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Application of the Hybridized Discontinuous Galerkin Method for Solving One-Dimensional Coupled Burgers Equations

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Abstract

This paper is devoted to proposing hybridized discontinuous Galerkin (HDG) approximations for solving a system of coupled Burgers equations (CBE) in a closed interval. The noncomplete discretized HDG method is designed for a nonlinear weak form of one-dimensional $x-$variable such that numerical fluxes are defined properly, stabilization parameters are applied, and broken Sobolev approximation spaces are exploited in this scheme. Having necessary conditions on the stabilization parameters, it is proven in a theorem and corollary that the proposed method is stable with imposed homogeneous Dirichlet and/or periodic boundary conditions to CBE. The desired HDG method is stated by using the Crank-Nicolson method for time-variable discretization and the Newton-Raphson method for solving nonlinear systems. Numerical experiences show that the optimal rate of convergence is gained for approximate solutions and their first derivatives.

Keywords: Coupled Burgers equations, Hybridized discontinuous Galerkin method, Stability analysis, Numerical flux, Stabilization parameters.

2020 Mathematics Subject Classification: 65M60, 65M12.

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1. Definitions and notations

This paper deals with the system of nonlinear CBE which has many applications such as modeling acoustic waves, heat conduction phenomena, and many other fields in physics and engineering. For all x in a closed interval $\Omega \subset \mathbb{R}$, $t \in [0, T]$, and with fixed values α, β, η, and γ, we consider the following CBE

$$\begin{cases} \mathfrak{u}_t + (-\mathfrak{u}_x + \frac{\eta}{2}\mathfrak{u}^2 + \alpha\mathfrak{u}\mathfrak{v})_x = 0, \\ \mathfrak{v}_t + (-\mathfrak{v}_x + \frac{\gamma}{2}\mathfrak{v}^2 + \beta\mathfrak{u}\mathfrak{v})_x = 0, \end{cases} \tag{1}$$

where α and β in (1) depend on different system parameters such as Peclet number. In general, a Peclet number is a dimensionless number that relates convective and diffusive transport phenomena together in a continuum. Due to gravity and Brownian diffusivity, the Peclet number is the Stokes velocity of particles in the theory of CBE. The coupled equations (1), that is introduced by Esipov [1], can be considered as modelling under the effect of gravity for sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids [2].

Nonlinear partial differential equations such as Burgers equations [3], KdV equations [4], and RLW equations [5], are applicable for modeling of science and engineering and so analytic or numerical solutions of the corresponding equations are a high degree of importance and concern for the detection of physical phenomena. Because the analytical solution of the system (1) is not expressed by an explicit form or this solution needs high operational costs, hence it is requested to design numerical methods for computing numerical solutions. Some significant numerical methods for solving (1) can be listed as the conjugate filter method [6], Chebyshev spectral collocation approach [7], the Adomian-Pade technique [8], a cubic B-spline collocation method [9], a generalized differential quadrature method [3], a fully implicit scheme and Crank-Nicolson scheme [10, 11], weak Galerkin finite element method [12], a high-order implicit hybridizable discontinuous Galerkin method [13], and local discontinuous Galerkin method [14]. It should be pointed out that solving linear and/or nonlinear partial differential equations with non sufficient smooth given data and/or spatial domain with non-adaptive methods fail to achieve the best possible rate with respect to the generated mesh. Indeed, for producing a sequence of approximation solutions with the best possible rate, an alternative is an adaptive method, e.g., wavelet adaptive method [15, 16].

The aim of this work depends on the DG and LDG methods. The DG method was utilized and proposed in [17]. To solve nonlinear time-dependent problems, the DG method was developed in [18, 19] some time-independent linear hyperbolic equations. Based on the DG method, the local discontinuous Galerkin (LDG) method was proposed for solving a second-order time-dependent convection-diffusion equation [20]. The significant part of the LDG method is based on the transformation of a high-order equation into a first-order system of equations and then solving the new system by the DG method. With the elimination of all auxiliary variables locally, all flexibilities of the DG method will be available for the LDG method.

The HDG method was introduced by mixing the DG method and continuous Galerkin (CG) method for steady-state problems [21–24]. In the HDG method, the DG method preserves the optimal convergence rate for approximate solutions and their derivatives. Moreover, the CG method is exploited in the HDG method for the high performance of the method. Numerical traces, as extra global unknowns, are entered into the HDG method. One of the main advantages of the HDG method is that the global unknowns produce a smaller global system of equations in comparison with other DG methods. This can be achieved by eliminating the local unknowns which are fundamental factors in the superiority of the method. The advantage of the HDG method in the comparison with DG method is to reduce the degree of freedom. Numerical fluxes of the HDG method are not defined uniquely, but the definitions of the numerical fluxes have to guarantee the stability of the desired method. Recently, the HDG method has been combined with other methods so that the new methods have applications in many areas of scientific fields of engineering for instance in biomechanics[25].

The rest of the paper is organized as follows. A semi-discrete HDG method for coupled Burgers equation, that is designed by the use of numerical fluxes and stabilization parameters, is proposed in Section 2. The stability of the proposed semi-discrete HDG method is investigated in Section 3. In fact, we prove that the method is stable in the L^2 norm under certain conditions on the stabilization parameters. To solve semi-discrete HDG approximation of problem (1), a full discretization approach is designed in Section 3 by exploiting the Crank-Nicolson method for time discretization and Newton-Raphson as a nonlinear solver. Numerical experiments in Section 4 show that the order of accuracy and motion of soliton waves are derived by the proposed method. Finally, the conclusion is given in Section 5.

2. Construction of the semi-discrete HDG method

To design an appropriate HDG method for solving (1), the initial step is to reformulate the coupled Burgers equation into a first-order system of equations. By defining $\mathfrak{p} = \mathfrak{u}_x$ and $\mathfrak{q} = \mathfrak{v}_x$, the corresponding first-order system of (1) is as follows:

$$\begin{cases} \mathfrak{u}_t + (-\mathfrak{p} + \dfrac{\eta}{2}\mathfrak{u}^2 + \alpha\mathfrak{u}\mathfrak{v})_x = 0, \\ \mathfrak{p} - \mathfrak{u}_x = 0, \\ \mathfrak{v}_t + (-\mathfrak{q} + \dfrac{\gamma}{2}\mathfrak{v}^2 + \beta\mathfrak{u}\mathfrak{v})_x = 0, \\ \mathfrak{q} - \mathfrak{v}_x = 0. \end{cases} \tag{2}$$

Having a corresponding conditionally well-posed problem of system (2), this system has to be equipped with some suitable initial and boundary conditions.

To design a system of weak formulation of the system (2), it is needed to define some notations and relevant approximate spaces for a desired HDG method. With final time T and for all $0 < t \leq T$, we consider the first-order system of equations (2) over the spatial domain $\Omega = [x_L, x_R] \subset \mathbb{R}$ with the following partitioning

$$x_L = x_{-\frac{1}{2}} < x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N-\frac{1}{2}} = x_R, \tag{3}$$

where N specifies number of subintervals, $h = \frac{x_R - x_L}{N}$ is the spatial step size, and $x_{j+\frac{1}{2}} = x_{j-\frac{1}{2}} + h$, for $j = 0, 1, \ldots, N-1$. With setting $\mathcal{K}_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, $j = 0, \ldots, N-1$, we consider the following non-overlaping elements and the set of the boundary points of the partition (3)

$$\mathscr{K}_h := \{\mathcal{K}_j\}_{j=1}^{N-1}, \qquad \partial\mathscr{K}_h = \{\partial\mathcal{K}_j = \{x^+_{j-\frac{1}{2}}, x^-_{j+\frac{1}{2}}\}\}_{j=1}^{N-1}.$$

Suppose that $\mathcal{F}_h^0$ and $\mathcal{F}_h^\partial$ indicate the set of sub intervals and boundary nodes, respectively, and $\mathcal{F}_h = \mathcal{F}_h^\partial \cup \mathcal{F}_h^0$ is the set of all nodes. Mean and jump of one variable function v on a given node $x_{j+\frac{1}{2}} \in \mathcal{F}_h^0$, respectively, are defined as:

$$\{\{v_{j+\frac{1}{2}}\}\} = \frac{1}{2}(v^-_{j+\frac{1}{2}} + v^+_{j+\frac{1}{2}}), \qquad [[v_{j+\frac{1}{2}}]] = v^-_{j+\frac{1}{2}} - v^+_{j+\frac{1}{2}},$$

where $v^+_{j+\frac{1}{2}} = \lim_{\epsilon\to 0^+} v(x_{j+\frac{1}{2}} + \epsilon)$ and $v^-_{j+\frac{1}{2}} = \lim_{\epsilon\to 0^-} v(x_{j+\frac{1}{2}} + \epsilon)$. Note that the mean and jump of function v at boundary faces $x_{-\frac{1}{2}}, x_{N-\frac{1}{2}}$ are determined as:

$$[[v_{-\frac{1}{2}}]] = -v^+_{-\frac{1}{2}}, \quad [[v_{N-\frac{1}{2}}]] = v^-_{N-\frac{1}{2}}, \quad \{\{v_{-\frac{1}{2}}\}\} = v^+_{-\frac{1}{2}}, \quad \{\{v_{N-\frac{1}{2}}\}\} = v^-_{N-\frac{1}{2}}.$$

Let $\mathbf{n}^-_{j+\frac{1}{2}} = +1$ and $\mathbf{n}^+_{j-\frac{1}{2}} = -1$ be corresponding outward unit normal vectors with respect to $\mathcal{K}_j$. Mean and jump of function v are

$$\{\{v\}\} = \begin{cases} (v^+ + v^-)/2, & e \in \mathcal{F}_h^0, \\ v, & e \in \mathcal{F}_h^\partial, \end{cases} \qquad [[v\mathbf{n}]] = \begin{cases} v^+\mathbf{n}^+ + v^-\mathbf{n}^-, & e \in \mathcal{F}_h^0, \\ v\mathbf{n}, & e \in \mathcal{F}_h^\partial. \end{cases}$$

It is needed to define suitable approximate spaces that fit the method for obtaining weak formulation associated with the first-order system (2). Based on the structure of DG method, broken Sobolev spaces are relevant spaces for approximating the solution of system (2) by an HDG method [26]. Discontinuous finite element space $S_{h,k}$, as a subspace of a broken Sobolev space, is defined by:

$$S_{h,k} = \Big\{ w \in L^2(\Omega) : w\mid_{\mathcal{K}} \in \mathcal{P}^k(\mathcal{K}), \forall \mathcal{K} \in \mathscr{K}_h \Big\},$$

where $\mathcal{P}^k(\mathcal{K})$ is the set of polynomials of degree at most k on the element $\mathcal{K} \in \mathscr{K}_h$. The approximation space of the broken Sobolev space over $\mathcal{F}_h$ is given as:

$$M_{h,k} = \{\mu \in L^2(\mathcal{F}_h) : \mu\mid_e \in \mathcal{P}^k(e), \forall e \in \mathcal{F}_h\}.$$

We need to define appropriate broken trace spaces w.r.t. boundary conditions. Suppose that $\Gamma_{\mathfrak{u}}$, $\Gamma_{\mathfrak{p}}$, $\Gamma_{\mathfrak{v}}$, and $\Gamma_{\mathfrak{q}}$ include boundary faces which boundary data are specified on $\mathfrak{u}$, $\mathfrak{p}$, $\mathfrak{v}$, and $\mathfrak{q}$, respectively. Regarding to the boundary conditions, we consider

$$M_{h,k}(l,\Gamma) := \{\mu \in M_{h,k} : \mu(x) = \Pi l(x),\ x \in \Gamma\},$$

such that Π is L^2 projection of the skeleton space restricted to the boundary of interval domain Ω. The inner products of $S_{h,k}$ and $M_{h,k}$ are defined as:

$$(w_1,w_2)_{\mathscr{K}_h} = \sum_{\mathcal{K}\in\mathscr{K}_h} (w_1,w_2)_{\mathcal{K}}, \quad \langle \mu_1,\mu_2\rangle_{\partial\mathscr{K}_h} = \sum_{\mathcal{K}\in\mathscr{K}_h} \langle \mu_1,\mu_2\rangle_{\partial\mathcal{K}},$$

where

$$(w_1,w_2)_{\mathcal{K}} = \int_{\mathcal{K}} w_1(x)w_2(x)dx, \quad \langle \mu_1,\mu_2\rangle_{\partial\mathcal{K}_j} = \mu^-_{1,j+\frac{1}{2}}\mu^-_{2,j+\frac{1}{2}} + \mu^+_{1,j-\frac{1}{2}}\mu^+_{2,j-\frac{1}{2}}. \tag{4}$$

