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A Simple Proof for the Being Eulerian of the Power Graphs $P_i(D)$ for $3 \leq i \leq 6$

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Abstract

In this article, we employ a novel and unique method to analyze the Eulerian nature of the power graphs $P_i(D)$ for $3 \leqslant i \leqslant 6$. Then, we will mention some applications of Eulerian power graphs in computer networks.

Keywords: Directed Euler trail, Directed Euler tour, Eulerian digraph, Directed power graph, Eulerian power graph.

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1. Introduction

The literature on Eulerian power graphs commenced with the introduction of the concept of power graphs in [1]. Six different categories of power graphs associated with a directed graph were initially presented in the article, by using the set theory. Then it was shown that these new power graph definitions are pairwise distinct by a few examples. The relationship between the Eulerian property of the base graph and the power graphs $P_1(D)$ and $P_2(D)$ was also explored by the authors, (see Subsections 3.1 and 3.2). Also, the relationship between being Eulerian or not being Eulerian $P_1(D)$ with $P_2(D)$ is given by Theorem 4.1 in Section 4. While

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Theorem 3.28 in Subsection 3.3 of the same article provided a theorem regarding the Eulerian nature of power graphs $P_i(D)$ for $3 \leqslant i \leqslant 6$ with an example. The main objective of this paper is to investigate the Eulerianness of power graphs $P_i(D)$ for $3 \leqslant i \leqslant 6$ in a distinct and more direct manner, with a particular focus on Section 3. In this article, we first, shall briefly review the being Eulerian of the power graphs $P_1(D)$ and $P_2(D)$ method. In such a way that $P_1(D)$ is Eulerian if power graph $P_1(D)$ consists of only one connected component. Furthermore, it is established that there exists just one directed Euler tour in the power graph $P_1(D)$. Specifically, if graph D is Eulerian, then an Euler tour exists in the power graph $P_1(D)$. Conversely, if the graph D is not Eulerian, then it is evident that the power graph $P_1(D)$ is also not Eulerian [1].
Being Eulerian $P_2(D)$ under several important theorems have been investigated. In the case of an Eulerian graph D with $n \geq 3$ vertices, if D is a simple cycle graph without loops or with exactly n loops, then the power graph $P_2(D)$ is Eulerian. However, if the simple cycle graph D has l loops, where $1 \leq l \leq n-1$, then the power graph $P_2(D)$ is not Eulerian. (For a graph D with $n = 2$ vertices and any number of loops, $P_2(D)$ is Eulerian). Additionally, if the graph D is Eulerian and cyclic (but not a simple cycle graph) with $n \geq 4$ vertices, then the power graph $P_2(D)$ is not Eulerian, as proven in Theorem 3.20. Examples 3.21 and 3.22 demonstrate that if the adjacency matrix in the Eulerian graph D is symmetric, then the power graph $P_2(D)$ is Eulerian, which is proven as Theorem 3.23. It is also noted that if D is a non-Eulerian graph, then $P_2(D)$ is not Eulerian, except in one specific case. Another scenario where the power graph $P_2(D)$ is Eulerian is when $E = I_v$. Ultimately, it is concluded that if the power graph $P_1(D)$ is Eulerian, then the power graph $P_2(D)$ is Eulerian as well. If $P_2(D)$ is not Eulerian, then $P_1(D)$ is definitely non-Eulerian [1].
The focus of this article remains on directed graphs, as indicated in the article on directed power graphs. The concept of power graphs is reviewed, starting with a graph $D = (V, E)$ where $\phi \neq E \subseteq V \times V$. The set of edges E and the definitions of E_i's for $1 \leq i \leq 6$ are provided, facilitating the introduction of the six power graphs $P_i(D)$ for $1 \leq i \leq 6$, which are defined as $P_i(D) = (V_i, E_i)$. It should be noted that $V_i \neq \phi$ and $V_i \subseteq V$ for $1 \leqslant i \leqslant 6$. The definitions of E_i's are presented in the following definition.

