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Pantograph System with Mixed Riemann-Liouville and Caputo-Hadamard Sequential Fractional Derivatives: Existence and Ulam-Stability

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Abstract

The pantograph equation improves the mathematical model of the system includes modeling the motion of the wire connected with the dynamics of the supports and modeling the dynamics of the pantograph. The subject of this paper is the existence and Ulam stability of solutions for a coupled system of sequential pantograph equations of fractional order involving both Riemann-Liouville and Caputo-Hadamard fractional derivative operators. By applying the classical theorems in nonlinear analysis, such as the Banach's fixed point theorem and Leray-Schauder nonlinear alternative, the uniqueness and existence of solutions are obtained. Furthermore, the Ulam stability results are also presented. Finally, we have shown the results in the applications section by presenting various examples to numerical effects which provided to support the theoretical findings.

Keywords: Fractional derivative, Nonlinear analysis, Existence, Fractional pantograph equation, Ulam stability.

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1. Introduction

It is well known that the fractional differential equations theory plays an important role in applied sciences and engineering. Recently, a significant development in this theory has been achieved, for more details, the readers can refer to [1–5] and the references therein. Moreover, existence theory and the analysis of the stability of the solutions is one of the most important areas concerning research on fractional differential equations, for more details, see [6–9].

The pantograph equation is a famous delay differential equation that has been known since 1971. Till the present day, the continuous and the discrete cases of the pantograph equation are well studied. In fact, the equation improves the mathematical model of the system includes modeling the motion of the wire connected with the dynamics of the supports and modeling the dynamics of the pantograph. Recently, the existence and the Ulam-stability problems have been attracted by many authors, we refer to the papers [10–15] and the references therein. So, in this current research work, we establish the existence and Ulam-stability of solutions of an important type of differential equation that has several applications in engineering and scientific disciplines, called pantograph equation (P$\mathbb{E}$). We note that the classical P$\mathbb{E}$ model, is

$$D^1\left[\mathfrak{h}(\wp)\right] = B_1\mathfrak{h}(\wp) + B_2\mathfrak{h}(\omega\wp), \qquad \wp \in \Lambda := [0,T],$$

under condition $\mathfrak{h}(0) = \mathfrak{h}_0$, where $0 < \omega < 1$ [16]. For more information, we refer the reader to the research papers [17–19]. In [16], the scholars considered the C-fractional version of the P$\mathbb{E}$s type,

$$_{\mathrm{C}}\mathscr{D}^{\eta}\left[\mathfrak{h}(\wp)\right] = \mathfrak{z}\left(\wp, \mathfrak{h}(\wp), \mathfrak{h}(\omega\wp)\right), \qquad \wp \in \Lambda, \quad 0 < \omega < 1,$$

with $\mathfrak{h}(0) = \mathfrak{h}_0$. The P$\mathbb{E}$s including different kinds of fractional operators such as Caputo (C), Riemann-Liouville (R.L), Hadamard (H), Katugampola-Hilfer (KH) and $q-$fractional operators, etc., have recently been studied by many researchers [20–26]. Also, by using different techniques of nonlinear analysis, several scholars have obtained results of the existence, uniqueness and Ulam-stability of solutions for different classes of fractional pantograph equations ($\mathbb{F}$P$\mathbb{E}$s). For more details see monographs [27–31]. In [32], the authors studied the existence, uniqueness and Ulam-stability for a class of C.H type fractional pantograph differential equations ($\mathbb{F}$P$\mathbb{D}\mathbb{E}$s) described as form,

$$_{\mathrm{C.H}}\mathscr{D}^{\eta}\mathfrak{h}(\wp) = \mathfrak{z}\left(\wp, \mathfrak{h}(\wp), \mathfrak{h}(\omega\wp)\right), \qquad \wp \in [1,T], \quad 0 < \omega < 1,$$

under the condition $\mathfrak{h}(1) = \mathfrak{h}_1 - \phi(\mathfrak{h}), \quad \mathfrak{h}_1 \in \mathbb{R}$, where order $0 < \eta \leq 1$ and $\mathfrak{z} : [1,T] \times \mathbb{R}^2 \to \mathbb{R}, \phi : C([1,T],\mathbb{R}) \to \mathbb{R}$ are given continuous functions. In [33], the authors considered a $\mathbb{F}$P$\mathbb{D}\mathbb{E}$s with R.L and two C fractional derivatives of the form

