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nX-Complementary Generations of the Chevalley Group $G_2(3)$

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Abstract

A finite non-abelian group G is said to be (l, m, n)-generated if it can be generated by two elements x and y such that $o(x) = l$, $o(y) = m$ and $o(xy) = n$. Also, G is said to be nX-complementary generated if given an arbitrary non-identity element $x \in G$, there exists an element $y \in nX$ such that $G = \langle x, y \rangle$. We studied the (p, q, r)-generation for the Chevalley group $G_2(3)$, where p, q and r are all the primes dividing the order of $G_2(3)$. In the current paper, we classify all the non-trivial conjugacy classes of $G_2(3)$ whether they are complementary generators or not. To achieve this, we mainly used the structure constant method together with other results applied to establish generation and non-generation of the group $G_2(3)$ by the (p, q, r) triples. Some particular algorithms, as well as the (Gap) programming tool, and the Atlas of finite groups have been exploited in our computations.

Keywords: Conjugacy classes, nX-Complementary generation, Structure constant, Chevalley group

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1. Introduction

The generation of finite groups is one of the most interesting problems in Group Theory and has a rich history. A finite group G can be generated in too many different ways. For example the probabilistic generation, $\frac{3}{2}$-generation, (p, q, r)-generations, ranks of non-trivial classes of G, nX-complementary generation, spread of G and many other methods. A finite group G is said to be (l, m, n)-*generated* if $G = \langle x, y \rangle$, with $o(x) = l$, $o(y) = m$ and $o(xy) = o(z) = n$. Here $[x] = lX$, $[y] = mY$ and $[z] = nZ$, where $[x]$ is the conjugacy class of lX in G containing elements of order l. The same applies to $[y]$ and $[z]$. In this case, G is also a quotient group of the triangular group $T(l, m, n)$ and, by definition of the triangular group, G is also a $(\sigma(l), \sigma(m), \sigma(n))$-generated group for any $\sigma \in S_3$. Therefore, we may assume that $l \leq m \leq n$. In the special case, we are more interested in the (p, q, r)-generations where p, q and r are prime numbers that divide the order of the group G. A consequence of the (p, q, r)-generation is the nX-complementary generation.

Definition 1.1. For a non-trivial conjugacy class nX of a finite non-abelian group G, we say that G is nX-*complementary generated* if there exists an element $y \in nX$ such that $G = \langle x, y \rangle$ for any $x \in G$. We say y is a *complementary*.

The motivation of studying this kind of generation comes from a conjecture by Brenner-Guralnick-Wiegold [1] that every finite simple group can be generated by an arbitrary non-trivial element together with another suitable element.

In a series of papers [2–9], the nX-complementary generations of the sporadic simple groups Th, Co_1, J_1, J_2, J_3, HS, McL, Co_3, Co_2 and F_{22} have been investigated.

In this paper, we intend to establish all the nX-complementary generations of an exceptional group of Lie type, namely, the Chevalley group $G_2(3)$, where nX is a non-trivial conjugacy class of elements of order n as in the Atlas [10]. We follow the methods used in the papers [11–19] and [20]. Note that, in general, if G is a $(2, 2, n)$-generated group then G is a dihedral group and therefore, G is not simple. Also by [21], if G is a non-abelian (l, m, n)-generated group then either $G \cong A_5$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Thus, for our purpose of establishing the nX-complementary generations of $G = G_2(3)$, the only cases we need to consider are when $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.

The following proposition gives a criterion for a group G to be nX-complementary generated or not, where nX is a non-trivial class of G.

Proposition 1.2. *A finite non-abelian group G is nX-complementary generated if and only if for each conjugacy class pY of G, where p is prime, there exists a conjugacy class $t_{pY}Z$, depending on pY, such that G is $(pY, nX, t_{pY}Z)$-generated. Moreover, if G is a finite simple group then G is not $2X$-complementary generated for any conjugacy class of involutions.*

Proof. See Lemma 2.3.8 of [22]. □

The main result on the nX-complementary generation of the group $G_2(3)$ can be summarized in Theorem 1.3. The proof will be established through a sequence of propositions that will be proved in Section 3.

Theorem 1.3. *The group $G_2(3)$ is nX-complementary generated if and only if $n \geq 6$ and $nX \notin \{6A, 6B\}$.*

2. Preliminaries

Let G be a finite group and $C_1, C_2, \ldots, C_k$ (not necessarily distinct) for $k \geq 3$ be conjugacy classes of G with $g_1, g_2, \ldots, g_k$ being representatives for these classes, respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i$, $1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1}) \in C_1 \times C_2 \times \ldots \times C_{k-1}$ such that $g_1 g_2 \ldots g_{k-1} = g_k$. This number is known as the *class algebra constant* or *structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_r\}$, the number Δ_G is easily calculated from the character table of G using Equation (1),

$$\Delta_G(C_1, C_2, \ldots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\ldots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}. \qquad (1)$$

Also, for a fixed $g_k \in C_k$, we denote by $\Delta_G^*(C_1, C_2, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1})$ satisfying

$$g_1 g_2 \ldots g_{k-1} = g_k \quad \text{and} \quad G = \langle g_1, g_2, \ldots, g_{k-1} \rangle. \qquad (2)$$

Definition 2.1. If $\Delta_G^*(C_1, C_2, \ldots, C_k) > 0$ then the group G is said to be $(C_1, C_2, \ldots, C_k)$**-generated**.

Furthermore, if H is any subgroup of G containing the fixed element $g_k \in C_k$, we let $\Sigma_H(C_1, C_2, \ldots, C_k)$ be the total number of distinct tuples $(h_1, h_2, \ldots, h_{k-1})$ which are in $C_1 \times C_2 \times \ldots \times C_{k-1}$ such that

$$h_1 h_2 \ldots h_{k-1} = g_k \quad \text{and} \quad \langle h_1, h_2, \ldots, h_{k-1} \rangle \leq H. \qquad (3)$$

The value of $\Sigma_H(C_1, C_2, \ldots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \ldots, c_k)$ of H-conjugacy classes $c_1, c_2, \ldots, c_k$ such that $c_i \subseteq H \cap C_i$.

Finally, for non-trivial conjugacy classes $c_1, c_2, \ldots, c_k$ of a proper subgroup H of G and a fixed $g_k \in c_k$, let $\Sigma_H^*(c_1, c_2, \ldots, c_k)$ represents the number of tuples $(h_1, h_2, \ldots, h_{k-1}) \in c_1 \times c_2 \times \ldots \times c_{k-1}$ such that $h_1 h_2 \ldots h_{k-1} = g_k$ and $\langle h_1, h_2, \ldots, h_{k-1} \rangle = H$.

When it is clear from the context which conjugacy classes of H are considered, we will use the notation $\Sigma(H)$ and $\Sigma^*(H)$ to denote $\Sigma_H(c_1, c_2, \ldots, c_k)$ and $\Sigma_H^*(c_1, c_2, \ldots, c_k)$, respectively.