Weak form of system (2) can be formulated by multiplying relevant test functions into each equation of (2), integrating over each element $\mathcal{K} \in \mathscr{K}_h$, and using integration by parts formula. To find $u,p,v,q \in S_{h,k}$ such that for all $\mathcal{K} \in \mathscr{K}_h$ and for all test functions $w_1,w_2,w_3,w_4 \in S_{h,k}$, the following system holds

$$\left\{\begin{array}{rl} (u_t,w_1)_{\mathcal{K}} - (f,w_{1x})_{\mathcal{K}} + \langle \widehat{f}\mathbf{n}, w_1\rangle_{\partial\mathcal{K}} = & 0, \\ (p,w_2)_{\mathcal{K}} + (u,w_{2x})_{\mathcal{K}} - \langle \hat{u}\mathbf{n}, w_2\rangle_{\partial\mathcal{K}} = & 0, \\ (v_t,w_3)_{\mathcal{K}} - (g,w_{3x})_{\mathcal{K}} + \langle \widehat{g}\mathbf{n}, w_3\rangle_{\partial\mathcal{K}} = & 0, \\ (q,w_4)_{\mathcal{K}} + (v,w_{4x})_{\mathcal{K}} - \langle \hat{v}\mathbf{n}, w_4\rangle_{\partial\mathcal{K}} = & 0, \end{array}\right. \tag{5}$$

where

- $f = -p + \frac{\eta}{2}u^2 + \alpha uv,\ g = -q + \frac{\gamma}{2}v^2 + \beta uv.$
- With global unknown pair $(\xi,\zeta) \in M_{h,k}(0,\Gamma_{\mathfrak{u}}) \times M_{h,k}(0,\Gamma_{\mathfrak{v}})$, numerical traces $\hat{u} \in M_{h,k}(b_{\mathfrak{u}},\Gamma_{\mathfrak{u}})$ and $\hat{v} \in M_{h,k}(b_{\mathfrak{v}},\Gamma_{\mathfrak{v}})$ can be defined on the faces $\partial\mathcal{K}$ as:

$$\hat{u} = \left\{\begin{array}{ll} b_{\mathfrak{u}}, & \Gamma_{\mathfrak{u}}, \\ \xi, & \mathcal{F}_h \setminus \Gamma_{\mathfrak{u}}, \end{array}\right. \qquad \hat{v} = \left\{\begin{array}{ll} b_{\mathfrak{v}}, & \Gamma_{\mathfrak{v}}, \\ \zeta, & \mathcal{F}_h \setminus \Gamma_{\mathfrak{v}}, \end{array}\right. \tag{6}$$

  In (6), $b_{\mathfrak{u}}$ and $b_{\mathfrak{v}}$ are added to numerical teaces $\hat{u}$ and $\hat{v}$, respectively, on $\Gamma_{\mathfrak{u}}$ and $\Gamma_{\mathfrak{v}}$. Since the CBE (1) can be equipped by the periodic boundary conditions and this type of boundary conditions are studied in the stability analysis, therefore it is required to specify relevant numerical fluxes for periodic type of boundary conditions. With periodic boundary conditions $u|_{x_L} = u|_{x_R}$ and $v|_{x_L} = v|_{x_R}$, the numerical traces $\hat{u}$ and $\hat{v}$ are defined as:

$$\hat{u} = \left\{\begin{array}{ll} \xi|_{x_L} = \xi|_{x_R}, & \mathcal{F}^{\partial}_h, \\ \xi, & \mathcal{F}^0_h, \end{array}\right. \qquad \hat{v} = \left\{\begin{array}{ll} \zeta|_{x_L} = \zeta|_{x_R}, & \mathcal{F}^{\partial}_h, \\ \zeta, & \mathcal{F}^0_h, \end{array}\right.$$

- Numerical fluxes $\widehat{f}$ and $\widehat{g}$, with non-unique definitions, are defined as:

$$\begin{aligned}\widehat{f} &= -p + \frac{\eta}{2}\hat{u}^2 + \frac{\alpha}{2}(u\hat{v} + \hat{u}\hat{v}) + \tau(u - \hat{u})\mathbf{n},\\ \widehat{g} &= -q + \frac{\gamma}{2}\hat{v}^2 + \frac{\beta}{2}(\hat{u}v + \hat{u}\hat{v}) + \sigma(v - \hat{v})\mathbf{n},\end{aligned} \tag{7}$$

  where τ and σ are stabilization parameters so that they will be determined somewhat more in the stability theorem of the next section. Also in (7), $\mathbf{n}$ is the outward unit normal vector where its value is determined in the definition of the inner products over the determined face in formula (4).

In order to balance the number of unknowns and the equations, it is needed to enforce the conservation of the fluxes by considering the following extra global equations

$$\begin{cases} p\mathbf{n} = b_{\mathfrak{p}}, & e \in \Gamma_{\mathfrak{p}},\\ [[\widehat{f}\mathbf{n}]] = 0, & e \in \mathcal{F}_h^0.\end{cases} \qquad \begin{cases} q\mathbf{n} = b_{\mathfrak{q}}, & e \in \Gamma_{\mathfrak{q}},\\ [[\widehat{g}\mathbf{n}]] = 0, & e \in \mathcal{F}_h^0,\end{cases} \tag{8}$$

global unknowns are obtained and then local unknowns u, p, v, and q, are computed by solving weak formulation (5) in each node $\mathcal{K} \in \mathscr{K}_h$. For the case of periodic boundary conditions, global equations (8) are considered as follows:

$$\begin{cases} p|_{x_L}\mathbf{n}^+_{x_L} = p|_{x_R}\mathbf{n}^-_{x_R}, & e \in \mathcal{F}_h^\partial,\\ [[\widehat{f}\mathbf{n}]] = 0, & e \in \mathcal{F}_h^0,\end{cases} \qquad \begin{cases} q|_{x_L}\mathbf{n}^+_{x_L} = q|_{x_R}\mathbf{n}^-_{x_R}, & e \in \mathcal{F}_h^\partial,\\ [[\widehat{g}\mathbf{n}]] = 0, & e \in \mathcal{F}_h^0.\end{cases}$$

3. Stability analysis

Now we are aiming to review the numerical stability of the weak formulation (5) w.r.t. the time variable $t \in [0, T]$. For this purpose, it is needed to impose boundary conditions to the CBE and we select periodic or homogeneous Dirichlet boundary conditions. The analysis is started by multiplying the first equation of (1) to $\mathfrak{u}$ and then we have

$$\frac{1}{2}\partial_t \mathfrak{u}^2 - \mathfrak{u}_{xx}\mathfrak{u} + \frac{\eta}{3}(\mathfrak{u}^3)_x + \alpha(\mathfrak{u}\mathfrak{v})_x\mathfrak{u} = 0.$$

Applying integration by parts over $\Omega = (x_L, x_R)$ to the above equation leads to

$$\frac{d}{dt}\int_\Omega \mathfrak{u}^2\, dx + 2\int_\Omega \mathfrak{u}_x^2\, dx - \alpha\Psi(\mathfrak{u}, \mathfrak{v}, \Omega) = 0, \tag{9}$$

where

$$\Psi(\mathfrak{r}, \mathfrak{s}, \Lambda) = \int_\Lambda (\mathfrak{r}^2)_x \mathfrak{s}\, dx.$$

Similar to (9), it holds that

$$\frac{d}{dt}\int_\Omega \mathfrak{v}^2\, dx + 2\int_\Omega \mathfrak{v}_x^2\, dx - \beta\Psi(\mathfrak{v}, \mathfrak{u}, \Omega) = 0. \tag{10}$$

Integrating (9) and (10) over $[0, t]$, where $0 < t \leq T$, leads to the following inequalities, respectively,

$$\begin{aligned} \|\mathfrak{u}(t)\|_{\Omega}^2 - \alpha \int_0^t \Psi(\mathfrak{u}, \mathfrak{v}, \Omega)\ dt &\leq \|\mathfrak{u}(0)\|_{\Omega}^2, \\ \|\mathfrak{v}(t)\|_{\Omega}^2 - \beta \int_0^t \Psi(\mathfrak{v}, \mathfrak{u}, \Omega)\ dt &\leq \|\mathfrak{v}(0)\|_{\Omega}^2, \end{aligned} \tag{11}$$

where $\|\cdot\|_{\Omega}$ is the L^2 standard norm with respect to domain Ω. Thanks to (11), we figure out that CBE (1) is well-posed in the sense of energy method under the following conditions

$$\int_0^t \Psi(\mathfrak{u}, \mathfrak{v}, \Omega)\ dt < 0, \quad \int_0^t \Psi(\mathfrak{v}, \mathfrak{u}, \Omega)\ dt < 0, \ \text{for all } t \in (0, T].$$

We need to propose some splitting for stabilization parameters σ and τ and functions f and g, and corresponding numerical fluxes, namely, $\hat{f}$ and $\hat{g}$. Let functions f and g be decomposed, respectively, as:

$$f = -p + f_1 + f_2, \qquad g = -q + g_1 + g_2,$$

where $f_1 = \frac{\eta}{2}u^2$, $f_2 = \alpha uv$, $g_1 = \frac{\gamma}{2}v^2$, and $g_2 = \beta uv$. Then, we propose following decompositions for fluxes $\widehat{f}$ and $\widehat{g}$, defined in (7), as:

$$\widehat{f} = \widehat{-p} + \widehat{f_1} + \widehat{f_2}, \qquad \widehat{g} = \widehat{-q} + \widehat{g_1} + \widehat{g_2},$$

where

$$\begin{aligned} &\widehat{-p} = -p + \tau_0(u - \hat{u})\mathbf{n},\ \widehat{f_1} = \tfrac{\eta}{2}\hat{u}^2 + \tau_1(u - \hat{u})\mathbf{n},\ \widehat{f_2} = \tfrac{\alpha}{2}(u\hat{v} + \hat{u}\hat{v}) + \tau_2(u - \hat{u})\mathbf{n}, \\ &\widehat{-q} = -q + \sigma_0(v - \hat{v})\mathbf{n},\ \widehat{g_1} = \tfrac{\gamma}{2}\hat{v}^2 + \sigma_1(v - \hat{v})\mathbf{n},\ \widehat{g_2} = \tfrac{\beta}{2}(\hat{u}v + \hat{u}\hat{v}) + \sigma_2(v - \hat{v})\mathbf{n}. \end{aligned} \tag{12}$$

In (12), the stabilization parameters τ and σ are split as $\tau = \tau_0 + \tau_1 + \tau_2$ and $\sigma = \sigma_0 + \sigma_1 + \sigma_2$, respectively.

Theorem 3.1. *We consider formulation* (5) *with periodic boundary conditions. Under the following necessary conditions on stabilization parameters*

$$\tau_0 > 0,\ \sigma_0 > 0,\ \tau_1 > \tilde{\tau_1},\ \tau_2 > \tilde{\tau_2},\ \sigma_1 > \tilde{\sigma_1},\ \sigma_2 > \tilde{\sigma_2}, \tag{13}$$

it is proven that the method of weak formulation (5) *is stable in the sense of energy method, i.e.,*

$$\|u(t)\|_{\mathscr{K}_h}^2 - \alpha \int_0^t \Psi(u, v, \mathscr{K}_h)\ dt \leq \|u(0)\|_{\mathscr{K}_h}^2, \tag{14}$$

$$\|v(t)\|_{\mathscr{K}_h}^2 - \beta \int_0^t \Psi(v, u, \mathscr{K}_h)\ dt \leq \|v(0)\|_{\mathscr{K}_h}^2, \tag{15}$$

where

$$\begin{aligned} \tilde{\tau_1}(u,\hat{u}) &:= \tfrac{1}{(u-\hat{u})^2}\int_{\hat{u}}^{u}(f_1(s)-f_1(\hat{u}))\mathbf{n}ds, \qquad \tilde{\tau_2}(\hat{v}) = |\alpha\hat{v}|/2, \\ \tilde{\sigma_1}(v,\hat{v}) &:= \tfrac{1}{(v-\hat{v})^2}\int_{\hat{v}}^{v}(g_1(r)-g_1(\hat{v}))\mathbf{n}dr, \qquad \tilde{\sigma_2}(\hat{u}) = |\beta\hat{u}|/2. \end{aligned} \tag{16}$$