$$\begin{cases} {}_{\mathrm{R.L}}\mathscr{D}^{\eta}\left[{}_{\mathrm{C}}\mathscr{D}^{\vartheta_1}\left[{}_{\mathrm{C}}\mathscr{D}^{\vartheta_2}\right]\right]\mathfrak{h}(\wp) = A\mathfrak{z}\left(\wp, s(\wp), \mathfrak{h}(\omega\wp)\right) \\ \qquad + B\,{}_{\mathrm{R.L}}\mathscr{I}^{\alpha}\left[\breve{\mathfrak{z}}\left(\wp, \mathfrak{h}(\wp), \mathfrak{h}(\varpi\wp)\right)\right], \\ \wp \in [0,1], \, 0 < \omega, \varpi < 1, \, 0 < \eta, \vartheta_1, \vartheta_2 \leq 1, \, \alpha \geq 0, \, A, B \in \mathbb{R}, \end{cases}$$

and $\lambda_1 \neq \lambda_2 \nu^{\eta+\vartheta_1+\vartheta_2-1}$, with the boundary value condition $\mathfrak{h}(0) = 0$, $\lambda_1 \mathfrak{h}(1) - \lambda_2 \mathfrak{h}(\nu) = \phi(\mathfrak{h})$, ${}_{\mathrm{C}}\mathscr{D}^{\eta_3}\mathfrak{h}(0) = 0$, $0 < \nu < 1$, λ_1, $\lambda_2 \in \mathbb{R}$, and $\mathfrak{z}, \breve{\mathfrak{z}} : [0,1] \times \mathbb{R}^2 \to \mathbb{R}, \phi : C([0,1], \mathbb{R}) \to \mathbb{R}$ are given continuous functions. Also in [34], the authors established the uniqueness and different kinds of Ulam stability for sequential $\mathbb{FP}$ $\mathbb{DE}$ involving Caputo $q-$fractional derivative given by,

$$\left[{}_{\mathrm{C}}\mathscr{D}_q^{\eta_1} + m\, {}_{\mathrm{C}}\mathscr{D}_q^{\eta_2}\right] \mathfrak{h}(\wp) = \mathfrak{z}\left(\wp, \mathfrak{h}(\wp), \mathfrak{h}(\omega\wp), {}_{\mathrm{C}}\mathscr{D}_q^{\eta_2}\mathfrak{h}(\omega\wp)\right), \qquad \wp \in \Lambda,$$

for $m \in \mathbb{R}_{\geq 0}$, $1 < \eta_1 \leq 2$, $0 < \eta_2 \leq 1$, $0 < q, \omega < 1$, $\pi_i \in \mathbb{R}$, $i = 1, 2$, with $\mathfrak{h}(0) = 0$, $\pi_1 \mathfrak{h}(T) = \pi_2 \mathfrak{h}(\varsigma) + A$, $0 < \eta < T$, $A \in \mathbb{R}$, where $\pi_1 T^{\eta_1 - \eta_2} \neq \pi_2 \varsigma^{\eta_1 - \eta_2}$ and $\mathfrak{z} \in C(\Lambda \times \mathbb{R}^3)$.

In this present work, we study the existence, uniqueness and Ulam-stability of solutions for the following coupled system of R.L and C.H sequential $\mathbb{FPE}$s:

$$\begin{cases} {}_{\mathrm{R.L}}\mathscr{D}^{\eta_1}\left[{}_{\mathrm{C.H}}\mathscr{D}^{\vartheta_1} + \beta_1\right] \mathfrak{h}_1(\wp) = \mathfrak{z}_1(\wp, \mathfrak{h}_1(\wp), \mathfrak{h}_1(\omega\wp), \mathfrak{h}_2(\wp)), \\ {}_{\mathrm{R.L}}\mathscr{D}^{\eta_2}\left[{}_{\mathrm{C.H}}\mathscr{D}^{\vartheta_2} + \beta_2\right] \mathfrak{h}_2(\wp) = \mathfrak{z}_2(\wp, \mathfrak{h}_1(\wp), \mathfrak{h}_2(\wp), \mathfrak{h}_2(\omega\wp)), \\ \wp \in \Lambda,\ 1 \leq \eta_k \leq 2,\ 0 \leq \vartheta_k \leq 1,\ 0 < \omega < 1,\ \beta_k,\ k = 1, 2, \end{cases} \tag{1}$$