Theorem 2.2. *Let G be a finite group and H be a subgroup of G containing a fixed element g such that* $\gcd(o(g), [N_G(H){:}H]) = 1$. *Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular,*

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|}, \tag{4}$$

where $x_1, x_2, \ldots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof. See [5] and [23, Theorem 2.1]. □

The above number $h(g, H)$ is useful in giving a lower bound for $\Delta^*_G(C_1, C_2, \ldots, C_k)$, namely $\Delta^*_G(C_1, C_2, \ldots, C_k) \geq \Theta_G(C_1, C_2, \ldots, C_k)$, where

$$\Theta_G(C_1, C_2, \ldots, C_k) = \Delta_G(C_1, C_2, \ldots, C_k) - \sum h(g_k, H)\Sigma_H(C_1, C_2, \ldots, C_k), \tag{5}$$

such that g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups of G containing elements of all the classes $C_1, C_2, \ldots, C_k$.

The following lemma in many cases will be very useful in establishing non-generation for finite groups.

Lemma 2.3 (e.g. see Lemma 2.7 of [17])**.** *Let G be a finite centerless group. If $\Delta^*_G(C_1, C_2, \ldots, C_k) < |C_G(g_k)|$ and $g_k \in C_k$ then $\Delta^*_G(C_1, C_2, \ldots, C_k) = 0$ and therefore, G is not $(C_1, C_2, \ldots, C_k)$-generated.*

3. The results on the nX-complementary generations of $G_2(3)$

In this section we apply the results discussed in Section 2 to the Chevalley group $G_2(3)$. We determine the non-trivial conjugacy classes nX such that $G_2(3)$ is nX-complementary generated. The group $G_2(3)$ is a simple group of order $4245696 = 2^6 \times 3^6 \times 7 \times 13$. By the Atlas [10], the group $G_2(3)$ has Schur multiplier isomorphic to $\mathbb{Z}_3$ and outer automorphism group that is isomorphic to $\mathbb{Z}_2$. Also, it has exactly 23 conjugacy classes of its elements and 10 conjugacy classes of its maximal subgroups. Representatives of these classes of maximal subgroups can be taken as in Table 1.

Throughout the paper, the notation for the conjugacy classes of elements and maximal subgroups of $G_2(3)$ will be as in the Atlas [10]. Using Equation (4) we calculated the values of $h(g, M_i)$, where g is a representative of a non-trivial

Table 1: Maximal subgroups of $G_2(3)$.

Maximal Subgroup	Order
$U_3(3):2 = M_1$	$12096 = 2^6 \times 3^3 \times 7$
$U_3(3):2 = M_2$	$12096 = 2^6 \times 3^3 \times 7$
$(3^2 \times 3^{1+2}):2S_4 = M_3$	$11664 = 2^4 \times 3^6$
$(3^2 \times 3^{1+2}):2S_4 = M_4$	$11664 = 2^4 \times 3^6$
$L_3(3):2 = M_5$	$11232 = 2^5 \times 3^3 \times 13$
$L_3(3):2 = M_6$	$11232 = 2^5 \times 3^3 \times 13$
$L_2(8):3 = M_7$	$1512 = 2^3 \times 3^3 \times 7$
$2^3.L_3(2) = M_8$	$1344 = 2^6 \times 3 \times 7$
$L_2(13) = M_9$	$1092 = 2^2 \times 3 \times 7 \times 13$
$2^{1+4}:3^2:2 = M_{10}$	$576 = 2^6 \times 3^2$

Table 2: The values $h(g, M_i)$, $1 \leq i \leq 10$ for non-identity classes and maximal subgroups of $G_2(3)$.

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}
$2A$	15	15	20	20	18	18	24	39	48	91
$3A$	27	0	13	40	0	54	0	0	0	81
$3B$	0	27	40	13	54	0	0	0	0	81
$3C$	0	0	13	13	0	0	27	0	0	0
$3D$	0	0	4	4	9	9	0	0	27	9
$3E$	9	9	4	4	0	0	18	27	0	9
$4A$	3	3	4	0	6	2	0	3	0	7
$4B$	3	3	0	4	2	6	0	3	0	7
$6A$	3	0	5	8	0	6	0	0	0	1
$6B$	0	3	8	5	6	0	0	0	0	1
$6C$	0	0	2	2	3	3	0	0	3	1
$6D$	3	3	2	2	0	0	6	3	0	1
$7A$	1	1	0	0	0	0	1	2	3	0
$8A$	1	1	2	0	2	0	0	1	0	1
$8B$	1	1	0	2	0	2	0	1	0	1
$9A$	0	0	1	1	0	0	3	0	0	0
$9B$	0	0	1	1	0	0	3	0	0	0
$9C$	0	0	1	1	0	0	3	0	0	0
$12A$	3	0	1	0	0	2	0	0	0	1
$12B$	0	3	0	1	2	0	0	0	0	1
$13A$	0	0	0	0	1	1	0	0	1	0
$13B$	0	0	0	0	1	1	0	0	1	0

conjugacy class of G, over all the maximal subgroups M_i of G. We list these values in Table 2.

Remark 1. It has been mentioned in Proposition 1.2 that a finite simple group G can not be $2X$-complementary generated, for if it were then there exists a conjugacy class nZ of G such that G is a $(2Y, 2X, nZ)$-generated group. We know that two involutions generate a dihedral group, which is not a simple group. Therefore, if G is a simple group then it is not $2X$-complementary generated for any conjugacy class $2X$ of involutions of G. Hence, the investigation of the nX-complementary generation in simple will be done when $n \geq 3$.

Proposition 3.1. *The group $G_2(3)$ is not $3X$-complementary generated for $X \in \{A, B\}$.*

Proof. For the case $(2A, 3X, tZ)$, we need only check the conjugacy classes of $G_2(3)$ with elements of orders greater than or equal to 7 because of the condition $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. From Proposition 6 of [24], we know that $G_2(3)$ is not $(2A, 3X, 7A)$-generated. Computations with GAP [25] yield $\Delta_{G_2(3)}(2A, 3X, tZ) = 0$ for $tZ \in \{8Y, 9A, 9B, 9C, 12X\}$, where $X, Y \in \{A, B\}$ and $X \neq Y$. We also find that $\Delta_{G_2(3)}(2A,\ 3X, 8Y) = 4 < 8 = |C_G(g)|,\ g \in 8Y$ and $\Delta_{G_2(3)}(2A, 3X, 12X) = 3 < 12 = |C_G(g)|,\ g \in 12X$, where $X, Y \in \{A, B\}$ and $X \neq Y$. Using Lemma 2.3, we see that $G_2(3)$ is not $(2A, 3X, tZ)$-generated for $X \in \{A, B\}$ and $tZ \in \{8A, 8B, 9A, 9B, 9C, 12A, 12B\}$. Finally, that $G_2(3)$ is not $(2A, 3X, 13Y)$-generated follows from Proposition 7 of [24]. Therefore, $G_2(3)$ is not $(2A, 3X, tZ)$-generated for every conjugacy class tZ of $G_2(3)$ and hence, it is not $3X$-complementary generated for $X \in \{A, B\}$. □