Proof. Setting $w_1 = u, w_2 = p, w_3 = v$, and $w_4 = q$ in (5) leads to

$$\begin{aligned} (u_t,u)_{\mathcal{K}_j} + (p - f_1 - f_2, u_x)_{\mathcal{K}_j} + \langle\widehat{-p}\mathbf{n},u\rangle_{\partial\mathcal{K}_j} + \langle\widehat{f_1}\mathbf{n},u\rangle_{\partial\mathcal{K}_j} + \langle\widehat{f_2}\mathbf{n},u\rangle_{\partial\mathcal{K}_j} &= 0, \\ (p,p)_{\mathcal{K}_j} + (u,p_x)_{\mathcal{K}_j} - \langle\hat{u}\mathbf{n},p\rangle_{\partial\mathcal{K}_j} &= 0, \\ (v_t,v)_{\mathcal{K}_j} + (q - g_1 - g_2, v_x)_{\mathcal{K}_j} + \langle\widehat{-q}\mathbf{n},v\rangle_{\partial\mathcal{K}_j} + \langle\widehat{g_1}\mathbf{n},v\rangle_{\partial\mathcal{K}_j} + \langle\widehat{g_2}\mathbf{n},v\rangle_{\partial\mathcal{K}_j} &= 0, \\ (q,q)_{\mathcal{K}_j} + (v,q_x)_{\mathcal{K}_j} - \langle\hat{v}\mathbf{n},q\rangle_{\partial\mathcal{K}_j} &= 0. \end{aligned} \tag{17}$$

Summing the first and second equation of (17), adding and subtracting $\langle\widehat{f_1},\hat{u}\rangle_{\partial\mathcal{K}_j}$ and $\langle\widehat{f_2},\hat{u}\rangle_{\partial\mathcal{K}_j}$, and finally by doing some simplifications and expanding boundary terms, we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|^2_{\mathcal{K}_j} - \frac{\alpha}{2}\Psi(u,v,\mathcal{K}_j) + \int_{\mathcal{K}_j} p^2 dx + E_{1,\mathcal{K}_j} + E_{2,\mathcal{K}_j} = 0, \tag{18}$$

where

$$\begin{aligned} E_{1,\mathcal{K}_j} = & \int_{\mathcal{K}_j}(up)_x dx + \langle(\widehat{-p})\mathbf{n},u\rangle_{\partial\mathcal{K}_j} - \langle\hat{u},p\mathbf{n}\rangle_{\partial\mathcal{K}_j} + \langle\widehat{f_1}\mathbf{n},\hat{u}\rangle_{\partial\mathcal{K}_j} + \langle\widehat{f_2}\mathbf{n},\hat{u}\rangle_{\partial\mathcal{K}_j} \\ = & \Big((\widehat{-p})^-_{j+\frac{1}{2}} + (p)^-_{j+\frac{1}{2}}\Big)u^-_{j+\frac{1}{2}} + \Big(-(p)^+_{j-\frac{1}{2}} - (\widehat{-p})^+_{j-\frac{1}{2}}\Big)u^+_{j-\frac{1}{2}} \\ & -\hat{u}_{j+\frac{1}{2}}(p)^-_{j+\frac{1}{2}} + \hat{u}_{j-\frac{1}{2}}(p)^+_{j-\frac{1}{2}} + \widehat{f_1}^-_{j+\frac{1}{2}}\hat{u}_{j+\frac{1}{2}} - \widehat{f_1}^+_{j-\frac{1}{2}}\hat{u}_{j-\frac{1}{2}} + \widehat{f_2}^-_{j+\frac{1}{2}}\hat{u}_{j+\frac{1}{2}} \\ & -\widehat{f_2}^+_{j-\frac{1}{2}}\hat{u}_{j-\frac{1}{2}}, \\ E_{2,\mathcal{K}_j} = & \langle\widehat{f_1}\mathbf{n},u\rangle_{\partial\mathcal{K}_j} + \langle\widehat{f_2}\mathbf{n},u\rangle_{\partial\mathcal{K}_j} - (f_1,u_x)_{\mathcal{K}_j} - \langle\widehat{f_1}\mathbf{n},\hat{u}\rangle_{\partial\mathcal{K}_j} - \langle\widehat{f_2}\mathbf{n},\hat{u}\rangle_{\partial\mathcal{K}_j}. \end{aligned}$$

We take sum the third and fourth equations of (17), add and subtract $\langle\widehat{g_1},\hat{v}\rangle_{\partial\mathcal{K}_j}$ and $\langle\widehat{g_2},\hat{v}\rangle_{\partial\mathcal{K}_j}$,and apply some simplifications and expanding boundary terms, then it holds that

$$\frac{1}{2}\frac{\partial}{\partial t}\|v\|^2_{\mathcal{K}_j} - \frac{\beta}{2}\Psi(v,u,\mathcal{K}_j) + \int_{\mathcal{K}_j} q^2 dx + E_{3,\mathcal{K}_j} + E_{4,\mathcal{K}_j} = 0, \tag{19}$$

where

$$\begin{aligned} E_{3,\mathcal{K}_j} = & \int_{\mathcal{K}_j}(vq)_x dx + \langle(\widehat{-q})\mathbf{n},v\rangle_{\partial\mathcal{K}_j} - \langle\hat{v},q\mathbf{n}\rangle_{\partial\mathcal{K}_j} + \langle\widehat{g_1}\mathbf{n},\hat{v}\rangle_{\partial\mathcal{K}_j} + \langle\widehat{g_2}\mathbf{n},\hat{v}\rangle_{\partial\mathcal{K}_j} \\ = & \Big((\widehat{-q})^-_{j+\frac{1}{2}} + (q)^-_{j+\frac{1}{2}}\Big)v^-_{j+\frac{1}{2}} + \Big(-(q)^+_{j-\frac{1}{2}} - (\widehat{-q})^+_{j-\frac{1}{2}}\Big)v^+_{j-\frac{1}{2}} - \\ & \hat{v}_{j+\frac{1}{2}}(q)^-_{j+\frac{1}{2}} + \hat{v}_{j-\frac{1}{2}}(q)^+_{j-\frac{1}{2}} + \widehat{g_1}^-_{j+\frac{1}{2}}\hat{v}_{j+\frac{1}{2}} - \widehat{g_1}^+_{j-\frac{1}{2}}\hat{v}_{j-\frac{1}{2}} \\ & +\widehat{g_2}^-_{j+\frac{1}{2}}\hat{v}_{j+\frac{1}{2}} - \widehat{g_2}^+_{j-\frac{1}{2}}\hat{v}_{j-\frac{1}{2}}, \\ E_{4,\mathcal{K}_j} = & \langle\widehat{g_1}\mathbf{n},v\rangle_{\partial\mathcal{K}_j} + \langle\widehat{g_2}\mathbf{n},v\rangle_{\partial\mathcal{K}_j} - (g_1,u_x)_{\mathcal{K}_j} - \langle\widehat{g_1}\mathbf{n},\hat{v}\rangle_{\partial\mathcal{K}_j} - \langle\widehat{g_2}\mathbf{n},\hat{v}\rangle_{\partial\mathcal{K}_j}. \end{aligned}$$

Numerical fluxes defined in (12) at $x_{j+\frac{1}{2}}$ and $x_{j+\frac{1}{2}}$ are

$$\begin{aligned}
(\widehat{-p})^-_{j+\frac{1}{2}} &= -p^-_{j+\frac{1}{2}} + \tau^-_{0,j+\frac{1}{2}}(u^-_{j+\frac{1}{2}} - \hat{u}_{j+\frac{1}{2}}),\\
(\widehat{-p})^+_{j-\frac{1}{2}} &= -p^+_{j-\frac{1}{2}} - \tau^+_{0,j-\frac{1}{2}}(u^+_{j-\frac{1}{2}} - \hat{u}_{j-\frac{1}{2}}),\\
(\widehat{-q})^-_{j+\frac{1}{2}} &= -q^-_{j+\frac{1}{2}} + \sigma^-_{0,j+\frac{1}{2}}(v^-_{j+\frac{1}{2}} - \hat{v}_{j+\frac{1}{2}}),\\
(\widehat{-q})^+_{j-\frac{1}{2}} &= -q^+_{j-\frac{1}{2}} - \sigma^+_{0,j-\frac{1}{2}}(v^+_{j-\frac{1}{2}} - \hat{v}_{j-\frac{1}{2}}).
\end{aligned} \tag{20}$$

Substituting all formulas (20) into $E_{1,\mathcal{K}_j}$ and $E_{3,\mathcal{K}_j}$ leads to

$$\begin{aligned}
E_{1,\mathcal{K}_j} = {}& \tau^-_{0,j+\frac{1}{2}}((u^-_{j+\frac{1}{2}})^2 - \hat{u}_{j+\frac{1}{2}}u^-_{j+\frac{1}{2}}) + \tau^+_{0,j-\frac{1}{2}}((u^+_{j-\frac{1}{2}})^2 - \hat{u}_{j-\frac{1}{2}}u^+_{j-\frac{1}{2}}) + \\
& \tau^-_{0,j+\frac{1}{2}}((\hat{u}_{j+\frac{1}{2}})^2 - \hat{u}_{j+\frac{1}{2}}u^-_{j+\frac{1}{2}}) + \tau^+_{0,j-\frac{1}{2}}((\hat{u}_{j-\frac{1}{2}})^2 - \hat{u}_{j-\frac{1}{2}}u^+_{j-\frac{1}{2}}) + \\
& (\widehat{f})^-_{j+\frac{1}{2}}\hat{u}_{j+\frac{1}{2}} - (\widehat{f})^+_{j-\frac{1}{2}}\hat{u}_{j-\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
E_{3,\mathcal{K}_j} = {}& \sigma^-_{0,j+\frac{1}{2}}((v^-_{j+\frac{1}{2}})^2 - \hat{v}_{j+\frac{1}{2}}v^-_{j+\frac{1}{2}}) + \sigma^+_{0,j-\frac{1}{2}}((v^+_{j-\frac{1}{2}})^2 - \hat{v}_{j-\frac{1}{2}}v^+_{j-\frac{1}{2}}) + \\
& \sigma^-_{0,j+\frac{1}{2}}((\hat{v}_{j+\frac{1}{2}})^2 - \hat{v}_{j+\frac{1}{2}}v^-_{j+\frac{1}{2}}) + \sigma^+_{0,j-\frac{1}{2}}((\hat{v}_{j-\frac{1}{2}})^2 - \hat{v}_{j-\frac{1}{2}}v^+_{j-\frac{1}{2}}) + \\
& (\widehat{g})^-_{j+\frac{1}{2}}\hat{v}_{j+\frac{1}{2}} - (\widehat{g})^+_{j-\frac{1}{2}}\hat{v}_{j-\frac{1}{2}}.
\end{aligned}$$

By taking sum over all elements of $\mathcal{K}_j$ on $E_{1,\mathcal{K}_j}$ and $E_{3,\mathcal{K}_j}$, using conservation of the fluxes, and applying periodic boundary conditions, we get

$$\begin{aligned}
E_1 &= \textstyle\sum_{\mathcal{K}_j\in\mathscr{K}_h} E_{1,\mathcal{K}_j} = \sum_j \tau^-_{0,j+\frac{1}{2}}\left(u^-_{j+\frac{1}{2}} - \hat{u}_{j+\frac{1}{2}}\right)^2 + \tau^+_{0,j-\frac{1}{2}}\left(u^+_{j-\frac{1}{2}} - \hat{u}_{j-\frac{1}{2}}\right)^2,\\
E_3 &= \textstyle\sum_{\mathcal{K}_j\in\mathscr{K}_h} E_{3,\mathcal{K}_j} = \sum_j \sigma^-_{0,j+\frac{1}{2}}\left(v^-_{j+\frac{1}{2}} - \hat{v}_{j+\frac{1}{2}}\right)^2 + \sigma^+_{0,j-\frac{1}{2}}\left(v^+_{j-\frac{1}{2}} - \hat{v}_{j-\frac{1}{2}}\right)^2.
\end{aligned}$$

If $\tau_0, \sigma_0 > 0$ then it is simply concluded that E_1 and E_3 are strictly positive values. Taking sum on $E_{2,\mathcal{K}_j}$ and $E_{4,\mathcal{K}_j}$ over all elements and then using

$$-(f_1(u), u_x)_{\mathscr{K}_h} = -\langle\int_{\hat{u}}^{u} f_1(s)ds, \mathbf{n}\rangle_{\partial\mathscr{K}_h}, \; -(g_1(v), v_x)_{\mathscr{K}_h} = -\langle\int_{\hat{v}}^{v} g_1(r)dr, \mathbf{n}\rangle_{\partial\mathscr{K}_h},$$

leads to

$$\begin{aligned}
E_2 = {}& -\langle\textstyle\int_{\hat{u}}^{u}(f_1(s) - f_1(\hat{u}))ds, \mathbf{n}\rangle_{\partial\mathscr{K}_h} - \langle\widehat{f_1(u)} - f_1(\hat{u}), (\hat{u} - u)\mathbf{n}\rangle_{\partial\mathscr{K}_h} + \\
& \langle\widehat{f_2}\mathbf{n}, u\rangle_{\partial\mathscr{K}_h} - \langle\widehat{f_2}\mathbf{n}, \hat{u}\rangle_{\partial\mathscr{K}_h},\\
E_4 = {}& -\langle\textstyle\int_{\hat{v}}^{v}(g_1(r) - g_1(\hat{v}))dr, \mathbf{n}\rangle_{\partial\mathscr{K}_h} - \langle\widehat{g_1(v)} - g_1(\hat{v}), (\hat{v} - v)\mathbf{n}\rangle_{\partial\mathscr{K}_h} + \\
& \langle\widehat{g_2}\mathbf{n}, v\rangle_{\partial\mathscr{K}_h} - \langle\widehat{g_2}\mathbf{n}, \hat{v}\rangle_{\partial\mathscr{K}_h}.
\end{aligned}$$