Definition 1.1. Denoted in [1], states that when considering subsets A and B of V, the definition of each E_i is as follows:

$$\begin{aligned}
AE_1B \quad if \quad \forall a \in A, \forall b \in B \quad ; \quad aEb,\\
AE_2B \quad if \quad \exists a \in A, \exists b \in B \quad ; \quad aEb,\\
AE_3B \quad if \quad \exists b \in B, \forall a \in A \quad ; \quad aEb,\\
AE_4B \quad if \quad \forall a \in A, \exists b \in B \quad ; \quad aEb,\\
AE_5B \quad if \quad \exists a \in A, \forall b \in B \quad ; \quad aEb,\\
AE_6B \quad if \quad \forall b \in B, \exists a \in A \quad ; \quad aEb.
\end{aligned}$$

The subsequent connections are established among the elements of E_i:

$$E_1 \subseteq E_3 \subseteq E_4 \subseteq E_2,$$
$$E_1 \subseteq E_5 \subseteq E_6 \subseteq E_2.$$

Prior to delving into the article, it is recommended to familiarize oneself with the following definitions to enhance comprehension of the content.
A directed graph, a directed trail, a directed Euler trail, a directed Euler tour [2], an Eulerian digraph, outdegree $od(v)$ of v (or $d^+(v)$) and indegree $id(v)$ of v (or $d^-(v)$)[3], a connected digraph [4], Euler's theorem [3].

2. Historical comments

Authors have frequently defined the power graph of a group G in various articles, where the corresponding power graph of a group G is constructed by considering the elements of G as vertices. In this graph, two distinct vertices are considered adjacent if one of them is a power of the other. Several articles have explored the Eulerian properties of these power graphs (refer to [5–8]). In a survey conducted by Jemal et al. in [9], they compiled all the results on power graphs of groups and semigroups, providing definitions for both directed and undirected power graphs. In Section 5 of their article, they also examined the Eulerian conditions of these power graphs. Additionally, in 2016, Bhuniya and Sudip introduced the power graph of a finite group G with a normal subgroup H [10].

3. Eulerian analysis of $P_i(D)$'s for $3 \leqslant i \leqslant 6$

We begin this section with a brief example of the "Eulerian conditions" of power graphs ($P_i(D)$'s for $3 \leqslant i \leqslant 6$) and then we examine the corresponding theorems types of power graphs. Being Eulerian $P_i(D)$'s for $3 \leqslant i \leqslant 6$ differently from being Eulerian $P_i(D)$'s for $i = 1, 2$ are reviewed in directed power graphs article.

3.1 Eulerian analysis of $P_3(D)$ and $P_4(D)$

Let $D = (V, E)$ be a directed graph and assume that $A, B \subseteq V$, AE_3B and also AE_4B. To illustrate Euler's theorem for power graph $P_3(D)$, let us provide an example.

Example 3.1. Consider the set $V = \{a, b, c\}$ of graph D. Suppose that we have $E = V \times V \backslash (a, a)$. We see a part of the power graph $P_3(D)$ in Figure 1. Let us consider the vertex $\{a\}$. The definition of E_3, clearly shows that, there is an output edge from vertex $\{a\}$ to all of the vertices containing b or c or both of them. Also, there is an input edge from vertices $\{b\}$ and $\{c\}$ and $\{b, c\}$ to vertex $\{a\}$, which implies that $P_3(D)$ contains at least a directed Euler tour does not. Therefore, $P_3(D)$ is non-Eulerian. By checking the degree of input and output of vertex $\{a\}$, it can be seen that the degree of output is equal to 7 and the degree of input is 4. So it is not an Eulerian graph $P_3(D)$.

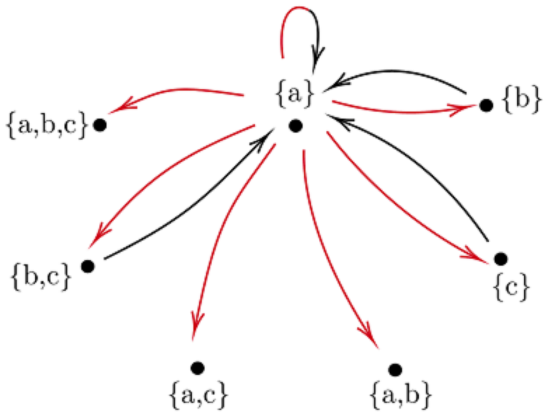

Figure 1: Display of input and output edges of vertex $\{a\}$ in non-Eulerian power graph $P_3(D)$.