Proposition 3.2. *The group $G_2(3)$ is not $3C$-complementary generated.*

Proof. Even though, by Proposition 7 of [24], the group $G_2(3)$ is $(2A, 3C, 13Y)$-generated we will prove that it is not $(3A, 3C, tZ)$-generated for all conjugacy classes tZ of $G_2(3)$. Firstly, we note that for this case the condition $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$ is satisfied when $t \geq 4$. Hence, $G_2(3)$ is not $(3A, 3C, tZ)$-generated for classes $tZ \in \{2A, 3A, 3B, 3C, 3D, 3E\}$. Now, computations with GAP yield $\Delta_{G_2(3)}(3A, 3C, tZ) = 0$ for $tZ \in \{4X, 6X, 6D, 7A, 8X, 9A, 12B\}$, where $X \in \{A, B\}$. We also find that

$$\begin{aligned}
\Delta_{G_2(3)}(3A, 3C, 6C) &= 6 < 18 = |C_{G_2(3)}(g)|,\ g \in 6C, \\
\Delta_{G_2(3)}(3A, 3C, 9X) &= 3 < 27 = |C_{G_2(3)}(g)|,\ g \in 9X, X \in \{B, C\}, \\
\Delta_{G_2(3)}(3A, 3C, 12A) &= 4 < 12 = |C_{G_2(3)}(g)|,\ g \in 12A.
\end{aligned}$$

It follows by Lemma 2.3 that $G_2(3)$ is neither $(3A, 3C, 6C)$-, $(3A, 3C, 9X)$-, nor $(3A, 3C, 12A)$-generated for $X \in \{B, C\}$. By Proposition 15 of [24], we have that $G_2(3)$ is not $(3A, 3C, 13X)$-generated for $X \in \{A, B\}$. We can see that $G_2(3)$ is not $(3A, 3C, tZ)$-generated for all conjugacy classes tZ of $G_2(3)$. Thus, $G_2(3)$ is not $3C$-complementary generated. □

Proposition 3.3. *The group $G_2(3)$ is not $3D$-complementary generated.*

Proof. We achieve the result by showing that $G_2(3)$ is not $(3A, 3D, tZ)$-generated for all conjugacy classes tZ of $G_2(3)$. Just as in the proof of the preceding proposition we note that the condition $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$ is satisfied when $t \geq 4$. Hence, $G_2(3)$

is not $(3A, 3D, tZ)$-generated for classes $tZ \in \{2A, 3A, 3B, 3C, 3D, 3E\}$. Now, computations with GAP give $\Delta_{G_2(3)}(3A, 3D, tZ) = 0$ for $tZ \in \{4A, 6B, 6C, 7A, 8A, 12A\}$. Further computations show that

$$\begin{aligned}
\Delta_{G_2(3)}(3A, 3D, 4B) &= 32 < 96 = |C_{G_2(3)}(g)|,\ g \in 4B,\\
\Delta_{G_2(3)}(3A, 3D, 6A) &= 24 < 72 = |C_{G_2(3)}(g)|,\ g \in 6A,\\
\Delta_{G_2(3)}(3A, 3D, 6D) &= 3 < 18 = |C_{G_2(3)}(g)|,\ g \in 6D,\\
\Delta_{G_2(3)}(3A, 3D, 9X) &= 3 < 27 = |C_{G_2(3)}(g)|,\ g \in 9X, X \in \{A, B, C\},\\
\Delta_{G_2(3)}(3A, 3D, 12B) &= 2 < 12 = |C_{G_2(3)}(g)|,\ g \in 12B.
\end{aligned}$$

By applying Lemma 2.3, non-generation of the group $G_2(3)$ by $(3A, 3D, tZ)$, where $tZ \in \{4B, 6A, 6D, 9A, 9B, 9C, 12B\}$, is obtained.

For $(3A, 3D, 8B)$, we find that only the maximal subgroups M_4, M_6 and M_{10} have conjugacy classes of elements of orders 3, 3 and 8 that fuse into classes $3A$, $3D$ and $8B$, respectively. We have $\Sigma(M_4) = 0 = \Sigma(M_{10})$ and $\Sigma(M_6) = 8 = \Sigma^*(M_{61})$, where M_{61} (isomorphic to $PSL(3,3)$) is the only maximal subgroup of M_6 with a contribution. Using Equation (4), we found that the number of conjugate subgroups of M_{61} in M_6 that contain a fixed element $z \in 8B$ is 1. Thus, $\Sigma^*(M_6) = \Sigma(M_6) - 1 \cdot \Sigma^*(M_{61}) = 8 - 8 = 0$. Non-generation follows from the computations

$$\begin{aligned}
\Delta^*_{G_2(3)}(3A, 3D, 8B) &= \Delta_{G_2(3)}(3A, 3D, 8B) - 2 \cdot \Sigma^*(M_6) - 1 \cdot \Sigma^*(M_{61})\\
&= 8 - 2(0) - 8 = 0.
\end{aligned}$$

Lastly, for $(3A, 3D, 13X)$ where $X \in \{A, B\}$, we have non-generation by Proposition 15 of [24]. □

Proposition 3.4. *The group $G_2(3)$ is not $3E$-complementary generated.*

Proof. We show that $G_2(3)$ is not $(3A, 3E, tZ)$-generated for all conjugacy classes tZ of $G_2(3)$. Again, applying the condition $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$ shows that t must be greater than or equals to 4. Hence, $G_2(3)$ is not $(3A, 3E, tZ)$-generated for classes $tZ \in \{2A, 3A, 3B, 3C, 3D, 3E\}$. Now, computations with GAP give $\Delta_{G_2(3)}(3A, 3E, tZ) = 0$ for $tZ \in \{4A, 4B, 6A, 6B, 8B, 13A, 13B\}$. Further computations show that

$$\begin{aligned}
\Delta_{G_2(3)}(3A, 3E, 6C) &= 3 < 18 = |C_G(g)|,\ g \in 6C,\\
\Delta_{G_2(3)}(3A, 3E, 6D) &= 6 < 18 = |C_G(g)|,\ g \in 6D,\\
\Delta_{G_2(3)}(3A, 3E, 9X) &= 3 < 27 = |C_G(g)|,\ g \in 9X, X \in \{A, B, C\},\\
\Delta_{G_2(3)}(3A, 3E, 12B) &= 6 < 12 = |C_G(g)|,\ g \in 12B.
\end{aligned}$$

Application of Lemma 2.3 reveals that $G_2(3)$ is not generated by the triples $(3A, 3E, tZ)$, where $tZ \in \{6C, 6D, 9A, 9B, 9C, 12B\}$.

By Proposition 11 of [24] the triple $(3A, 3E, 7A)$ does not generate $G_2(3)$.