Using the definitions of $\widehat{f_1}$, $\widehat{f_2}$, $\widehat{g_1}$, $\widehat{g_2}$ in (12) and (16), it can be shown that:

$$\begin{aligned}
E_2 &= \langle\tau_1 - \tilde{\tau}_1, (u - \hat{u})^2\rangle_{\partial\mathscr{K}_h} + \tfrac{\alpha}{2}\langle\hat{v}\mathbf{n}, u^2\rangle_{\partial\mathscr{K}_h} + \langle\tau_2, (u - \hat{u})^2\rangle_{\partial\mathscr{K}_h},\\
E_4 &= \langle\sigma_1 - \tilde{\sigma}_1, (v - \hat{v})^2\rangle_{\partial\mathscr{K}_h} + \tfrac{\beta}{2}\langle\hat{u}\mathbf{n}, v^2\rangle_{\partial\mathscr{K}_h} + \langle\sigma_2, (v - \hat{v})^2\rangle_{\partial\mathscr{K}_h}.
\end{aligned}$$

Based on the definitions of E_2 and E_4, we define the corresponding auxiliary terms

$$\begin{array}{l}\tilde{E}_2 = \langle \tau_1 - \tilde{\tau_1}, (u-\hat{u})^2\rangle_{\partial\mathscr{K}_h} - \langle \tilde{\tau_2}, u^2\rangle_{\partial\mathscr{K}_h} + \langle \tau_2, (u-\hat{u})^2\rangle_{\partial\mathscr{K}_h},\\ \tilde{E}_4 = \langle \sigma_1 - \tilde{\sigma_1}, (v-\hat{v})^2\rangle_{\partial\mathscr{K}_h} - \langle \tilde{\sigma_2}, v^2\rangle_{\partial\mathscr{K}_h} + \langle \sigma_2, (v-\hat{v})^2\rangle_{\partial\mathscr{K}_h}.\end{array} \tag{21}$$

Due to $\langle -|\frac{\alpha}{2}\hat{v}|, u^2\rangle_{\partial\mathscr{K}_h} \le \langle \frac{\alpha}{2}\hat{v}\boldsymbol{n}, u^2\rangle_{\partial\mathscr{K}_h}$ and $\langle -|\frac{\beta}{2}\hat{u}|, v^2\rangle_{\partial\mathscr{K}_h} \le \langle \frac{\beta}{2}\hat{u}\boldsymbol{n}, v^2\rangle_{\partial\mathscr{K}_h}$, one can conclude the following significant results

$$\tilde{E}_2 \le E_2, \quad \tilde{E}_4 \le E_4.$$

With considering assumptions (13) and two possible cases in the following, we show that $E_2, E_4 > 0$.
• The case when $(u-\hat{u})^2 \ge u^2$ and $(v-\hat{v})^2 \ge v^2$. From (21), we get

$$\begin{aligned}\tilde{E}_2 &= \langle \tau_1 - \tilde{\tau_1}, (u-\hat{u})^2\rangle_{\partial\mathscr{K}_h} - \langle\langle \tfrac{\tilde{\tau_2}u^2}{(u-\hat{u})^2}, (u-\hat{u})^2\rangle_{\partial\mathscr{K}_h} + \langle \tau_2, (u-\hat{u})^2\rangle_{\partial\mathscr{K}_h}\\ &= \langle \tau_1 - \tilde{\tau_1}, (u-\hat{u})^2\rangle_{\partial\mathscr{K}_h} \langle \tau_2 - \tfrac{\tilde{\tau_2}u^2}{(u-\hat{u})^2}, (u-\hat{u})^2\rangle_{\partial\mathscr{K}_h} \le E_2,\\ \tilde{E}_4 &= \langle \sigma_1 - \tilde{\sigma_1}, (v-\hat{v})^2\rangle_{\partial\mathscr{K}_h} - \langle \tfrac{\tilde{\sigma_2}v^2}{(v-\hat{v})^2}, (v-\hat{v})^2\rangle_{\partial\mathscr{K}_h} + \langle \sigma_2, (v-\hat{v})^2\rangle_{\partial\mathscr{K}_h}\\ &= \langle \sigma_1 - \tilde{\sigma_1}, (v-\hat{v})^2\rangle_{\partial\mathscr{K}_h} + \langle \sigma_2 - \tfrac{\tilde{\sigma_2}v^2}{(v-\hat{v})^2}, (v-\hat{v})^2\rangle_{\partial\mathscr{K}_h} \le E_4.\end{aligned}$$

Under assumptions (13), one can show that $\tilde{E}_2$ and $\tilde{E}_4$ are strictly positive and it results that E_2 and E_4 are strictly positive.
• The case when $(u-\hat{u})^2 \le u^2$ and $(v-\hat{v})^2 \le v^2$. Taking into account (21), we have

$$\begin{aligned}\tilde{E}_2 &= \langle \tau_1 - \tilde{\tau_1}, (u-\hat{u})^2\rangle_{\partial\mathscr{K}_h} - \langle \tilde{\tau_2}, u^2\rangle_{\partial\mathscr{K}_h} + \langle \tfrac{\tau_2(u-\hat{u})^2}{u^2}, u^2\rangle_{\partial\mathscr{K}_h}\\ &= \langle \tau_1 - \tilde{\tau_1}, (u-\hat{u})^2\rangle_{\partial\mathscr{K}_h} \langle \tilde{\tau_2} - \tfrac{\tau_2(u-\hat{u})^2}{u^2}, u^2\rangle_{\partial\mathscr{K}_h} \le E_2,\\ \tilde{E}_4 &= \langle \sigma_1 - \tilde{\sigma_1}, (v-\hat{v})^2\rangle_{\partial\mathscr{K}_h} - \langle \tilde{\sigma_2}, v^2\rangle_{\partial\mathscr{K}_h} + \langle \tfrac{\sigma_2(v-\hat{v})^2}{v^2}, v^2\rangle_{\partial\mathscr{K}_h}\\ &= \langle \sigma_1 - \tilde{\sigma_1}, (v-\hat{v})^2\rangle_{\partial\mathscr{K}_h} + \langle \tfrac{\sigma_2(v-\hat{v})^2}{v^2} - \tilde{\sigma_2}, v^2\rangle_{\partial\mathscr{K}_h} \le E_4.\end{aligned}$$

Similar to the previous case, with assuming (13), we can conclude that E_2 and E_4 are also strictly positive.
Hence, from (18) and (19) we get

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|^2_{\mathscr{K}_h} - \frac{\alpha}{2}\Psi(u, v, \mathscr{K}_h) \le 0, \qquad \frac{1}{2}\frac{\partial}{\partial t}\|v\|^2_{\mathscr{K}_h} - \frac{\beta}{2}\Psi(v, u, \mathscr{K}_h) \le 0,$$

Finally by integrating the above formulae over the time interval $[0, t]$, the results (14) and (15) can be obtained. □

Stability of the method for homogeneous Dirichlet boundary conditions is proved in the following corollary.

Corollary 3.2. *Let CBE (1) be equipped by the following homogeneous Dirichlet boundary conditions*

$$\mathfrak{u}(x_L, \cdot) = \mathfrak{u}(x_R, \cdot) = \mathfrak{v}(x_L, \cdot) = \mathfrak{v}(x_R, \cdot) = 0.$$

Then the relations (14) and (15) hold under the necessary conditions $\tau_0, \sigma_0 > 0$, $\tau_1 > \tilde{\tau_1}$, $\tau_2 > \tilde{\tau_2}$, $\sigma_1 > \tilde{\sigma_1}$, and $\sigma_2 > \tilde{\sigma_2}$. In other words, if the semi-discrete proposed HDG method is stable over time interval $[0, T]$ in the sense of energy method, then the mentioned specific conditions on the stabilization parameters will be fulfilled.

Proof. The proof is almost similar to the proof of Theorem 3.1. The only difference is that the following terms

$$(\widehat{f})^-_{j+\frac{1}{2}}\hat{u}_{j+\frac{1}{2}} - (\widehat{f})^+_{j-\frac{1}{2}}\hat{u}_{j-\frac{1}{2}}, \quad (\widehat{g})^-_{j+\frac{1}{2}}\hat{v}_{j+\frac{1}{2}} - (\widehat{g})^+_{j-\frac{1}{2}}\hat{v}_{j-\frac{1}{2}}, \tag{22}$$

over all inner faces become zero since the conservation of the fluxes $\widehat{f}, \widehat{g}$ hold on the inner faces. Due to zero boundary conditions for $\hat{u}, \hat{v}$, the terms in (22) become again zero on both sides of the domain Ω. The rest of the proof follows the proof of the theorem. □

Corollary 3.3. *Using the mean value theorem, we get*

$$\begin{aligned}
\tilde{\tau_1} &= \tfrac{1}{(u-\hat{u})^2}\int_{\hat{u}}^{u}(f_1(s) - f_1(\hat{u}))\mathbf{n}ds \\
&= \tfrac{1}{(u-\hat{u})^2}\int_{\hat{u}}^{u}\tfrac{\partial f_1}{\partial u}(\xi)(s-\hat{u})\mathbf{n}ds \leq \tfrac{1}{2}\sup_{s\in I_u}|\tfrac{\partial f_1}{\partial u}(s)|, \\
\tilde{\sigma_1} &= \tfrac{1}{(v-\hat{v})^2}\int_{\hat{v}}^{v}(g_1(r) - g_1(\hat{v}))\mathbf{n}dr \\
&= \tfrac{1}{(v-\hat{v})^2}\int_{\hat{v}}^{v}\tfrac{\partial g_1}{\partial v}(\mu_1)(s-\hat{v})\mathbf{n}ds \leq \tfrac{1}{2}\sup_{r\in I_v}|\tfrac{\partial g_1}{\partial v}(r)|,
\end{aligned}$$

where

$$I_u = [\min\{u, \hat{u}\}, \max\{u, \hat{u}\}], \qquad I_v = [\min\{v, \hat{v}\}, \max\{v, \hat{v}\}].$$

Thus, the stabilization parameters τ_1 and σ_1 satisfy the condition $\tau_1 > \tilde{\tau_1}$ and $\sigma_1 > \tilde{\sigma_1}$, respectively, when

$$\tau_1 > \frac{1}{2}\sup_{s\in I_u}|\frac{\partial f_1}{\partial u}(s)|, \quad \sigma_1 > \frac{1}{2}\sup_{r\in I_v}|\frac{\partial g_1}{\partial v}(r)|.$$

Thus, it is easier to choose the stabilization parameters τ_1 and σ_1 regarding to above formulas than conditions in Theorem 3.1.

4. Full discretezation of HDG method

To derive a full discretization HDG method for CBE (1), we apply the Crank-Nicolson method, as a standard time-discretization approach, to the weak formulation (5). With $\Delta t = T/J$ and $t_n = n\Delta t$, for $n = 0, \ldots, J$, weak form (5) becomes

$$\left\{\begin{aligned}
\frac{1}{\Delta t}(u^n, w_1)_{\mathcal{K}} - \frac{1}{2}(f^n, w_{1x})_{\mathcal{K}} + \frac{1}{2}\langle\widehat{f}^n\mathbf{n}, w_1\rangle_{\partial\mathcal{K}} &= l_1(w_1), \\
(p^n, w_2)_{\mathcal{K}} + (u^n, w_{2x})_{\mathcal{K}} - \langle\hat{u}^n\mathbf{n}, w_2\rangle_{\partial\mathcal{K}} &= 0, \\
\frac{1}{\Delta t}(v^n, w_3)_{\mathcal{K}} - \frac{1}{2}(g^n, w_{3x})_{\mathcal{K}} + \frac{1}{2}\langle\widehat{g}^n\mathbf{n}, w_3\rangle_{\partial\mathcal{K}} &= l_3(w_3), \\
(q^n, w_4)_{\mathcal{K}} + (v^n, w_{4x})_{\mathcal{K}} - \langle\hat{v}^n\mathbf{n}, w_4\rangle_{\partial\mathcal{K}} &= 0,
\end{aligned}\right. \tag{23}$$

where

$$l_1(w_1) = \frac{1}{\Delta t}(u^{n-1}, w_1)_K + \frac{1}{2}(f^{n-1}, w_{1x})_K - \frac{1}{2}\langle \widehat{f}^{n-1}\mathbf{n}, w_1\rangle_{\partial K},$$
$$l_3(w_3) = \frac{1}{\Delta t}(v^{n-1}, w_3)_K + \frac{1}{2}(g^{n-1}, w_{3x})_K - \frac{1}{2}\langle \widehat{g}^{n-1}\mathbf{n}, w_3\rangle_{\partial K},$$

where superscripts $m = n, n-1$ stand for the values at time levels t_m.

