqrt{t \sum_{i=1}^{t} \sum_{v_j \in N_i'} \frac{a_{ij}^3}{\Delta_i}},$$

*where $\Delta_i = \sum_{v_j \in N_i'} a_{ij}$ for $i = 1, \cdots, t$.*

*Proof.* We define symmetric matrices $\mathcal{A}_1, \cdots, \mathcal{A}_t$ as follows:

$$a_{rs}^{(i)} = \begin{cases} a_{ij}, & \text{if } r = i \ \& \ s \in N_i', \\ 0, & \text{otherwise}, \end{cases}$$

where $i = 1, \cdots, t$ and $1 \leq r, s \leq n$. We show that $\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_t$.

Suppose that $a_{ij} \neq 0$ and without loss of generality suppose that $i < j$; by the assumption $v_i \in C$ or $v_j \in C$. There are two cases:

I . $v_i \in C$ and $v_j \notin C$, then it is clear that $a_{ij} = a_{ij}^{(i)}$ and $a_{ij}^{(k)} = 0$ for all $k \neq i$.

II . $v_i \in C$ and $v_j \in C$, then it is clear that $a_{ij} = a_{ij}^{(i)}$ and $a_{ij}^{(k)} = 0$ for all $k \neq i$. Note that $a_{ij}^{(j)} = 0$, since $v_i \in \{v_1, \cdots, v_{j-1}\}$ and then $v_i \notin N(v_j) \setminus \{v_1, \cdots, v_{j-1}\}$.

Then we have $\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_t$, and thus with the notations in the proof of Theorem 3.4, $\mathcal{R} = \mathcal{R}_1 + \cdots + \mathcal{R}_t$, in which $\mathcal{R}_l = \tilde{\mathcal{D}}^{-\frac{1}{2}}\mathcal{A}_l\tilde{\mathcal{D}}^{-\frac{1}{2}}$ for $l = 1, \cdots, t$.

Now by Lemma 4.9, $ER(\mathcal{H}) \leq E(\mathcal{R}_1) + \cdots + E(\mathcal{R}_t)$. On the other hand by Lemma 4.5, $E(\mathcal{R}_i) = 2\sqrt{\sum_{v_j \in N'_i} \frac{a_{ij}^3}{\Delta_i}}$ (note that $\mathcal{R}_i$ is the Randić matrix of the multistar of size $\Delta_i$ and the heart $v_i$ together with some isolated vertices). Therefore by using Cauchy-Schwarz inequality we have:

$$\begin{aligned} ER(\mathcal{H}) &\leq 2\sqrt{\sum_{v_j \in N'_1} \frac{a_{1j}^3}{\Delta_1}} + \cdots + 2\sqrt{\sum_{v_j \in N'_t} \frac{a_{tj}^3}{\Delta_t}} \\ &= 2\sum_{i=1}^{t} \sqrt{\sum_{v_j \in N'_i} \frac{a_{ij}^3}{\Delta_i}} \\ &\leq 2\sqrt{t\sum_{i=1}^{t} \sum_{v_j \in N'_i} \frac{a_{ij}^3}{\Delta_i}}, \end{aligned}$$

and this completes the proof. □

**Lemma 4.11.** *Let $a_i$ $(1 \leq i \leq n)$ be real numbers and $b_i$ $(1 \leq i \leq n)$ be non-negative real numbers such that $\sum_{i=1}^n b_i = 1$. Then we have:*

$$0 \leq \sum_{i=1}^{n} a_i^2 b_i - \big(\sum_{i=1}^{n} a_i b_i\big)^2 \leq \frac{1}{2}(M - m)\sum_{i=1}^{n} b_i|a_i - \sum_{i=1}^{n} a_i b_i|,$$

*where $m = min_{1\leq i\leq n} a_i$ and $m = max_{1\leq i\leq n} a_i$.*

*Proof.* See [31]. □

**Theorem 4.12.** *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a $k$-uniform hypergraph of order $n$ and $1 = \rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ be the eigenvalues of its Randić matrix. Then we have:*

$$ER(\mathcal{H}) \geq 2\sqrt{\sum_{v_i \sim v_j} \frac{a_{ij}^2}{d_i d_j} - \frac{n}{2(n-1)}}.$$

*Proof.* By using Lemma 4.11 and setting $a_i := \rho_i$ and $b_i := \frac{1}{n-1}$ for $i = 2, \cdots, n$, we have:

$$\frac{1}{n-1}\sum_{i=2}^{n} \rho_i^2 - \frac{1}{(n-1)^2}\big(\sum_{i=2}^{n} \rho_i\big)^2 \leq \frac{\rho_2 - \rho_n}{2(n-1)} \sum_{i=2}^{n} \left|\rho_i - \frac{1}{n-1}\sum_{i=2}^{n} \rho_i\right|.$$

Hence,

$$\frac{1}{n-1}\left(2\sum_{v_i\sim v_j}\frac{a_{ij}^2}{d_id_j}-1\right)-\frac{1}{(n-1)^2}\leq\frac{\rho_2-\rho_n}{2(n-1)}\sum_{i=2}^{n}\left|\rho_i+\frac{1}{n-1}\right|$$
$$\leq\frac{\rho_2-\rho_n}{2(n-1)}\sum_{i=2}^{n}\left(|\rho_i|+\frac{1}{n-1}\right)$$
$$=\frac{\rho_2-\rho_n}{2(n-1)}ER(\mathcal{H}).$$

Now by Theorem 3.7 and simple calculations we have:

$$2(n-1)\sum_{v_i\sim v_j}\frac{a_{ij}^2}{d_id_j}-n\leq(n-1)\sqrt{\sum_{v_i\sim v_j}\frac{a_{ij}^2}{d_id_j}-\frac{n}{2(n-1)}}ER(\mathcal{H}).$$

Therefore,

$$ER(\mathcal{H})\geq\frac{2(n-1)\sum_{v_i\sim v_j}\frac{a_{ij}^2}{d_id_j}-n}{(n-1)\sqrt{\sum_{v_i\sim v_j}\frac{a_{ij}^2}{d_id_j}-\frac{n}{2(n-1)}}}$$
$$=2\sqrt{\sum_{v_i\sim v_j}\frac{a_{ij}^2}{d_id_j}-\frac{n}{2(n-1)}}.$$

□

## 5. Conclusions

In recent years, the matrix representation of hypergraphs has very much attracted researcher's attention. In this paper, we define the Randić matrix of a uniform hypergraph as a generalization of the Randić matrix of a graph and study some its spectral properties. We also define the Randić energy of a uniform hypergraph and determine some upper and lower bounds for it.

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