For $(3A, 3E, 8A)$, we have only the maximal subgroups M_1, M_3 and M_{10} with conjugacy classes of elements of orders 3, 3 and 8 that fuse into classes $3A, 3E$ and $8A$, respectively. Computations give $\Delta_{G_2(3)}(3A, 3E, 8A) = 8$, $\Sigma(M_3) = 0 = \Sigma(M_{10})$ and $\Sigma(M_1) = 8 = \Sigma^*(M_{11})$, where M_{11} (isomorphic to $PSU(3, 3)$) is the only maximal subgroup of M_1 which makes a contribution. Thus, $\Sigma^*(M_1) = \Sigma(M_1) - 1 \cdot \Sigma^*(M_{11}) = 8 - 8 = 0$. Therefore,

$$\begin{aligned} \Delta^*_{G_2(3)}(3A, 3E, 8A) &= \Delta_{G_2(3)}(3A, 3E, 8A) - 1 \cdot \Sigma^*(M_1) - 1 \cdot \Sigma^*(M_{11}) \\ &= 8 - 1(0) - 8 = 0, \end{aligned}$$

showing that $G_2(3)$ is not $(3A, 3E, 8A)$-generated.

For the last case $(3A, 3E, 12A)$ we still have only the maximal subgroups M_1, M_3 and M_{10} with conjugacy classes of elements of orders 3, 3 and 12 that fuse into classes $3A, 3E$ and $12A$, respectively. We get $\Sigma(M_3) = 0 = \Sigma(M_{10})$ and $\Sigma(M_1) = 12 = \Sigma^*(M_{11})$, where M_{11} is the only maximal subgroup of M_1 which makes a contribution. Thus, $\Sigma^*(M_1) = \Sigma(M_1) - 1 \cdot \Sigma^*(M_{11}) = 12 - 12 = 0$. Non-generation follows from the computation

$$\begin{aligned} \Delta^*_{G_2(3)}(3A, 3E, 12A) &= \Delta_{G_2(3)}(3A, 3E, 12A) - 3 \cdot \Sigma^*(M_1) - 1 \cdot \Sigma^*(M_{11}) \\ &= 12 - 3(0) - 12 = 0. \end{aligned}$$

Therefore, $G_2(3)$ is not $(3A, 3E, tZ)$-generated for every conjugacy class tZ of $G_2(3)$ and hence, it is not $3E$-complementary generated. □

Proposition 3.5. *The group $G_2(3)$ is not $4X$-complementary generated for $X \in \{A, B\}$.*

Proof. Direct computations with GAP yield $\Delta_{G_2(3)}(3X, 4Y, tZ) = 0$ for all $tZ \in \{3Y, 3C, 3E, 6Y, 6C, 12A, 12B\}$, where $X, Y \in \{A, B\}$ with $X \neq Y$, from which we deduce non-generation. The condition $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$ implies non-generation of $G_2(3)$ by $(3X, 4Y, 2A)$. Furthermore,

$$\begin{aligned} \Delta_{G_2(3)}(3X, 4Y, 3X) &= 243 < 5832 = |C_G(g)|, \ g \in 3X, \\ \Delta_{G_2(3)}(3X, 4Y, 3D) &= 54 < 162 = |C_G(g)|, \ g \in 3D, \\ \Delta_{G_2(3)}(3X, 4Y, 4X) &= 24 < 96 = |C_G(g)|, \ g \in 4X, \\ \Delta_{G_2(3)}(3X, 4Y, 4Y) &= 32 < 96 = |C_G(g)|, \ g \in 4Y, \\ \Delta_{G_2(3)}(3X, 4Y, 6X) &= 27 < 72 = |C_G(g)|, \ g \in 6X, \\ \Delta_{G_2(3)}(3X, 4Y, 8X) &= 2 < 8 = |C_G(g)|, \ g \in 8X, \\ \Delta_{G_2(3)}(3X, 4Y, 9Z) &= 9 < 27 = |C_G(g)|, \ g \in \{9A, 9B, 9C\}, \end{aligned}$$

for all $X, Y \in \{A, B\}$ and $X \neq Y$. It follows from Lemma 2.3 that $G_2(3)$ is not generated by $(3X, 4Y, tZ)$ for $tZ \in \{3X, 3D, 4X, 4Y, 6X, 8X, 9A, 9B, 9C\}$ with $X, Y \in \{A, B\}$ and $X \neq Y$.

We now consider the case $(3X, 4Y, 6D)$ for $X, Y \in \{A, B\}$ and $X \neq Y$. Let

$$(i, j) \quad = \quad \begin{cases} (4,1), & \text{if } X = A, \\ (3,2), & \text{if } X = B. \end{cases}$$

We have $\Delta_{G_2(3)}(3X, 4Y, 6D) = 18$ and $\Sigma(M_i) = 18 = \Sigma(M_{i1}) = 18$, where M_{i1} is isomorphic to the group $((3^2 \times ((3^2{:}3)){:}Q_8)){:}3$ for $i = 3, 4$. Maximal subgroups M_j and M_{10} make no contribution since $\Sigma(M_j) = 0 = \Sigma(M_{10})$. Thus, $\Sigma^*(M_i) = \Sigma(M_i) - 1 \cdot \Sigma(M_{i1}) = 18 - 18 = 0$ and so,

$$\begin{aligned} \Delta^*_{G_2(3)}(3X, 4Y, 6D) &= \Delta_{G_2(3)}(3X, 4Y, 6D) - 2 \cdot \Sigma(M_i) - 1 \cdot \Sigma^*(M_{i1}) \\ &= 18 - 2(0) - 18 = 0, \end{aligned}$$

showing non-generation for $X, Y \in \{A, B\}$ and $X \neq Y$.

For the case $(3X, 4Y, 7A)$, we have $\Delta_{G_2(3)}(3X, 4Y, 7A) = 7$ for $X, Y \in \{A, B\}$ and $X \neq Y$. Further, $\Sigma^*(M_i) = \Sigma(M_i) = 7$ with $i = 1$ when $X = A$ and $i = 2$ when $X = B$. Now,

$$\Delta^*_{G_2(3)}(3X, 4Y, 7A) = \Delta_{G_2(3)}(3X, 4Y, 7A) - 1 \cdot \Sigma(M_i) = 7 - 7 = 0,$$

and non-generation by the triple $(3X, 4Y, 7A)$ follows, where $X \in \{A, B\}$ and $X \neq Y$.

Now, we deal with the case $(3X, 4Y, 8Y)$, where $X, Y \in \{A, B\}$ and $X \neq Y$. Let

$$(i, j) \quad = \quad \begin{cases} (5,1) \text{ or } (5,4), & \text{if } X = A, \\ (6,2) \text{ or } (6,3), & \text{if } X = B. \end{cases}$$

We find that $\Delta_{G_2(3)}(3X, 4Y, 8Y) = 8$, $\Sigma(M_i) = \Sigma^*(M_{i1}) = 8$ and $\Sigma(M_j) = 0 = \Sigma(M_{10})$. Now, $\Sigma^*(M_i) = \Sigma(M_i) - 1 \cdot \Sigma(M_i) = 8 - 8 = 0$. The non-generation by $(3X, 4Y, 8Y)$ follows since

$$\begin{aligned} \Delta^*_{G_2(3)}(3X, 4Y, 8Y) &= \Delta_{G_2(3)}(3X, 4Y, 8Y) - 2 \cdot \Sigma(M_i) - 1 \cdot \Sigma^*(M_{i1}) \\ &= 8 - 2(0) - 8 = 0, \end{aligned}$$

for $X, Y \in \{A, B\}$ with $X \neq Y$.