Regarding to the structure of approximate spaces, the aim is to seek $u^n, p^n, v^n, q^n \in S_{h,k}$, and $(\xi^n, \zeta^n) \in M_{h,k}(0, \Gamma_{\mathfrak{u}}) \times M_{h,k}(0, \Gamma_{\mathfrak{v}})$, such that (8) and (23) are satisfied for $n = 1, 2, \ldots, J$. By summing over all elements, inserting the fluxes (7) into (8) and (23) at t_n and imposing boundary conditions (6), we get

$$\begin{aligned}
l_1(w_1) &= \frac{1}{\Delta t}(u^n, w_1)_{\mathscr{K}_h} - \frac{1}{2}(p_x^n, w_1)_{\mathscr{K}_h} - \frac{\eta}{4}((u^2)^n, w_{1x})_{\mathscr{K}_h} - \frac{\alpha}{2}(u^n v^n, w_{1x})_{\mathscr{K}_h} \\
&\quad + \tfrac{\eta}{4}\langle (\xi^2)^n \mathbf{n}, w_1\rangle_{\partial\mathscr{K}_h\setminus\Gamma_{\mathfrak{u}}} + \frac{\alpha}{4}\langle \zeta^n u^n \mathbf{n}, w_1\rangle_{\partial\mathscr{K}_h} + \frac{\alpha}{4}\langle \zeta^n \xi^n \mathbf{n}, w_1\rangle_{\partial\mathscr{K}_h\setminus\Gamma_{\mathfrak{u}}} \\
&\quad + \tfrac{1}{2}\langle \tau^n u^n, w_1\rangle_{\partial\mathscr{K}_h} - \tfrac{1}{2}\langle \tau^n \xi^n, w_1\rangle_{\partial\mathscr{K}_h\setminus\Gamma_{\mathfrak{u}}}, \\
l_2(w_2 &= (p^n, w_2)_{\mathscr{K}_h} + (u^n, w_{2x})_{\mathscr{K}_h} - \langle \xi^n \mathbf{n}, w_2\rangle_{\partial\mathscr{K}_h\setminus\Gamma_{\mathfrak{u}}}, \\
l_3(w_3) &= \frac{1}{\Delta t}(v^n, w_3)_{\mathscr{K}_h} - \frac{1}{2}(q_x^n, w_3)_{\mathscr{K}_h} - \frac{\gamma}{4}((v^2)^n, w_{3x})_{\mathscr{K}_h} - \frac{\beta}{2}(u^n v^n, w_{3x})_{\mathscr{K}_h} \\
&\quad + \tfrac{\gamma}{4}\langle (\zeta^2)^n \mathbf{n}, w_3\rangle_{\partial\mathscr{K}_h\setminus\Gamma_{\mathfrak{v}}} + \frac{\beta}{4}\langle \xi^n v^n \mathbf{n}, w_3\rangle_{\partial\mathscr{K}_h} + \frac{\beta}{4}\langle \xi^n \zeta^n \mathbf{n}, w_3\rangle_{\partial\mathscr{K}_h\setminus\Gamma_{\mathfrak{v}}} \\
&\quad + \tfrac{1}{2}\langle \sigma^n v^n, w_3\rangle_{\partial\mathscr{K}_h} - \tfrac{1}{2}\langle \sigma^n \zeta^n, w_3\rangle_{\partial\mathscr{K}_h\setminus\Gamma_{\mathfrak{v}}}, \\
l_4(w_4) &= (q, w_4)_{\mathscr{K}_h} + (v, w_{4x})_{\mathscr{K}_h} - \langle \zeta \mathbf{n}, w_4\rangle_{\partial\mathscr{K}_h\setminus\Gamma_{\mathfrak{v}}}, \\
l_5(\mu_1) &= \frac{\eta}{2}\langle \xi^2 \mathbf{n}, \mu_1\rangle_{\partial\mathscr{K}_h\setminus\mathcal{F}_h^{\partial}} + \frac{\alpha}{2}\langle \zeta u \mathbf{n}, \mu_1\rangle_{\partial\mathscr{K}_h\setminus\mathcal{F}_h^{\partial}} + \frac{\alpha}{2}\langle \xi\zeta \mathbf{n}, \mu_1\rangle_{\partial\mathscr{K}_h\setminus\mathcal{F}_h^{\partial}} + \\
&\quad \langle \tau u, \mu_1\rangle_{\partial\mathscr{K}_h\setminus\mathcal{F}_h^{\partial}} - \langle \tau\xi, \mu_1\rangle_{\partial\mathscr{K}_h\setminus\mathcal{F}_h^{\partial}} - \langle p\mathbf{n}, \mu_1\rangle_{\partial\mathscr{K}_h}, \\
l_6(\mu_2) &= \frac{\gamma}{2}\langle \zeta^2 \mathbf{n}, \mu_2\rangle_{\partial\mathscr{K}_h\setminus\mathcal{F}_h^{\partial}} + \frac{\beta}{2}\langle \xi v \mathbf{n}, \mu_2\rangle_{\partial\mathscr{K}_h\setminus\mathcal{F}_h^{\partial}} + \frac{\beta}{2}\langle \xi\zeta \mathbf{n}, \mu_2\rangle_{\partial\mathscr{K}_h\setminus\mathcal{F}_h^{\partial}} + \\
&\quad \langle \sigma v, \mu_2\rangle_{\partial\mathscr{K}_h\setminus\mathcal{F}_h^{\partial}} - \langle \sigma\zeta, \mu_2\rangle_{\partial\mathscr{K}_h\setminus\mathcal{F}_h^{\partial}} - \langle q\mathbf{n}, \mu_2\rangle_{\partial\mathscr{K}_h},
\end{aligned} \tag{24}$$

where $\mu_1 \in M_{h,k}^{\mathfrak{u}}(0)$, $\mu_2 \in M_{h,k}^{\mathfrak{v}}(0)$, and

$$\begin{aligned}
l_1(w) &= -\frac{\eta}{4}\langle (b_{\mathfrak{u}}^2)^n \mathbf{n}, w\rangle_{\partial\mathscr{K}_h\cap\Gamma_{\mathfrak{u}}} - \frac{\alpha}{4}\langle b_{\mathfrak{v}}^n b_{\mathfrak{u}}^n \mathbf{n}, w\rangle_{\partial\mathscr{K}_h\cap\Gamma_{\mathfrak{u}}} + \frac{1}{2}\langle \tau^n b_u^n, w\rangle_{\partial\mathscr{K}_h\cap\Gamma_{\mathfrak{u}}} \\
&\quad + \tfrac{1}{\Delta t}(u^{n-1}, w_1)_{\mathscr{K}_h} + \frac{1}{2}(p_x^{n-1}, w_1)_{\mathscr{K}_h} + \frac{\eta}{4}((u^2)^{n-1}, w_{1x})_{\mathscr{K}_h} + \\
&\quad \tfrac{\alpha}{2}(u^{n-1}v^{n-1}, w_{1x})_{\mathscr{K}_h} - \tfrac{\eta}{4}\langle (\xi^2)^{n-1}\mathbf{n}, w_1\rangle_{\partial\mathscr{K}_h\setminus\Gamma_{\mathfrak{u}}} - \frac{\alpha}{4}\langle \zeta^{n-1}u^{n-1}\mathbf{n}, w_1\rangle_{\partial\mathscr{K}_h} \\
&\quad - \tfrac{\alpha}{4}\langle \zeta^{n-1}\xi^{n-1}\mathbf{n}, w_1\rangle_{\partial\mathscr{K}_h\setminus\Gamma_{\mathfrak{u}}} - \tfrac{1}{2}\langle \tau^{n-1}u^{n-1}, w_1\rangle_{\partial\mathscr{K}_h} + \\
&\quad \frac{1}{2}\langle \tau^{n-1}\lambda_u^{n-1}, w_1\rangle_{\partial\mathscr{K}_h\setminus\Gamma_{\mathfrak{u}}},
\end{aligned}$$

$$
\begin{aligned}
l_3(w) = &-\frac{\gamma}{4}\langle (b_{\mathfrak{v}}^2)^n\mathbf{n}, w\rangle_{\partial\mathscr{K}_h\cap\Gamma_{\mathfrak{v}}} - \frac{\beta}{4}\langle b_{\mathfrak{u}}^n b_{\mathfrak{v}}^n\mathbf{n}, w\rangle_{\partial\mathscr{K}_h\cap\Gamma_{\mathfrak{v}}} + \frac{1}{2}\langle \sigma^n b_v^n, w\rangle_{\partial\mathscr{K}_h\cap\Gamma_{\mathfrak{v}}}\\
&+\tfrac{1}{\Delta t}(v^{n-1}, w_3)_{\mathscr{K}_h} + \frac{1}{2}(q_x^{n-1}, w_3)_{\mathscr{K}_h} + \frac{\gamma}{4}((v^2)^{n-1}, w_{3x})_{\mathscr{K}_h} +\\
&\tfrac{\beta}{2}(u^{n-1}v^{n-1}, w_{3x})_{\mathscr{K}_h} - \tfrac{\gamma}{4}\langle (\zeta^2)^{n-1}\mathbf{n}, w_3\rangle_{\partial\mathscr{K}_h\setminus\Gamma_{\mathfrak{v}}} - \frac{\beta}{4}\langle \xi^{n-1}v^{n-1}\mathbf{n}, w_3\rangle_{\partial\mathscr{K}_h}\\
&-\tfrac{\beta}{4}\langle \xi^{n-1}\zeta^{n-1}\mathbf{n}, w_3\rangle_{\partial\mathscr{K}_h\setminus\Gamma_{\mathfrak{v}}} - \tfrac{1}{2}\langle \sigma^{n-1}v^{n-1}, w_3\rangle_{\partial\mathscr{K}_h}\\
&+\frac{1}{2}\langle \sigma^{n-1}\lambda_v^{n-1}, w_3\rangle_{\partial\mathscr{K}_h\setminus\Gamma_{\mathfrak{v}}},
\end{aligned}
$$

and also

$$
\begin{aligned}
&l_2(w) = \langle b_{\mathfrak{u}}^n\mathbf{n}, w\rangle_{\partial\mathscr{K}_h\cap\Gamma_{\mathfrak{u}}}, \quad l_4(w) = \langle b_{\mathfrak{v}}^n\mathbf{n}, w\rangle_{\partial\mathscr{K}_h\cap\Gamma_{\mathfrak{v}}},\\
&l_5(\mu) = \langle b_{\mathfrak{p}}^n, \mu\rangle_{\Gamma_{\mathfrak{p}}}, \quad l_6(\mu) = \langle b_{\mathfrak{q}}^n, \mu\rangle_{\Gamma_{\mathfrak{q}}}.
\end{aligned}
$$