If aEa, then according to E_3, all vertices containing an element $a \in V$ are connected to the vertex $\{a\}$, satisfying the condition for Euler's theorem in power graph $P_3(D)$. We note that Figure 2 is a part of power graph $P_3(D)$.

In the following theorem, we determine the conditions for graph D, that the power graph $P_3(D)$ will be Eulerian.

Theorem 3.2. *The power graph $P_3(D)$ is Eulerian if and only if the relation E contains all ordered pairs that are constructed with respect to the members of the set V, ($E = V \times V$).*

Proof. If $E = V \times V$, then $P_3(D) = (2^V \setminus \{\phi\}) \times (2^V \setminus \{\phi\})$ and obviously $P_3(D)$ is Eulerian.
We assume that $P_3(D)$ is an Eulerian power graph. To prove $E = V \times V$, it is sufficient to show that $d^-(v) = n$; $\forall v \in V$ or $(d^+(v) = n$; $\forall v \in V)$.
Consider $v \in V$ constant. $B \subseteq V$ is a minimal dependent set of vertices V if :
Note that if we have $\{v\}E_3B$, then there exists an element $b \in B$ such that $\{v\}E_3\{b\}$. So, minimal dependent set of $\{v\}$ has just one element. Now suppose

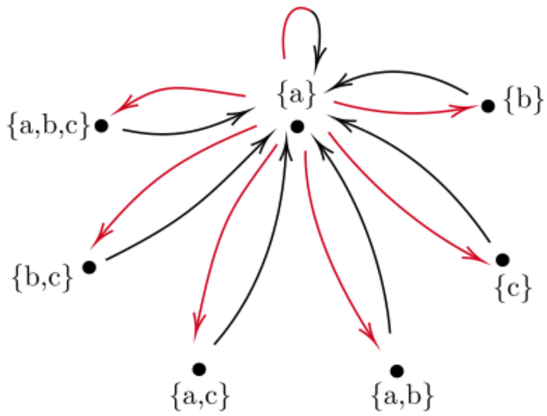

Figure 2: Display of input and output edges of vertex $\{a\}$ in Eulerian power graph $P_3(D)$.

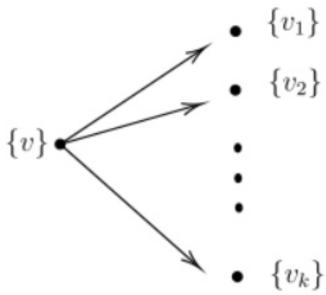

Figure 3: Minimal dependent set of $\{v\}$

minimal dependent set of $\{v\}$ are $\{v_1\}$,$\{v_2\}$,...,$\{v_k\}$. Note that $k = d^-(v)$.
By definition $P_3(D)$, for any $B \subseteq V$ that $\{v\}E_3B$, satisfies $B \cap \{v_1, v_2, ..., v_k\} \neq \phi$. For every $B \subseteq V$ that $\{v\}E_3B$, B can be decomposed into two disjoint subsets of B_1 and B_2 ($B = B_1 \cup B_2$), and we have $\phi \neq B_1 \subseteq \{v_1, ..., v_k\}$ and $B_2 \subseteq V \backslash \{v_1, ..., v_k\}$. Clearly, the number of these sets are $2^{n-k} \times (2^k - 1)$. Therefore, in a power graph $P_3(D)$ we have $d^-_{P_3(D)}(\{v\}) = 2^{n-k}(2^k - 1)$. Now we consider the set A such that $AE_3\{v\}$. Note that if $A_1E_3\{v\}$ and $A_2E_3\{v\}$, then $A_1 \cup A_2E_3\{v\}$. Also if $AE_3\{v\}$, then we have aEv for every $a \in A$. By puting $A = \cup_{CE_3\{v\}}C$, it results that A is only maximal element with a condition $AE_3\{v\}$. If the number of A is equal to l, then $l = d^+(v)$. On the other hand, for every C such that $CE_3\{v\}$, thus $C \subseteq A$. So, the number of nonempty subsets A is equal to with $d^+_{P_3(D)}(\{v\}) = 2^l - 1$. Since $P_3(D)$ is Eulerian, it proves that:

$$d^-(\{v\}) = d^+(\{v\}),$$
$$2^{n-k}(2^k - 1) = 2^l - 1.$$

This relationship is established when $2^{n-k} = 1$. So $n = k$ and then $d^-(v) = k = n$. As the same way $d^+(v) = k = n$ and the proof is complete. □

Now, let us refer to Example 3.27 in [1] to observe an Eulerian power graph $P_4(D)$.

Theorem 3.3. *The power graph $P_4(D)$ is Eulerian if and only if the relation E contains all ordered pairs that are constructed with respect to the members of the set V, ($E = V \times V$).*

Proof. The proof is done by a similar method of Theorem 3.2. □

Example 3.4. Cyclopropenone (C_3H_6) is a molecule of exceptional stability. In this molecule, intramolecular forces such as covalent bonds hold atoms together in molecules and atomic ions. We see the structure of this ring molecule in the basic graph in Figure 4, where in every carbon is labeled with C_1, C_2 and C_3. It is clear that the Eulerian condition holds for power graphs $P_3(D)$ and $P_4(D)$.

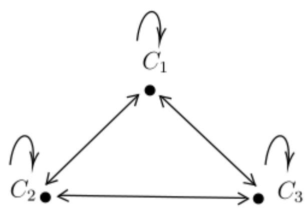

Figure 4: Basic graph of Cyclopropenone molecule.

Now we can see power graph $P_i(D)$ for $i = 3$ or $i = 4$ in Figure 5.

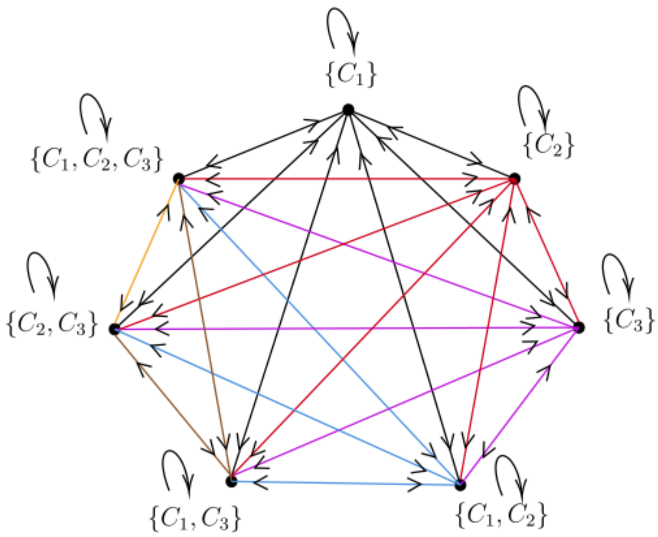

Figure 5: Eulerian power graph $P_i(D)$ for i = 3 or i = 4.

3.2 Eulerian analysis of $P_5(D)$ and $P_6(D)$

In this section, we present definitions that by using these definitions and previous discussions, we can easily obtain the results that $P_5(D)$ and $P_6(D)$ are Eulerian.

Definition 3.5. Defines the reverse or transpose of a directed graph D as another directed graph with the same set of vertices, where all edges are reversed compared to their orientation in D.

$$V(D^R) = V(D)(= V),$$
$$E(D^R) = \{(u, v) \in V \times V \quad : \quad (v, u) \in E(D)\}.$$

Example 3.6. Pay attention to the graphs below.

Lemma 3.7. *If D be a directed graph, then we have:*

1. $P_5(D) = P_3(D)^R$,
2. $P_6(D) = P_4(D)^R$,
3. $(P_1(D))^R = P_1(D)^R$,
4. $(P_2(D))^R = P_2(D)^R$.