Finally, we consider the cases $(3X, 4Y, 13Z)$ for $X, Y, Z \in \{A, B\}$ and $X \neq Y$. We deal with the case $(3B, 4A, 13Z)$, $Z \in \{A, B\}$. Only M_5 and its maximal subgroup M_{51} contribute to $\Delta_{G_2(3)}(3B, 4A, 13Z)$. We have $\Sigma(M_{51}) = 13$ which implies that $\Sigma^*(M_{51}) = 13$. Thus, $\Sigma^*(M_5) = \Sigma(M_5) - 1 \cdot \Sigma^*(M_{51}) = 13 - 13$. It follows that

$$\begin{aligned} \Delta^*_{G_2(3)}(3B, 4A, 13Z) &= \Delta_{G_2(3)}(3B, 4A, 13Z) - 1 \cdot \Sigma^*(M_5) - 1 \cdot \Sigma^*(M_{51}) \\ &= 13 - 0 - 13 = 0, \end{aligned}$$

for $Z \in \{A, B\}$ showing non-generation. Similarly, with M_5 and M_{51} replaced by M_6 and M_{61} respectively, we obtain $\Delta^*_{G_2(3)}(3A, 4B, 13Z) = 0$ for $Z \in \{A, B\}$.

We deduce that $G_2(3)$ is not $(3X, 4Y, 13Z)$-generated for $X, Y, Z \in \{A, B\}$ and $X \neq Y$. Therefore, that $G_2(3)$ is not $4Y$-complementary generated follows as a consequence of non-generation of $G_2(3)$ by $(3X, 4Y, tZ)$ for $X, Y \in \{A, B\}$, $X \neq Y$ and all classes tZ of $G_2(3)$. □

Proposition 3.6. *The group $G_2(3)$ is not $6X$-complementary generated for $X \in \{A, B\}$.*

Proof. For $X \in \{A, B\}$, the triple $(3X, 6X, 2A)$ does not generate G since it does not satisfy the condition $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Direct computations with GAP give $\Delta_{G_2(3)}(3X, 6X, tZ) = 0$ for $tZ \in \{3Y, 3C, 3E, 4Y, 6C, 12X\}$, where $X, Y \in \{A, B\}$ and $X \neq Y$. We also have

$$\begin{aligned}
\Delta_{G_2(3)}(3X, 6X, 3X) &= 243 < 5832 = |C_G(g)|,\ g \in 3X,\\
\Delta_{G_2(3)}(3X, 6X, 3D) &= 54 < 162 = |C_G(g)|,\ g \in 3D,\\
\Delta_{G_2(3)}(3X, 6X, 4X) &= 36 < 96 = |C_G(g)|,\ g \in 4X,\\
\Delta_{G_2(3)}(3X, 6X, 6Y) &= 24 < 72 = |C_G(g)|,\ g \in 6Y,\\
\Delta_{G_2(3)}(3X, 6X, 6X) &= 41 < 72 = |C_G(g)|,\ g \in 6X,\\
\Delta_{G_2(3)}(3X, 6X, 6D) &= 6 < 18 = |C_G(g)|,\ g \in 6D,\\
\Delta_{G_2(3)}(3X, 6X, 9Z) &= 9 < 27 = |C_G(g)|,\ g \in 9Z, \text{ where } Z \in \{A, B, C\},
\end{aligned}$$

for all $X, Y \in \{A, B\}$ and $X \neq Y$. Using Lemma 2.3, we conclude that G is not $(3X, 6X, tZ)$-generated for $tZ \in \{3X, 3D, 4X, 6Y, 6X, 6D, 9A, 9B, 9C\}$ for all $X, Y \in \{A, B\}$ and $X \neq Y$.

Computations with GAP give $\Delta_{G_2(3)}(3A, 6A, 7A) = 7, \Sigma(M_1) = 7 = \Sigma^*(M_{11})$, where M_{11} is the maximal subgroup of M_1 making a contribution. Thus, non-generation by the triple $(3A, 6A, 7A)$ follows from the computations $\Sigma^*(M_1) = \Sigma(M_1) - 1 \cdot \Sigma^*(M_{11}) = 7 - 7 = 0$ and consequently,

$$\begin{aligned}
\Delta^*_{G_2(3)}(3A, 6A, 7A) &= \Delta_{G_2(3)}(3A, 6A, 7A) - 1 \cdot \Sigma^*(M_1) - 1 \cdot \Sigma^*(M_{11})\\
&= 7 - 0 - 7 = 0.
\end{aligned}$$

Similarly, replacing M_1 and M_{11} by M_2 and M_{21} respectively, we get that, $G_2(3)$ is not $(3B, 6B, 7A)$-generated.

For the triple $(3X, 6X, 8X)$, $X \in \{A, B\}$, the involved subgroups of $G_2(3)$ in the computations of $\Delta^*_{G_2(3)}(3X, 6X, 8X)$ will be M_1, M_3, M_{11}, M_{12}, M_{15} and M_{10} if $X = A$ and M_2, M_4, M_{21}, M_{22}, M_{25} and M_{10} if $X = B$. We have $\Delta_{G_2(3)}(3X, 6X, 8X) = 8$, $\Sigma(M_i) = 8 = \Sigma(M_{i1})$ and $\Sigma(M_j) = \Sigma(M_{10}) = \Sigma(M_{i2}) = \Sigma(M_{i5}) = 0$, for $i \in \{1, 2\}$ and $j \in \{3, 4\}$. It follows that $\Sigma^*(M_i) = \Sigma(M_i) - 1 \cdot \Sigma^*(M_{i1}) = 8 - 8 = 0$, for $i \in \{1, 2\}$. Therefore,

$$\begin{aligned}
\Delta^*_{G_2(3)}(3X, 6X, 8X) &= \Delta_{G_2(3)}(3X, 6X, 8X) - 1 \cdot \Sigma^*(M_1) - 1 \cdot \Sigma^*(M_{11})\\
&= 8 - 0 - 8 = 0.
\end{aligned}$$

Thus, the group $G_2(3)$ is not $(3X, 6X, 8X)$-generated for $X \in \{A, B\}$.