The variational formulation (24), that is arisen by using the Crank-Nicolson approach in (5), is nonlinear and so a technique with the order of at least two is requested for solving (24). Regarding the nonlinear weak form (24), one has to convert this weak form to a linear variational form by an appropriate iterative method. Among iterative methods with the order of at least two, we apply the Newton-Raphson method to nonlinear variational formulation (24). We set

$$
W^n = (\bar{u}^n, \bar{p}^n, \bar{v}^n, \bar{q}^n, \bar{\xi}^n, \bar{\zeta}^n) \in S_{h,k}^4 \times M_{h,k}(0,\Gamma_{\mathfrak{u}}) \times M_{h,k}(0,\Gamma_{\mathfrak{v}}),
$$

where $(\bar{u}^n, \bar{p}^n, \bar{v}^n, \bar{q}^n, \bar{\xi}^n, \bar{\zeta}^n)$ is the exact solution vector of problem (24) and $(u^n, p^n, v^n, q^n, \xi^n, \zeta^n)$ is corresponding approximate solution of problem (24) at time t_n. With appropriate initial guess $W_{n,0}$, the following sequence of solution vectors is created

$$
W_{n,i} = W_{n,i-1} + \delta W_{n,i}, \quad i = 1, 2, \dots,
$$

where $W_{n,i}$ tends to analytical solution of the problem (24) W, at t_n where i goes to infinity, and

$$
\delta W_{n,i} = (\delta\bar{u}_{n,i}, \delta\bar{p}_{n,i}, \delta\bar{v}_{n,i}, \delta\bar{q}_{n,i}, \delta\bar{\xi}_{n,i}, \delta\bar{\zeta}_{n,i}),
$$

is approximated by the Newton-Raphson method. By applying the Newton-Raphson method to (24), $\delta W_{n,i}$ is the solution of following linear weak form such that all $w_1, w_2, w_3, w_4 \in S_{h,k}$ and $(\mu_1, \mu_2) \in M_{h,k}(0,\Gamma_{\mathfrak{u}}) \times M_{h,k}(0,\Gamma_{\mathfrak{v}})$ satisfy

$$
\begin{aligned}
&\tilde{\mathsf{l}}_1(w_1) = \tilde{\mathsf{a}}_1(\delta u_{n,i}, w_1) - \tfrac{1}{2}\tilde{\mathsf{b}}_1^T(\delta p_{n,i}, w_1) - \tilde{\mathsf{a}}_2(\delta v_{n,i}, w_1) + \tilde{\mathsf{a}}_3(\delta\xi_{n,i}, w_1)\\
&\qquad\quad +\tilde{\mathsf{a}}_4(\delta\zeta_{n,i}, w_1),\\
&\tilde{\mathsf{l}}_2(w_2) = \tilde{\mathsf{b}}_1(\delta u_{n,i}, w_2) + \tilde{\mathsf{b}}_2(\delta p_{n,i}, w_2) - \tilde{\mathsf{b}}_3(\delta\lambda\mathsf{u}_{n,i}, w_2),\\
&\tilde{\mathsf{l}}_3(w_3) = -\tilde{\mathsf{c}}_1(\delta u_{n,i}, w_3) + \tilde{\mathsf{c}}_2(\delta v_{n,i}, w_3) - \tfrac{1}{2}\tilde{\mathsf{b}}_1^T(\delta q_{n,i}, w_3) + \tilde{\mathsf{c}}_3(\delta\xi_{n,i}, w_3)\\
&\qquad\quad +\tilde{\mathsf{c}}_4(\delta\zeta_{n,i}, w_3),\\
&\tilde{\mathsf{l}}_4(w_4) = \tilde{\mathsf{b}}_1(\delta v_{n,i}, w_4) + \tilde{\mathsf{b}}_2(\delta q_{n,i}, w_4) - \tilde{\mathsf{b}}_4(\delta\zeta_{n,i}, w_4),\\
&\tilde{\mathsf{l}}_5(\mu_1) = \tilde{\mathsf{d}}_1(\delta u_{n,i}, \mu_1) - \tilde{\mathsf{d}}_2(\delta p_{n,i}, \mu_1) + \tilde{\mathsf{d}}_3(\delta\xi_{n,i}, \mu_1) + \tilde{\mathsf{d}}_4(\delta\zeta_{n,i}, \mu_1),\\
&\tilde{\mathsf{l}}_6(\mu_2) = \tilde{\mathsf{e}}_1(\delta v_{n,i}, \mu_2) - \tilde{\mathsf{d}}_2(\delta q_{n,i}, \mu_2) + \tilde{\mathsf{e}}_2(\delta\xi_{n,i}, \mu_2) + \tilde{\mathsf{e}}_3(\delta\zeta_{n,i}, \mu_2),
\end{aligned}
\tag{25}
$$

such that the multilinear forms and linear functionals in (25) are defined as:

$$\begin{aligned}
\tilde{\mathsf{a}}_1(\delta u_{n,i}, w) =& \tfrac{1}{\Delta t}(\delta u_{n,i}, w)_{\mathscr{K}_h} - \tfrac{\eta}{2}(u_{n,i-1}\delta u_{n,i}, w_x)_{\mathscr{K}_h} - \tfrac{\alpha}{2}(v_{n,i-1}\delta u_{n,i}, w_x)_{\mathscr{K}_h} + \\
& \tfrac{\alpha}{4}\langle \zeta_{n,i-1}\delta u_{n,i}\mathbf{n}, w\rangle_{\partial\mathscr{K}_h} + \tfrac{1}{2}\langle \tau_{n,i-1}\delta u_{n,i}, w\rangle_{\partial\mathscr{K}_h},\\
\tilde{\mathsf{a}}_2(\delta v_{n,i}, w) =& \tfrac{\alpha}{2}(u_{n,i-1}\delta v_{n,i}, w_x)_{\mathscr{K}_h},\\
\tilde{\mathsf{a}}_3(\delta \xi_{n,i}, w) =& \tfrac{\eta}{2}\langle \xi_{n,i-1}\delta \xi_{n,i}\mathbf{n}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{u}}} + \tfrac{\alpha}{4}\langle \zeta_{n,i-1}\delta \xi_{n,i}\mathbf{n}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{u}}}\\
& -\tfrac{1}{2}\langle \tau_{n,i-1}\delta \xi_{n,i}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{u}}} + \tfrac{1}{2}\langle u_{n,i-1}(\tfrac{\partial\tau}{\partial\xi})_{n,i-1}\delta \xi_{n,i}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{u}}}\\
& -\tfrac{1}{2}\langle \xi_{n,i-1}(\tfrac{\partial\tau}{\partial\xi})_{n,i-1}\delta \xi_{n,i}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{u}}},\\
\tilde{\mathsf{a}}_4(\delta \zeta_{n,i}, w) =& \tfrac{\alpha}{4}\langle \xi_{n,i-1}\delta \zeta_{n,i}\mathbf{n}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{v}}} + \tfrac{\alpha}{4}\langle u_{n,i-1}\delta \zeta_{n,i}\mathbf{n}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{v}}} + \\
& \tfrac{1}{2}\langle u_{n,i-1}(\tfrac{\partial\tau}{\partial\zeta})_{n,i-1}\delta \zeta_{n,i}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{v}}} - \\
& \tfrac{1}{2}\langle \xi_{n,i-1}(\tfrac{\partial\tau}{\partial\zeta})_{n,i-1}\delta \zeta_{n,i}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{v}}},\\
\tilde{\mathsf{b}}_1(\delta u_{n,i}, w) =& (u_{n,i}, w_x)_{\mathscr{K}_h}, \qquad \tilde{\mathsf{b}}_2(\delta p_{n,i}, w) = (\delta p_{n,i}, w)_{\mathscr{K}_h},\\
\tilde{\mathsf{b}}_3(\delta \xi_{n,i}, w) =& \langle \delta \xi_{n,i}\mathbf{n}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{u}}}, \qquad \tilde{\mathsf{b}}_4(\delta \zeta_{n,i}, w) = \langle \delta \zeta_{n,i}\mathbf{n}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{v}}},\\
\tilde{\mathsf{c}}_2(\delta v_{n,i}, w) =& \tfrac{1}{\Delta t}(\delta v_{n,i}, w)_{\mathscr{K}_h} - \tfrac{\gamma}{2}(v_{n,i-1}\delta v_{n,i}, w_x)_{\mathscr{K}_h} - \tfrac{\beta}{2}(u_{n,i-1}\delta v_{n,i}, w_x)_{\mathscr{K}_h} + \\
& \tfrac{\beta}{4}\langle \xi_{n,i-1}\delta v_{n,i}, w\rangle_{\partial\mathscr{K}_h} + \tfrac{1}{2}\langle \sigma_{n,i-1}\delta v_{n,i}, w\rangle_{\partial\mathscr{K}_h},\\
\tilde{\mathsf{c}}_3(\delta \xi_{n,i}, w) =& \tfrac{\beta}{4}\langle \zeta_{n,i-1}\delta \xi_{n,i}\mathbf{n}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{u}}} + \tfrac{\beta}{4}\langle v_{n,i-1}\delta \xi_{n,i}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{u}}} + \\
& \tfrac{1}{2}\langle v_{n,i-1}(\tfrac{\partial\sigma}{\partial\xi})_{n,i-1}\delta \xi_{n,i}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{u}}} - \\
& \tfrac{1}{2}\langle \zeta_{n,i-1}(\tfrac{\partial\sigma}{\partial\xi})_{n,i-1}\delta \xi_{n,i}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{u}}},\\
\tilde{\mathsf{c}}_4(\delta \zeta_{n,i}, w) =& \tfrac{\gamma}{2}\langle \zeta_{n,i-1}\delta \zeta_{n,i}\mathbf{n}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{v}}} + \tfrac{\beta}{4}\langle \xi_{n,i-1}\delta \zeta_{n,i}\mathbf{n}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{v}}} - \\
& \tfrac{1}{2}\langle \sigma_{n,i-1}\delta \zeta_{n,i}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{v}}} + \tfrac{1}{2}\langle v_{n,i-1}(\tfrac{\partial\sigma}{\partial\zeta})_{n,i-1}\delta \zeta_{n,i}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{v}}}\\
& -\tfrac{1}{2}\langle \zeta_{n,i-1}(\tfrac{\partial\sigma}{\partial\zeta})_{n,i-1}\delta \zeta_{n,i}, w\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{v}}},\\
\tilde{\mathsf{c}}_1(\delta u_{n,i}, w) =& \tfrac{\beta}{2}(v_{n,i-1}\delta u_{n,i}, w_x)_{\mathscr{K}_h}, \qquad \tilde{\mathsf{d}}_2(\delta p_{n,i}, \mu) = \langle \delta p_{n,i}\mathbf{n}, \mu\rangle_{\partial\mathscr{K}_h},\\
\tilde{\mathsf{d}}_1(\delta u_{n,i}, \mu) =& \tfrac{\alpha}{2}\langle \zeta_{n,i-1}\delta u_{n,i}\mathbf{n}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}} + \langle \tau_{n,i-1}\delta u_{n,i}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}},\\
\tilde{\mathsf{d}}_3(\delta \xi_{n,i}, \mu) =& \eta\langle \xi_{n,i-1}\delta \xi_{n,i}\mathbf{n}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}} + \tfrac{\alpha}{2}\langle \zeta_{n,i-1}\delta \xi_{n,i}\mathbf{n}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}} - \\
& \langle \tau_{n,i-1}\delta \xi_{n,i}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}} + \langle u_{n,i-1}(\tfrac{\partial\tau}{\partial\xi})_{n,i-1}\delta \xi_{n,i}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}}\\
& -\langle \xi_{n,i-1}(\tfrac{\partial\tau}{\partial\xi})_{n,i-1}\delta \xi_{n,i}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}},\\
\tilde{\mathsf{d}}_4(\delta \zeta_{n,i}, \mu) =& \tfrac{\alpha}{2}\langle \xi_{n,i-1}\delta \zeta_{n,i}\mathbf{n}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}} + \tfrac{\alpha}{2}\langle u_{n,i-1}\delta \zeta_{n,i}\mathbf{n}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}} + \\
& \langle u_{n,i-1}(\tfrac{\partial}{\partial\zeta})\tau_{n,i-1}\delta \zeta_{n,i}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}} - \\
& \langle \xi_{n,i-1}(\tfrac{\partial\tau}{\partial\zeta})_{n,i-1}\delta \zeta_{n,i}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}},\\
\tilde{\mathsf{e}}_1(\delta v_{n,i}, \mu) =& \tfrac{\beta}{2}\langle \xi_{n,i-1}\delta v_{n,i}\mathbf{n}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}} + \langle \sigma_{n,i-1}\delta v_{n,i}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}},\\
\tilde{\mathsf{e}}_2(\delta \xi_{n,i}, \mu) =& \tfrac{\beta}{2}\langle \zeta_{n,i-1}\delta \xi_{n,i}\mathbf{n}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}} + \tfrac{\beta}{2}\langle v_{n,i-1}\delta \xi_{n,i}\mathbf{n}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}} + \\
& \langle v_{n,i-1}(\tfrac{\partial\sigma}{\partial\xi})_{n,i-1}\delta \xi_{n,i}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}} - \\
& \langle \zeta_{n,i-1}(\tfrac{\partial\sigma}{\partial\xi})_{n,i-1}\delta \xi_{n,i}, \mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^{\partial}},
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathsf{e}}_3(\delta\zeta_{n,i},\mu) = {} & \gamma\langle \zeta_{n,i-1}\delta\zeta_{n,i}\mathbf{n},\mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^\partial} + \tfrac{\beta}{2}\langle \xi_{n,i-1}\delta\zeta_{n,i}\mathbf{n},\mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^\partial} - \\
& \langle \sigma_{n,i-1}\delta\zeta_{n,i},\mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^\partial} + \langle v_{n,i-1}(\tfrac{\partial\sigma}{\partial\zeta})_{n,i-1}\delta\zeta_{n,i},\mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^\partial} \\
& -\langle \zeta_{n,i-1}(\tfrac{\partial\sigma}{\partial\zeta})_{n,i-1}\delta\zeta_{n,i},\mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^\partial},
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathsf{l}}_1(w) = {} & -\tfrac{1}{\Delta_t}(u_{n,i-1},w)_{\mathscr{K}_h} + \tfrac{1}{2}((p_x)_{n,i-1},w)_{\mathscr{K}_h} + \tfrac{\eta}{4}(u^2_{n,i-1},w_x)_{\mathscr{K}_h} \\
& +\tfrac{\alpha}{2}(u_{n,i-1}v_{n,i-1},w_x)_{\mathscr{K}_h} - \tfrac{\alpha}{4}\langle \zeta_{n,i-1}\xi_{n,i-1}\mathbf{n},w\rangle_{\partial\mathscr{K}_h} - \\
& \tfrac{\alpha}{4}\langle u_{n,i-1}\zeta_{n,i-1}\mathbf{n},w\rangle_{\partial\mathscr{K}_h} - \tfrac{\eta}{4}\langle(\xi)^2_{n,i-1}\mathbf{n},w\rangle_{\partial\mathscr{K}_h} - \tfrac{1}{2}\langle \tau_{n,i-1}u_{n,i-1},w\rangle_{\partial\mathscr{K}_h} \\
& +\tfrac{1}{2}\langle \tau_{n,i-1}\xi_{n,i-1},w\rangle_{\partial\mathscr{K}_h} + \tfrac{1}{\Delta t}(u^{n-1},w_1)_{\mathscr{K}_h} + \tfrac{1}{2}(p_x^{n-1},w_1)_{\mathscr{K}_h} + \\
& \tfrac{\eta}{4}((u^2)^{n-1},w_{1x})_{\mathscr{K}_h} + \tfrac{\alpha}{2}(u^{n-1}v^{n-1},w_{1x})_{\mathscr{K}_h} - \tfrac{\eta}{4}\langle(\xi^2)^{n-1}\mathbf{n},w_1\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{u}}} - \\
& \tfrac{\alpha}{4}\langle \zeta^{n-1}u^{n-1}\mathbf{n},w_1\rangle_{\partial\mathscr{K}_h} - \tfrac{\alpha}{4}\langle \zeta^{n-1}\xi^{n-1}\mathbf{n},w_1\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{u}}} - \\
& \tfrac{1}{2}\langle \tau^{n-1}u^{n-1},w_1\rangle_{\partial\mathscr{K}_h} + \tfrac{1}{2}\langle \tau^{n-1}\lambda_u^{n-1},w_1\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{u}}},
\end{aligned}$$