Proof. It is obviously according to the definition of power graphs. □

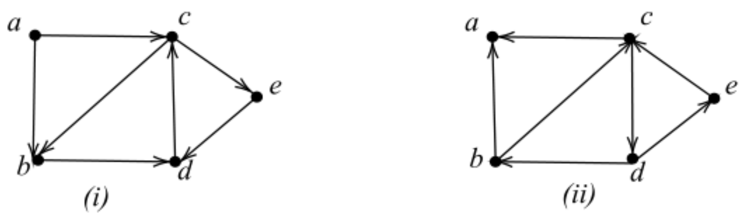

Figure 6: Figure (i) is the graph D and Figure (ii) is the transpose graph of the given graph.

Theorem 3.8. *If D is a directed graph, then D is Eulerian if and only if D^R is Eulerian.*

Proof. According to the definition of the reverse of the graph and Eulerianity of the graph, the result is obtained immediately. □

Theorem 3.9. *The power graph $P_5(D)$ is Eulerian if and only if the relation E contains all ordered pairs that are constructed with respect to the members of the set V, ($E = V \times V$).*

Proof. According to Lemma 3.7 (Part 1), Theorem 3.8, and Theorem 3.2, the proof is obvious. □

Theorem 3.10. *The power graph $P_6(D)$ is Eulerian if and only if the relation E contains all ordered pairs that are constructed with respect to the members of the set V, ($E = V \times V$).*

Proof. According to Lemma 3.7 (Part 2), Theorem 3.8, and Definition 3.5, the proof is obvious. □

Corollary 3.11. *States that $P_i(D)$ is Eulerian if and only if $P_j(D)$ is Eulerian (for $3 \leqslant i, j \leqslant 6$).*

Example 3.12. In this example, we consider a chemical graph, which represents a type of intermolecular force that can be seen in water, methanol, propanol, and cyclopropane molecules. There are positive and negative poles which are characterized by dipole-dipole interactions. We consider water, methanol, propanol, and cyclopropane molecules as the vertices of the basic graph and intermolecular forces as the edges of the graph. We can observe a power graph corresponding to the intermolecular forces for $i = 5$ or $i = 6$ (also for $i = 3, 4$) in Figure 8. It is worth noting that in Figure 8, due to the large number of edges, the Eulerian condition is provided for one of the vertices of $P_i(D)$ to facilitate better understanding of Eulerian power graphs. However, the results hold for other vertices as well.

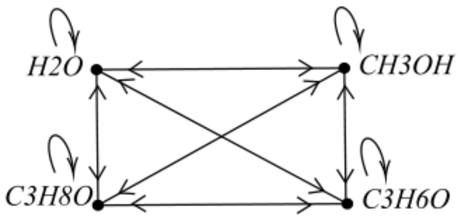

Figure 7: Intermolecular forces that arise as a result of the interaction between positively and negatively charged particles.