Now, we consider the case $(3X, 6X, 8Y)$. For $X, Y \in \{A, B\}$ and $X \neq Y$, there are 16 pairs $(x, y) \in (3X, 6X)$ such that $xy = z$, where $z \in 8Y$ is a fixed element (i.e., $\Delta_{G_2(3)}(3X, 6X, 8Y) = 16$). Now, these 16 pairs generate groups that are all isomorphic to the group $(3^2{:}Q_8){:}S_3$, which has order 432. Hence, none of these 16 pairs generate $G_2(3)$. Therefore, $G_2(3)$ is not $(3X, 6X, 8Y)$-generated for $X, Y \in \{A, B\}$ and $X \neq Y$.

With $\Delta_{G_2(3)}(3A, 6A, 12B) = 12$ and contributions by the involved subgroups as $\Sigma(M_4) = 12 = \Sigma(M_{41})$ and $\Sigma(M_{10}) = \Sigma(M_{42}) = \Sigma(M_{44}) = 0$, non-generation by the triple $(3A, 6A, 12B)$ follows from $\Sigma^*(M_4) = \Sigma(M_4) - 1 \cdot \Sigma^*(M_{41}) = 12 - 12 = 0$ and

$$\begin{aligned}\Delta^*_{G_2(3)}(3A, 6A, 12B) &= \Delta_{G_2(3)}(3A, 6A, 12B) - 1 \cdot \Sigma^*(M_4) - 1 \cdot \Sigma^*(M_{41}) \\ &= 12 - 0 - 12 = 0.\end{aligned}$$

In the above, if we replace M_4 and M_{41} by M_3 and M_{31}, respectively then non-generation of $G_2(3)$ by $(3B, 6B, 12A)$ follows in a very similar way to that of $(3A, 6A, 12B)$.

Finally, for the triple $(3X, 6X, 13Y)$, where $X, Y \in \{A, B\}$, we have the structure constant $\Delta_{G_2(3)}(3X,\ 6X, 13Y) = 13$. The involved subgroups in the computations of $\Delta^*_{G_2(3)}(3X, 6X, 13Y)$ are M_6 and M_{61} if $X = A$ and M_5 and M_{51} if $X = B$. We have $\Sigma(M_i) = 13 = \Sigma^*(M_{i1})$ for $i \in \{5, 6\}$. Thus $\Sigma^*(M_i) = \Sigma(M_i) - 1 \cdot \Sigma^*(M_{i1}) = 13 - 13 = 0$ for $i \in \{5, 6\}$. It follows that

$$\begin{aligned}\Delta^*_{G_2(3)}(3X, 6X, 13Y) &= \Delta_{G_2(3)}(3X, 6X, 13Y) - 1 \cdot \Sigma^*(M_i) - 1 \cdot \Sigma^*(M_{i1}) \\ &= 13 - 0 - 13 = 0,\end{aligned}$$

for $i \in \{5, 6\}$, showing the non-generation of $G_2(3)$ by $(3X, 6X, 13Y)$, where $X, Y \in \{A, B\}$.

Now, since our group $G_2(3)$ is not $(3X, 6X, tZ)$-generated for all the conjugacy classes tZ of G, we deduce that G is not $6X$-complementary generated for $X \in \{A, B\}$. □

For the remaining classes $nX \in \{6C, 6D, 7A, 8A, 8B, 9A, 9B, 9C, 12A, 12B, 13A, 13B\}$ we show that $G_2(3)$ is nX-complementary generated.

Proposition 3.7. *The group $G_2(3)$ is $6C$-complementary generated.*

Proof. Here, we show that $G_2(3)$ is $(pY, 6C, 7A)$-generated for all conjugacy classes pY of $G_2(3)$, where p is a prime number that divides $|G_2(3)|$. From Table 2, we can see that the only maximal subgroup of $G_2(3)$ that contains elements from $6C$ and $7A$ together is M_9 and the corresponding value of h is 3. For the case $(2A, 6C, 7A)$, we have $\Delta_{G_2(3)}(2A, 6C, 7A) = 336$ and $\Sigma^*(M_9) = \Sigma(M_9) = 14$. Therefore,

$$\begin{aligned}\Delta^*_{G_2(3)}(2A, 6C, 7A) &= \Delta_{G_2(3)}(2A, 6C, 7A) - 3 \cdot \Sigma^*(M_9) \\ &= 336 - 3(14) = 294,\end{aligned}$$

showing that $G_2(3)$ is $(2A, 6C, 7A)$-generated. For the case $(3D, 6C, 7A)$, we have $\Delta_{G_2(3)}(3D, 6C, 7A) = 1344$ and $\Sigma^*(M_9) = \Sigma(M_9) = 28$. Therefore,

$$\begin{aligned}\Delta^*_{G_2(3)}(3D, 6C, 7A) &= \Delta_{G_2(3)}(3D, 6C, 7A) - 3 \cdot \Sigma^*(M_9) \\ &= 1344 - 3(28) = 1260,\end{aligned}$$

and thus $G_2(3)$ is $(3D, 6C, 7A)$-generated. For the case $(7A, 6C, 7A)$, we have $\Delta_{G_2(3)}(7A, 6C, 7A) = 34776$ and $\Sigma^*(M_9) = \Sigma(M_9) = 28+28+28 = 84$. Therefore,

$$\begin{aligned}\Delta^*_{G_2(3)}(7A, 6C, 7A) &= \Delta_{G_2(3)}(7A, 6C, 7A) - 3 \cdot \Sigma^*(M_9) \\ &= 34776 - 3(84) = 34524,\end{aligned}$$

showing that $G_2(3)$ is $(7A, 6C, 7A)$-generated. Lastly, with $X \in \{A, B\}$, we find that $G_2(3)$ is $(13X, 6C, 7A)$-generated since

$$\begin{aligned}\Delta^*_{G_2(3)}(13X, 6C, 7A) &= \Delta_{G_2(3)}(13X, 6C, 7A) - 3 \cdot \Sigma^*(M_9) \\ &= 18144 - 3(14) = 18102.\end{aligned}$$

For the remaining cases $3A, 3B, 3C$ and $3E$, we note lack of the required fusion hence generation arise because

$$\begin{aligned}\Delta^*_{G_2(3)}(3X, 6C, 7A) &= \Delta_{G_2(3)}(3X, 6C, 7A) = 28, \text{ for } X \in \{A, B\}, \\ \Delta^*_{G_2(3)}(3C, 6C, 7A) &= \Delta_{G_2(3)}(3C, 6C, 7A) = 280, \text{ and} \\ \Delta^*_{G_2(3)}(3E, 6C, 7A) &= \Delta_{G_2(3)}(3E, 6C, 7A) = 1596.\end{aligned}$$

Since $G_2(3)$ is $(pY, 6C, 7A)$-generated for all the conjugacy classes pY of $G_2(3)$ (p is a prime), it follows by Proposition 1.2 that $G_2(3)$ is $6C$-complementary generated. □

Proposition 3.8. *The group $G_2(3)$ is $6D$-complementary generated.*

Proof. We show that $G_2(3)$ is $(pY, 6D, 13X)$-generated for all conjugacy classes pY of elements of prime order p, where $X \in \{A, B\}$. From GAP, we get the structure constants in Table 3. Now, from Table 3, we see that $\Delta_{G_2(3)}(pY, 6D, 13X) > 0$