$$\tilde{\mathsf{l}}_2(w) = \quad -(p_{n,i-1},w)_{\mathscr{K}_h} - (u_{n,i-1},w_x)_{\mathscr{K}_h} + \langle \xi_{n,i-1}\mathbf{n},w\rangle_{\partial\mathscr{K}_h},$$

$$\begin{aligned}
\tilde{\mathsf{l}}_3(w) = {} & -\tfrac{1}{\Delta_t}(v_{n,i-1},w)_{\mathscr{K}_h} + \tfrac{1}{2}((q_x)_{n,i-1},w)_{\mathscr{K}_h} + \tfrac{\gamma}{4}(v^2_{n,i-1},w_x)_{\mathscr{K}_h} + \\
& \tfrac{\beta}{2}(u_{n,i-1}v_{n,i-1},w_x)_{\mathscr{K}_h} - \tfrac{\beta}{4}\langle \zeta_{n,i-1}\xi_{n,i-1}\mathbf{n} + v_{n,i-1}\xi_{n,i-1}\mathbf{n},w\rangle_{\partial\mathscr{K}_h} \\
& -\tfrac{\gamma}{4}\langle(\zeta)^2_{n,i-1}\mathbf{n},w\rangle_{\partial\mathscr{K}_h} - \tfrac{1}{2}\langle \sigma_{n,i-1}v_{n,i-1},w\rangle_{\partial\mathscr{K}_h} + \tfrac{1}{2}\langle \sigma_{n,i-1}\zeta_{n,i-1},w\rangle_{\partial\mathscr{K}_h} \\
& +\tfrac{1}{\Delta t}(v^{n-1},w_3)_{\mathscr{K}_h} + \tfrac{1}{2}(q_x^{n-1},w_3)_{\mathscr{K}_h} + \tfrac{\gamma}{4}((v^2)^{n-1},w_{3x})_{\mathscr{K}_h} + \\
& \tfrac{\beta}{2}(u^{n-1}v^{n-1},w_{3x})_{\mathscr{K}_h} - \tfrac{\gamma}{4}\langle(\zeta^2)^{n-1}\mathbf{n},w_3\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{v}}} - \tfrac{\beta}{4}\langle \xi^{n-1}v^{n-1}\mathbf{n},w_3\rangle_{\partial\mathscr{K}_h} \\
& -\tfrac{\beta}{4}\langle \xi^{n-1}\zeta^{n-1}\mathbf{n},w_3\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{v}}} - \tfrac{1}{2}\langle \sigma^{n-1}v^{n-1},w_3\rangle_{\partial\mathscr{K}_h} \\
& +\tfrac{1}{2}\langle \sigma^{n-1}\lambda_v^{n-1},w_3\rangle_{\partial\mathscr{K}_h\backslash\Gamma_{\mathfrak{v}}},
\end{aligned}$$

$$\tilde{\mathsf{l}}_4(w) = \quad -(q_{n,i-1},w)_{\mathscr{K}_h} - (v_{n,i-1},w_x)_{\mathscr{K}_h} + \langle \zeta_{n,i-1}\mathbf{n},w\rangle_{\partial\mathscr{K}_h},$$

$$\begin{aligned}
\tilde{\mathsf{l}}_5(\mu) = {} & \langle (b_{\mathfrak{p}})_{n,i-1},\mu\rangle_{\eta_{\mathfrak{p}}} - \tfrac{\gamma}{2}\langle(\xi)^2_{n,i-1}\mathbf{n},\mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^\partial} - \tfrac{\alpha}{2}\langle \xi_{n,i-1}\zeta_{n,i-1}\mathbf{n},\mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^\partial} + \\
& \langle p_{n,i-1}\mathbf{n},\mu\rangle_{\partial\mathscr{K}_h} - \tfrac{\alpha}{2}\langle u_{n,i-1}\zeta_{n,i-1}\mathbf{n},\mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^\partial} - \langle \tau_{n,i-1}u_{n,i-1},\mu\rangle_{\partial\mathcal{T}_h\backslash\mathcal{F}_h^\partial} \\
& +\langle \tau_{n,i-1}\xi_{n,i-1},\mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^\partial},
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathsf{l}}_6(\mu) = {} & \langle (b_{\mathfrak{q}})_{n,i-1},\mu\rangle_{\Gamma_{\mathfrak{q}}} - \tfrac{\gamma}{2}\langle(\zeta)^2_{n,i-1}\mathbf{n},\mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^\partial} - \tfrac{\beta}{2}\langle \xi_{n,i-1}\zeta_{n,i-1}\mathbf{n},\mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^\partial} + \\
& \langle q_{n,i-1}\mathbf{n},\mu\rangle_{\partial\mathscr{K}_h} - \tfrac{\beta}{2}\langle v_{n,i-1}\xi_{n,i-1}\mathbf{n},\mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^\partial} - \langle \sigma_{n,i-1}v_{n,i-1},\mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^\partial} \\
& +\langle \sigma_{n,i-1}\zeta_{n,i-1},\mu\rangle_{\partial\mathscr{K}_h\backslash\mathcal{F}_h^\partial}.
\end{aligned}$$

To solve the huge and sparse linear weak form (25), we decompose it to the corresponding linear systems with smaller size by applying the Schur complement scheme. It can be shown that (25) can be reformulated to the following vector-matrix equations

$$\begin{cases} M_{11}X_{n,i} + M_{12}Y_{n,i} = \mathcal{R}_1, \\ M_{21}X_{n,i} + M_{22}Y_{n,i} = \mathcal{R}_2, \end{cases} \tag{26}$$

where $X_{n,i} = [\delta\bar{u}_{n,i}\ \delta\bar{p}_{n,i}\ \delta\bar{v}_{n,i}\ \delta\bar{q}_{n,i}]^T$, $Y_{n,i} = [\delta\bar{\xi}_{n,i}\ \delta\bar{\zeta}_{n,i}]^T$,

$$M_{11} = \begin{bmatrix} \tilde{A}_1 & -\frac{1}{2}B_1^T & -\tilde{A}_2 & 0 \\ \tilde{B}_1 & \tilde{B}_2 & 0 & 0 \\ -\tilde{C}1 & 0 & \tilde{C}2 & -\frac{1}{2}B_1^T \\ 0 & 0 & \tilde{B}_1 & \tilde{B}2 \end{bmatrix},\ M_{12} = \begin{bmatrix} \tilde{A}_4 & \tilde{A}_5 \\ -\tilde{B}3 & 0 \\ \tilde{C}_3 & \tilde{C}_4 \\ 0 & -\tilde{B}4 \end{bmatrix},$$

$$M_{22} = \begin{bmatrix} \tilde{D}_3 & \tilde{D}_4 \\ \tilde{E}_2 & \tilde{E}_3 \end{bmatrix}, \; M_{21} = \begin{bmatrix} \tilde{D}_1 & -\tilde{D}_2 & -E_3 & 0 \\ 0 & 0 & \tilde{E}_1 & -\tilde{D}_2 \end{bmatrix},$$

$$\mathcal{R}_1 = \begin{bmatrix} L_1 & L_2 & L_3 & L_4 \end{bmatrix}, \qquad \mathcal{R}_2 = \begin{bmatrix} L_5 & L_6 \end{bmatrix},$$

In the above decompositions, capital letters are the matrix and vector representation of multi-linear forms and linear functionals which are given in (25) to standard bases functions. Regarding to (26) and the Schur complement technique, the following system of equations has to be solved

$$(M_{22} - M_{21}M_{11}^{-1}M_{12})Y_{n,i} = \mathcal{R}_2 - M_{21}M_{11}^{-1}\mathcal{R}_1, \tag{27}$$

then $X_{n,k}$ can be gained by

$$X_{n,i} = M_{11}^{-1}\mathcal{R}_1 - M_{11}^{-1}M_{12}Y_{n,i}, \tag{28}$$

and finally set $\delta W_{n,i} = [X_{n,i}, Y_{n,i}]^T$.
Regarding the HDG method, Newton-Raphson approach, and Schur complement decomposition, one can design numerical HDG scheme.

5. Numerical experiments

In this section, we intend to demonstrate the validation and efficiency of the proposed HDG method. Numerical results of the HDG method are shown in four examples so that the CBE with specific boundary conditions are considered in [12, 14]. In the first example, we intend to investigate the numerical spatial and temporal order of accuracy for the HDG method. In the second example, we test the HDG method on the non-homogeneous CBE. In the third example, the motion of the single soliton wave will be tested. And finally, in the fourth example, the behaviour of the solution of the nonlinear system of equations (1) will be verified with different coefficients.

Example 5.1. We consider nonlinear system of equations (1) over $\Omega = [-\pi, \pi]$, with $\alpha = \beta = 1$, and $\eta = \gamma = -2$ where exact solutions are $\mathfrak{u}(x,t) = \mathfrak{v}(x,t) = \exp(-t)\sin(x)$ [14]. It is clear that $\mathfrak{u}(x,t)$ and $\mathfrak{v}(x,t)$ satisfy homogeneous Dirichlet boundary conditions at $x = -\pi, \pi$ and $\mathfrak{u}_0(x) = \mathfrak{v}_0(x) = \sin(x)$ are the initial conditions. By setting appropriate small time step sizes and $\tau_0 = \sigma_0 = 2$, L^2 error norms and corresponding numerical spatial orders of accuracy of u, v, and their derivatives at the final time $T = 0.1$ are reported in Table 1 for different mesh sizes and approximate polynomials of degree $k = 0, 1, 2, 3$. It is observed that the approximate solutions, derived by the HDG method, converge to exact solutions with $k+1$ spatial order of accuracy. Also in Table 2, the temporal order of accuracy are shown for approximate polynomial of degree three and different time step sizes. As expected, due to utilization the Newton-Raphson and the Crank-Nicolson methods, the temporal order of accuracy is two.