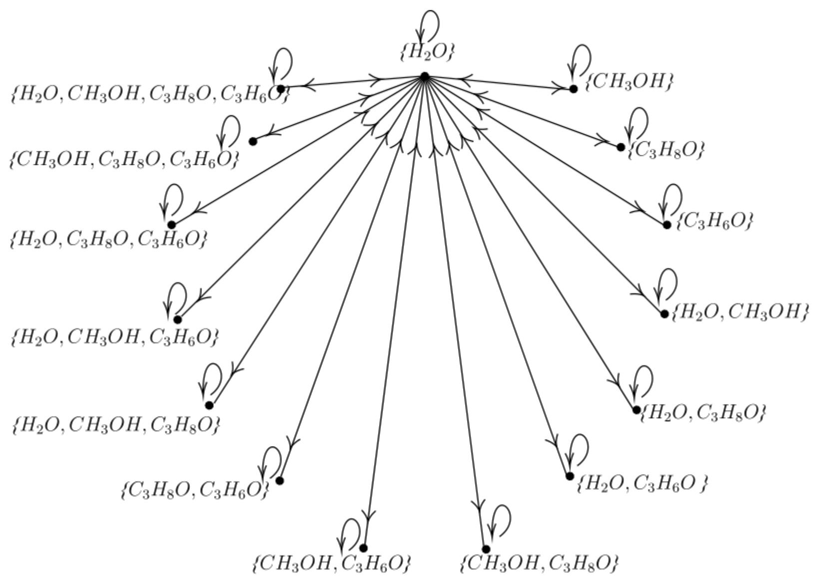

Figure 8: Drawing Eulerian power graph $P_i(D)$, $for\ i = 3, 4, 5, 6$ on the vertex of $\{a\}$ only.

Give another example to show that power graphs, $(P_i(D); for\ i = 3, 4, 5, 6)$ are Eulerian if $E = V \times V$. Based on the theorems presented in Section 3, we can derive conclusive results regarding the relationship between the base graph being Eulerian and the Eulerian properties of power graphs [1].

4. Applications of Eulerian power graphs

Here we examine some applications of power graphs in computer networks [11, 12]. Network topology refers to the arrangement of a network, consisting of nodes and connecting lines, facilitating communication between senders and receivers. Network topology defines the physical and logical structure of the network and like a map, it shows how network nodes are connected and how data flows between nodes. Definitely, with the help of such a map, you can see how information moves in the network. The various network topologies are Mesh Topology, Star Topology, Bus Topology, Ring Topology, Tree Topology, Hybrid Topology, and Line Topology. In computer networks, network topology is divided into two categories, Physical topology and Logical topology. Physical topology is like a map that shows the physical structure of connections between network nodes (wired or wireless), but it does not deal with how network traffic flows between nodes, how to address, how to transfer data from one device to another in the network and it only mentions the physical structure of the network. Logical topology shows how data is transferred between network nodes and it specifies how and through what paths data is sent from one node to another.
There are several physical topologies for connecting network equipment to each other. We see some common examples of these connections.
In a mesh topology, each device is interconnected with other devices through dedicated channels.
All-to-all communication is a method in computer communication where every sender transmits messages to all receivers within a group. This can be achieved through broadcasting or multicasting, in contrast to point-to-point communication where each sender communicates with a single receiver. In general, broadcast communication is a one-to-all communication. In fact, suppose we want to send an information packet from one point to several destinations so that all elements of the network receive this message and on the other hand, the information packet should pass through each path only once (to avoid traffic in the network). Broadcasting is a general communication method involving simultaneous message delivery to all recipients. It is a resource-intensive process that may require multiple messages and the engagement of numerous network devices. Broadcasting can be performed as all scatter, where each sender individually sends distinct messages to each receiver, or as all broadcast, where the same message is broadcasted to all recipients.

Example 4.1. We consider the internet system of a university. Suppose all parts of the university form a computer network. We also know that all these parts have