Table 3: The structure constants $\Delta_{G_2(3)}(pY, 6D, 13X)$.

pY	$2A$	$3A$	$3B$	$3C$	$3D$	$3E$	$7A$	$13A$	$13B$
$\Delta_{G_2(3)}(pY, 6D, 13X)$	312	52	52	208	1716	1248	33696	19656	19656

for all pY. From Table 2, we can see that no maximal subgroup of G contains elements from $6D$ and $13X$ together for $X \in \{A, B\}$. Therefore, maximal subgroups of $G_2(3)$ make no contribution in the calculations of $\Delta^*_{G_2(3)}(pY, 6D, 13X)$. Hence, $\Delta^*_{G_2(3)}(pY, 6D, 13X) = \Delta_{G_2(3)}(pY, 6D, 13X) > 0$. Therefore, $G_2(3)$ is $(pY, 6D, 13X)$-generated for all the conjugacy classes pY, where p is a prime. It follows by Proposition 1.2 that $G_2(3)$ is $6D$-complementary generated. □

Proposition 3.9. *The group $G_2(3)$ is $7A$-complementary generated.*

Proof. By Propositions 8, 19 and 22 of [24], the triple $(pY, 7A, 7A)$ generates the group $G_2(3)$ for all $pY \in \{2A, 3A, 3B, 3C, 3D, 3E, 7A\}$. Also, by Proposition 24 of [24], $G_2(3)$ is $(7A, 13X, 13Y)$-generated for $X, Y \in \{A, B\}$. Now, if $G_2(3)$ is $(7A, 13X, 13Y)$-generated then it is also $(13X, 7A, tZ)$-generated for some class tZ of $G_2(3)$. It follows that $G_2(3)$ is a $(pY, 7A, tZ)$-generated group for all the conjugacy classes pY containing elements of prime orders. Hence, $G_2(3)$ is $7A$-complementary generated. □

Proposition 3.10. *The group $G_2(3)$ is $8X$-complementary generated for $X \in \{A, B\}$.*

Proof. We show that $G_2(3)$ is $(pY, 8X, 13A)$- and $(pY, 8X, 7A)$-generated for pY in the sets $\{2A, 3A, 3B, 3C, 3E, 7A\}$ and $\{3D, 13A, 13B\}$, respectively. From Table 2, we see that M_5 (respectively, M_6) is the only maximal subgroup of $G_2(3)$ with classes of elements of orders 2, 3, 8 and 13 that fuse into classes $2A$, $3B$, $8A$ (respectively, $8B$) and $13A$ of $G_2(3)$. Let $i = 5$ when $X = A$ and $i = 6$ when $X = B$. Now, for the case $(2A, 8X, 13A)$, we find $\Delta_{G_2(3)}(2A, 8X, 13A) = 910$, $\Sigma(M_i) = 26 + 52 = 78$ and $\Sigma(M_{i1}) = 13 + 13 = 26$. Thus, $\Sigma^*(M_i) = \Sigma(M_i) - 1 \cdot \Sigma^*(M_{i1}) = 78 - 26 = 52$. Therefore,

$$\begin{aligned}\Delta^*_{G_2(3)}(2A, 8X, 13A) &= \Delta_{G_2(3)}(2A, 8X, 13A) - 1 \cdot \Sigma^*(M_i) - 1 \cdot \Sigma^*(M_{i1}) \\ &= 910 - 52 - 26 = 832,\end{aligned}$$

showing generation of $G_2(3)$ by the triple $(2A, 8X, 13A)$ for $X \in \{A, B\}$.

Again, we let $i = 5$ when $X = A$, $i = 6$ when $X = B$ and consider the case $(3Y, 8X, 13A)$, $Y \in \{A, B\}$ and $Y \neq X$. We have $\Delta_{G_2(3)}(3Y, 8X, 13A) = 91$, $\Sigma(M_i) = 26$ and $\Sigma(M_{i1}) = 13 + 13 = 26$. So, $\Sigma^*(M_i) = \Sigma(M_i) - 1 \cdot \Sigma^*(M_{i1}) = 26 - 26 = 0$. Therefore,

$$\begin{aligned}\Delta^*_{G_2(3)}(3Y, 8X, 13A) &= \Delta_{G_2(3)}(3Y, 8X, 13A) - 1 \cdot \Sigma^*(M_i) - 1 \cdot \Sigma^*(M_{i1}) \\ &= 91 - 26 = 65,\end{aligned}$$

also showing generation of $G_2(3)$ by the triple $(3Y, 8X, 13A)$ for $Y, X \in \{A, B\}$ with $Y \neq X$.

For all the remaining cases, we note that no maximal subgroup make a contribution in the calculations of $\Delta^*_{G_2(3)}(pY, 8X, 13A)$, where $pY \in \{3A, 3B, 3C, 3E, 7A\}$. So, $\Delta^*_{G_2(3)}(pY, 8X, 13A) = \Delta_{G_2(3)}(pY, 8X, 13A)$. Similarly, $\Delta^*_{G_2(3)}(pY, 8X, 7A) =$

$\Delta_{G_2(3)}(pY, 8X, 7A)$ for $pY \in \{3D, 13A, 13B\}$. From GAP, we find that

$$\begin{aligned}
\Delta^*_{G_2(3)}(3X, 8X, 13A) &= \Delta_{G_2(3)}(3X, 8X, 13A) = 91,\\
\Delta^*_{G_2(3)}(3C, 8X, 13A) &= \Delta_{G_2(3)}(3C, 8X, 13A) = 728,\\
\Delta^*_{G_2(3)}(3E, 8X, 13A) &= \Delta_{G_2(3)}(3E, 8X, 13A) = 3276,\\
\Delta^*_{G_2(3)}(7A, 8X, 13A) &= \Delta_{G_2(3)}(7A, 8X, 13A) = 75712,\\
\Delta^*_{G_2(3)}(3D, 8X, 7A) &= \Delta_{G_2(3)}(3D, 8X, 7A) = 3276,\\
\Delta_{G_2(3)}(13A, 8X, 7A) &= \Delta_{G_2(3)}(13A, 8X, 7A) = 40768,\\
\Delta_{G_2(3)}(13B, 8X, 7A) &= \Delta_{G_2(3)}(13B, 8X, 7A) = 40768,
\end{aligned}$$

where $X \in \{A, B\}$. Generation by all these triples follows since $\Delta^*(G_2(3)) > 0$. Consequently, $G_2(3)$ is $8X$-complementary generated for $X \in \{A, B\}$. ☐

Proposition 3.11. *The group $G_2(3)$ is $9X$-complementary generated for $X \in \{A, B, C\}$.*

Proof. Let $T := \{2A, 3A, 3B, 3C, 3D, 3E, 7A, 13A, 13B\}$. We show that $G_2(3)$ is $(pY, 9X, 13Z)$-generated for all $pY \in T$, where $X \in \{A, B, C\}$ and $Z \in \{A, B\}$. Direct computations with GAP yield the structure constants in Table 4. Now, from