Table 1: L^2 error norms and corresponding spatial orders of accuracy for Example 5.1.

k	N	$\|u-\mathfrak{u}\|_\Omega$	$\|v-\mathfrak{v}\|_\Omega$	order	$\|p-\mathfrak{p}\|_\Omega$	$\|q-\mathfrak{q}\|_\Omega$	order
0	10	2.9820 E-1	2.9820 E-1		4.0312 E-1	4.0312 E-1	
	20	1.4916 E-1	1.4916 E-1	1.00	1.9685 E-1	1.9685 E-1	1.03
	40	7.4476 E-2	7.4476 E-2	1.00	9.7111 E-2	9.7111 E-2	1.02
	80	3.7199 E-2	3.7199 E-2	1.00	4.8271 E-2	4.8271 E-2	1.01
1	10	3.5395 E-2	3.5395 E-2		7.4563 E-2	7.4563 E-2	
	20	7.2460 E-3	7.2460 E-3	2.29	1.9767 E-2	1.9767 E-2	1.92
	40	1.7083 E-3	1.7083 E-3	2.08	5.1862 E-3	5.1862 E-3	1.93
	80	4.2109 E-4	4.2109 E-4	2.02	1.3317 E-3	1.3317 E-3	1.96
2	10	1.9436 E-3	1.9436 E-3		6.0240 E-3	6.0240 E-3	
	20	1.7369 E-4	1.7369 E-4	3.48	4.3302 E-4	4.3302 E-4	3.80
	40	2.1382 E-5	2.1382 E-5	3.02	5.6864 E-5	5.6864 E-5	2.93
	80	2.6667 E-6	2.6667 E-6	3.00	7.2313 E-6	7.2313 E-6	2.98
3	10	9.7931 E-5	9.7931 E-5		3.3861 E-4	3.3861 E-4	
	20	3.3945 E-6	3.3945 E-6	4.85	9.7606 E-6	9.7606 E-6	5.11
	40	2.1353 E-7	2.1353 E-7	3.99	6.2487 E-7	6.2487 E-7	3.97
	80	1.3281 E-8	1.3281 E-8	4.01	3.9355 E-8	3.9355 E-8	3.99

Table 2: L^2 error norms and corresponding temporal orders of accuracy for Example 5.1 with 60 number of elements and approximate polynomial of degree three.

Δt	$\|u-\mathfrak{u}\|_\Omega$	$\|v-\mathfrak{v}\|_\Omega$	order	$\|p-\mathfrak{p}\|_\Omega$	$\|q-\mathfrak{q}\|_\Omega$	order
$\frac{1}{10}$	1.336454 E-4	1.336454 E-4		1.336456 E-4	1.336456 E-4	
$\frac{1}{20}$	3.332776 E-5	3.332776 E-5	2.0036	3.332802 E-5	3.332802 E-5	2.0036
$\frac{1}{40}$	8.306256 E-6	8.306256 E-6	2.0044	8.307134 E-6	8.307134 E-6	2.0043
$\frac{1}{80}$	2.065220 E-6	2.065220 E-6	2.0079	2.066753 E-6	2.066753 E-6	2.0070
$\frac{1}{160}$	5.127025 E-7	5.127025 E-7	2.0101	5.166183 E-7	5.166183 E-7	2.0020

Example 5.2. ([14]). In this example, HDG solutions for simulating motion of two single soliton waves will be examined. Consider the coupled Burgers equation (1) with $\alpha = \beta = -2$, and $\gamma = \eta = 5/2$. The exact solutions of this coupled system is

$$\mathfrak{u}(x,t) = \mathfrak{v}(x,t) = \lambda\Big(1 - \tanh(\frac{3}{2}\lambda(x - 3\lambda t))\Big).$$

It is clear that the boundary and initial conditions can be extracted from the exact solution. Here $\Omega = [-20, 20]$ and we set $\tau_0 = \sigma_0 = 20$. The curve of numerical and analytical solutions for $\lambda = 1$ and $\lambda = 10$ are plotted in Figures 1 and 2, respectively with approximate polynomial of degree one and 500 number of elements.

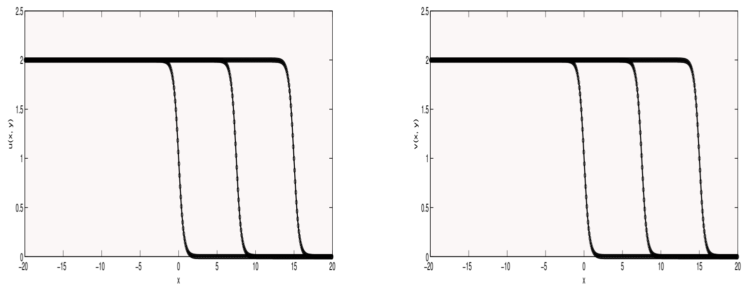

Figure 1: Exact and numerical solutions (lines and dots respectively) of Example 5.2 for Equation (1) with $\eta = \gamma = -2$ and $\alpha = \beta = 5/2$ at $T = 0, 2.5, 5$. The results are reported for $\lambda = 1$, degree of polynomials one and 500 number of elements.

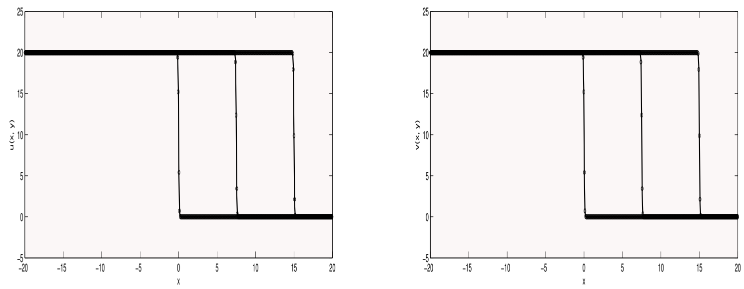

Figure 2: Exact and numerical solutions (lines and dots respectively) of Example 5.2 for Equation (1) with $\eta = \gamma = -2$ and $\alpha = \beta = 5/2$ at $T = 0, 0.25, 0.5$. The results are reported for $\lambda = 10$, degree of polynomials one and 500 number of elements.

Example 5.3. ([12]). In this example, we aim to achieve numerical solution of (1) with zero boundary condition and the following initial conditions

$$\mathfrak{u}(x,0)=\begin{cases}\sin(2\pi x), & 0\le x\le 0.5,\\ 0, & 0.5<x\le 1,\end{cases}\quad , \qquad \mathfrak{v}(x,0)=\begin{cases}0, & 0\le x\le 0.5,\\ -\sin(2\pi x), & 0.5<x\le 1.\end{cases}$$

By setting $\Delta t = 0.001$, the number of partitions as 300, and $\tau_0 = \sigma_0 = 20$ it can be seen the numerical solutions in Figures 3 and 4 for $\eta = \gamma = 2$, $\alpha = \beta = 10$ and $\eta = \gamma = 2$, $\alpha = \beta = 100$, respectively. In addition, in Figures 5 and 6, we show the results for $\alpha = \beta = 10$ and $\eta = \gamma = 20, 200$. It can be concluded that the solution decays to zero with increasing time levels and with increasing the values of η and γ.

6. Conclusion

In this paper, the system of CBE with order two has been studied such that initial and boundary conditions with periodic conditions are enforced to the desired system. By proposing suitable broken Sobolev spaces, the semi-discrete variational formulation of the CBE is set up when the initial system is converted to a first-order system of PDEs. The proposed HDG method is formed by introducing numerical traces and uses for the reduced system from second order to first order. Numerical traces are global unknowns and also depend on boundary conditions that are periodic and homogeneous Dirichlet boundary conditions. Numerical traces depend on the form of the CBE, numerical traces, and stabilization parameters. By defining numerical traces properly and adding enough global equations, the L2-stability of the desired HDG method has been investigated while considering mild conditions on the stability parameters. To gain a fully discrete method, the Crank-Nicolson method is used for temporal discretization. After applying the Crank-Nicolson scheme, the Newton-Raphson method has been exploited to the nonlinear terms discrete weak form. By using the Schur complement idea, the final matrix-vector the equation is split into smaller matrix-vector equations. A similar HDG method has been applied to CBE in two dimensions [27]. L^2-stability of CBE in one dimension, in this work, is more complicated than L^2-stability of CBE in two dimensions. Also, the implementation issues of CBE in one and two dimensions are considerably different. For verifying the applicability of the proposed HDG method, the method is applied to some problems. In fact, in an example, we show that for a mesh with an approximate polynomial of degree k, approximate solutions and their first derivatives converge with the best possible rate $k + 1$. Moreover, the HDG method has dealt with some physical concepts such as the motion of single soliton waves.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

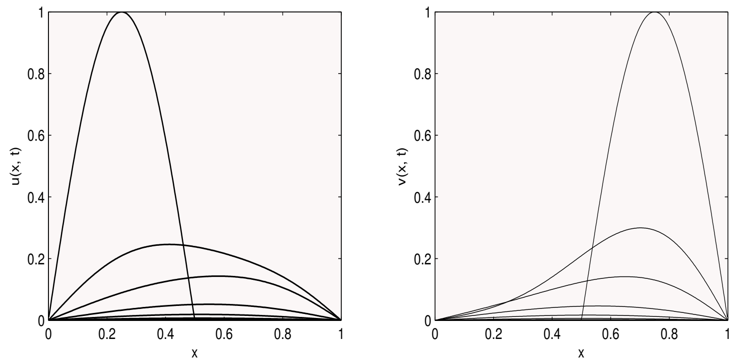

Figure 3: Numerical solutions for Example 5.3 with $\gamma = \eta = 2$ and $\alpha = \beta = 10$ at times $T = 0, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5$ (respectively from top to bottom in the boxes).

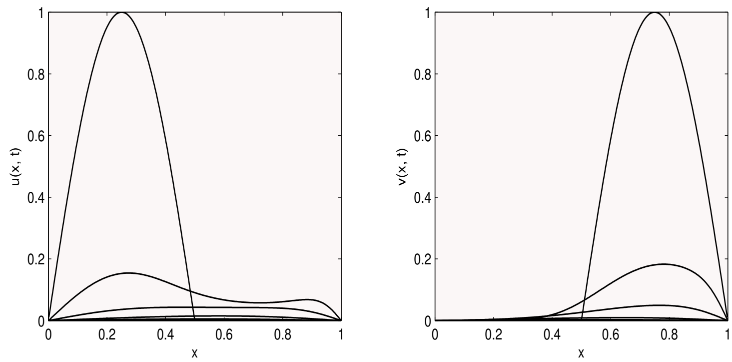

Figure 4: Numerical solutions for Example 5.3 with $\gamma = \eta = 2$ and $\alpha = \beta = 100$ at times $T = 0, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5$ (respectively from top to bottom in the boxes).

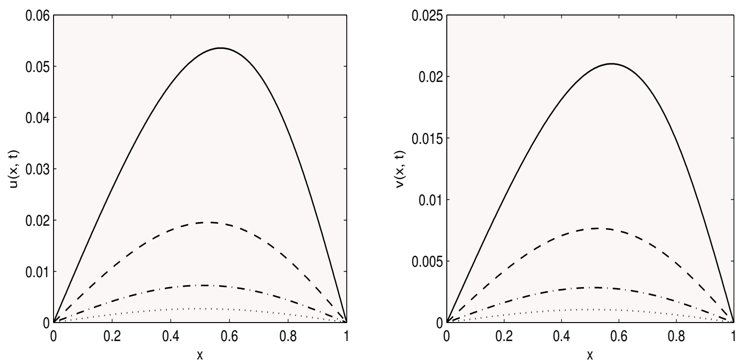

Figure 5: Numerical solutions for Example 5.3 with $\gamma = \eta = 20$ and $\alpha = \beta = 10$ at times $T = 0.2, 0.3, 0.4, 0.5$ (respectively from top to bottom in the boxes).

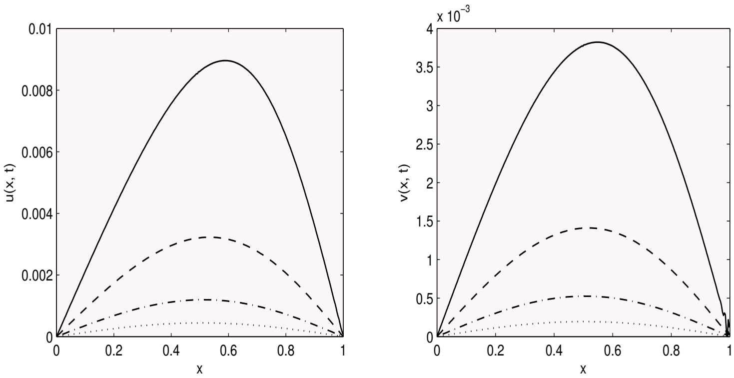

Figure 6: Numerical solutions for Example 5.3 with $\gamma = \eta = 200$ and $\alpha = \beta = 10$ at times $T = 0.2, 0.3, 0.4, 0.5$ (respectively from top to bottom in the boxes).

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