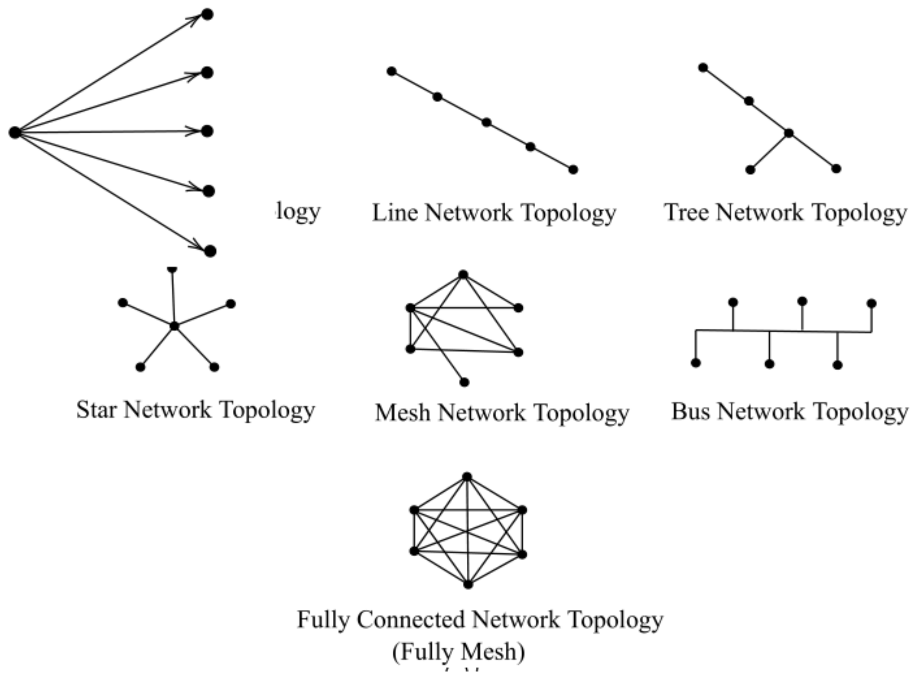

Figure 9: Network topologies.

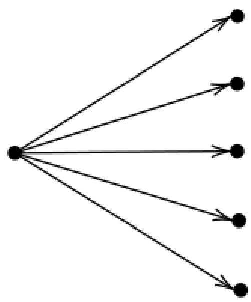

Figure 10: Broadcast (one to all).

access to each other through a private internet. We want to check these connections and how to transfer information in the university network in power graphs so that we consider each part of the university as one of the vertices of the power graph and the elements of the set vertices as the information and data that are transferred between the parts of the university. Let's assume that a circular from the university president's department is sent to all parts of the university so that all parts receive it (data broadcasting technique) and those parts send your answer to the university president. In the same way, each part of the university sends its information and messages to all parts of the university (Fully Mesh topology connection in Figure 9), we consider that each part also shares information internally with other employees.
For every Eulerian graph D where the edge set can be $E = V \times V$, there exist the Eulerian power graphs $P_i(D)$ for $1 \leq i \leq 6$.
For example, if we consider the Eulerian power graph $P_4(D)$, it is clear that for every data like a from the university president's part (as a set A) to Graduate (as a set B), there is a data like b to be sent from the Graduate part to the university president's part as a response. The main condition to avoid traffic in the network is to be an Eulerian power graph.

Ping, the command-line utility available on various operating systems with network connectivity, serves as a tool to test the reachability of a networked device. By sending a request to a specific device, the ping command expects a response from the target computer. The term "Ping" originates from sonar terminology, where a ping refers to an audible sound wave made to detect objects. If the sound wave encounters an object, it reflects back or echoes back to the source. Analyzing sound wave direction and time can provide information on the object's location and distance. Similarly, the ping command sends an echo request and awaits an echo reply from the target system. Essentially, Ping measures the round-trip time for messages sent from the originating host to a destination computer and echoed back to the source. It draws an analogy to sending pulses of sound and listening for echoes to detect objects underwater.
We consider power graph $P_i(D)$ for $3 \leqslant i \leqslant 6$ with any number of vertices. It is clear that from each vertex as set A (beginning vertex), several information packets can be sent to other vertices as set B (final vertex) and receive the reply. Also, due to the Eulerian of these power graphs, it is clear that the problem of slow speed or internet interruption in these graphs is zero and the quality of connection and availability of information is done in the best possible way.

5. Conclusion

This article aims to highlight the various applications of Eulerian power graphs, specifically targeting young readers. Additionally, power graphs have proven to be useful in numerous fascinating scenarios across various scientific disciplines. In bioinformatics, Euler graphs are used for DNA sequencing by hybridization

[13, 14]. In many articles, they investigated the role of Eulerian graphs in the process of determining the sequence of DNA by hybridization [15, 16]. In the fourteenth chapter of the book of cellular-molecular bioinformatics, the introduction of biological systems and types of biological networks and their analysis using graph theory has been discussed [17]. Our new approach is to investigate the functional aspects of the biological system in power graphs. In the continuation of our work, we are going to investigate the role of Eulerian power graphs in determining DNA sequence. The interested readers are invited to refer to us and available sources for more information. We also define and analyze a new category of graphs that can have important applications in social sciences, computer sciences and economic sciences .

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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