Table 4: The structure constants $\Delta_{G_2(3)}(pY, 9X, 13Z)$.

pY	$2A$	$3A$	$3B$	$3C$	$3D$	$3E$	$7A$	$13A$	$13B$
$\Delta_{G_2(3)}(pY, 9A, 13Z)$	195	52	52	91	1131	897	21762	14040	14391
$\Delta_{G_2(3)}(pY, 9B, 13Z)$	312	13	13	286	897	1014	22815	11232	10881
$\Delta_{G_2(3)}(pY, 9C, 13Z)$	312	13	13	286	897	1014	22815	11232	10881

Table 4, we see that $\Delta^*_{G_2(3)}(pY, 9X, 13Z) > 0$ for all pY, where $X \in \{A, B, C\}$ and $Z \in \{A, B\}$. From Table 2, we can see that no maximal subgroup of G contains elements of orders 9 and 13 at the same time. Thus, maximal subgroups of $G_2(3)$ do not make any contribution in the calculations of $\Delta^*_{G_2(3)}(pY, 9X, 13Z)$. Thus, $\Delta^*_{G_2(3)}(pY, 9X, 13Z) = \Delta_{G_2(3)}(pY, 9A, 13Z) > 0$. Therefore, $G_2(3)$ is $(pY, 9X, 13Z)$-generated for all the conjugacy classes pY containing elements of prime orders. It follows that $G_2(3)$ is $9X$-complementary generated for $X \in \{A, B, C\}$. ☐

Proposition 3.12. *The group $G_2(3)$ is $12X$-complementary generated for $X \in \{A, B\}$.*

Proof. We show that $G_2(3)$ is $(pY, 12X, 13A)$- and $(pY, 12X, 7A)$-generated for pY in the sets $\{2A, 3A, 3B, 3C, 3E, 7A\}$ and $\{3D, 13A, 13B\}$, respectively. From Table 2, we see that M_5 (respectively, M_6) is the only maximal subgroup of $G_2(3)$ with classes of elements of orders 2, 3, 12 and 13 that fuse into classes $2A$, $3B$,

$12A$ (respectively, $12B$) and $13A$ of $G_2(3)$. Let $i = 5$ when $X = B$ and $i = 6$ when $X = A$. Now, for the case $(2A, 12X, 13A)$, we find $\Delta_{G_2(3)}(2A, 12X, 13A) = 702$ for $X \in \{A, B\}$ and $\Sigma(M_i) = 39 + 39 = 78$. No subgroup of M_i makes a contribution hence $\Sigma^*(M_i) = \Sigma(M_i) = 78$. Thus,

$$\begin{aligned} \Delta^*_{G_2(3)}(2A, 12X, 13A) &= \Delta_{G_2(3)}(2A, 12X, 13A) - 1 \cdot \Sigma^*(M_i) \\ &= 702 - 78 = 624, \end{aligned}$$

showing generation of $G_2(3)$ by the triple $(2A, 12X, 13A)$ for $X \in \{A, B\}$.

Again we let $i = 5$ when $X = B$, $i = 6$ when $X = A$ and consider the case $(3X, 12X, 13A)$, $X \in \{A, B\}$. We have $\Delta_{G_2(3)}(3X, 12X, 13A) = 39$, and $\Sigma^*(M_i) = \Sigma(M_i) = 0$. Therefore,

$$\begin{aligned} \Delta^*_{G_2(3)}(3X, 12X, 13A) &= \Delta_{G_2(3)}(3X, 12X, 13A) - 1 \cdot \Sigma^*(M_i) \\ &= 39 - 0 = 39, \end{aligned}$$

also showing generation of $G_2(3)$ by the triple $(3X, 12X, 13A)$ for $X \in \{A, B\}$.

For all the remaining cases, namely $(pY, 12X, 13A)$ and $(pY, 12X, 7A)$ for pY in the sets $\{3C, 3E, 7A\}$ and $\{3D, 13A, 13B\}$, respectively, we note that no maximal subgroup make a contribution in the calculations of $\Delta^*_{G_2(3)}(pY, 12X, 13A)$ or $\Delta^*_{G_2(3)}(pY, 12X, 7A)$ for pY in the sets $\{3C, 3E, 7A\}$ and $\{3D, 13A, 13B\}$. Therefore, $\Delta^*_{G_2(3)}(pY, 12X, 13A) = \Delta_{G_2(3)}(pY, 12X, 13A)$ and $\Delta^*_{G_2(3)}(pY, 12X, 7A) = \Delta_{G_2(3)}(pY, 12X, 7A)$. The computations reveal that

$$\begin{aligned} \Delta^*_{G_2(3)}(3Y, 12X, 13A) &= \Delta_{G_2(3)}(3Y, 12X, 13A) = 39, \\ \Delta^*_{G_2(3)}(3C, 12X, 13A) &= \Delta_{G_2(3)}(3C, 12X, 13A) = 624, \\ \Delta^*_{G_2(3)}(3E, 12X, 13A) &= \Delta_{G_2(3)}(3E, 12X, 13A) = 2340, \\ \Delta^*_{G_2(3)}(7A, 12X, 13A) &= \Delta_{G_2(3)}(7A, 12X, 13A) = 50544, \\ \Delta^*_{G_2(3)}(3D, 12X, 7A) &= \Delta_{G_2(3)}(3D, 12X, 7A) = 2268, \\ \Delta_{G_2(3)}(13A, 12X, 7A) &= \Delta_{G_2(3)}(13A, 12X, 7A) = 27216, \\ \Delta_{G_2(3)}(13B, 12X, 7A) &= \Delta_{G_2(3)}(13B, 12X, 7A) = 27216, \end{aligned}$$

where $X, Y \in \{A, B\}$ and $X \neq Y$. The desired result follows since $\Delta^*(G_2()3) > 0$. Consequently $G_2(3)$ is $12X$-complementary generated for $X \in \{A, B\}$. □

Proposition 3.13. *The group $G_2(3)$ is $13X$-complementary generated for $X \in \{A, B\}$.*

Proof. By Propositions 10, 21, 24 and 25 of [24], the group $G_2(3)$ is $(pY, 13X, 13Z)$-generated for $pY \in \{2A, 3A, 3B, 3C, 3D, 3E, 7A, 13X\}$ and $X, Z \in \{A, B\}$. It follows that $G_2(3)$ is a $(pY, 13X, tZ)$-generated group for all the conjugacy classes pY containing elements of prime order p. Hence, $G_2(3)$ is $13X$-complementary generated for $X \in \{A, B\}$. □

We conclude this paper by mentioning that collecting the results in Remark 1 and Proposition 3.1 to 3.13 show that the Chevalley group $G_2(3)$ is nX-complementary generated if and only if $n \geq 6$ and $nX \notin \{6A, 6B\}$, proving Theorem 1.3.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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