supplemented with coupled conditions

$$\begin{cases} \left[{}_{\mathrm{C.H}}\mathscr{D}^{\vartheta_1} + \beta_1\right] \mathfrak{h}_1(0) = 0,\ \mathfrak{h}_1(T) = {}_{\mathrm{R.L}}\mathscr{I}^{\alpha_1}\left[\mathfrak{h}_1(\theta_1)\right], \quad \mathfrak{h}_1(0) = 0, \\ \left[{}_{\mathrm{C.H}}\mathscr{D}^{\vartheta_2} + \beta_2\right] \mathfrak{h}_2(0) = 0,\ \mathfrak{h}_2(T) = {}_{\mathrm{R.L}}\mathscr{I}^{\alpha_2}\left[\mathfrak{h}_2(\theta_2)\right], \quad \mathfrak{h}_2(0) = 0, \end{cases} \tag{2}$$

where $0 < \theta_k < T$ and $\mathfrak{z}_k : \Lambda \times \mathbb{R}^3 \to \mathbb{R}$, $k = 1, 2$ are given continuous functions.

We recall auxiliary concepts and definitions about fractional calculus in the next section. In Sections 3 and 4, the existence of a solution for the proposed coupled system R.L and C.H sequential $\mathbb{FPE}$ (1) is presented by using suitable theorems in the existence of its solution and the Ulam stability of the system is defined and is studied. Some applications to show the validity of the existence results in Section 5, are presented. At the end, the conclusion is expressed to introduce future works in Section 6.

2. Prelimeneris of fractional calculus

Definition 2.1 ([3, 4, 35])**.** The operator ${}_{\mathrm{R.L}}\mathscr{D}^{\eta}$ is the fractional derivative in the sense of R.L, defined by

$${}_{\mathrm{R.L}}\mathscr{D}^{\eta}\left[\mathfrak{h}\right](\wp) = \left(\tfrac{\mathrm{d}}{\mathrm{d}\wp}\right)^{\imath} \int_0^{\wp} (\wp - \tilde{p})^{\imath - \eta - 1} \tfrac{1}{\Gamma(\imath - \eta)} \mathfrak{h}(\tilde{p})\, \mathrm{d}\tilde{p}, \qquad \imath = [\eta] + 1,$$

where $\Gamma(\cdot)$ is the Euler Gamma function.

Definition 2.2 ([3])**.** The operator ${}_{\mathrm{C.H}}\mathscr{D}^{\vartheta}$ is the C.H fractional derivative defined by,

$${}_{\mathrm{C.H}}\mathscr{D}^{\vartheta}\left[\mathfrak{h}\right](\wp) = \int_0^{\imath} \left(\ln \tfrac{\wp}{\tilde{p}}\right)^{\imath - \vartheta - 1} \delta^{\imath} \tfrac{\mathfrak{h}(\tilde{p})}{\tilde{p}} \tfrac{\mathrm{d}\tilde{p}}{\Gamma(\imath - \vartheta)},$$

where $\grave{\imath}-1<\vartheta<\grave{\imath}$, $\grave{\imath}=[\vartheta]+1$, $\delta=\wp\frac{\mathrm{d}}{\mathrm{d}\wp}$ and $[\vartheta]$ denotes the integer part of ϑ.

The R.L and H fractional integral of order $\rho>0$, are defined by

$$
\begin{aligned}
{}_{\mathrm{R.L}}\mathscr{I}^{\rho}\left[\mathfrak{h}\right](\wp) &= \int_0^{\wp}(\wp-\tilde{p})^{\vartheta-1}\,\mathfrak{h}(\tilde{p})\,\tfrac{\mathrm{d}\tilde{p}}{\Gamma(\vartheta)}, \qquad \rho>0,\\
{}_{\mathrm{H}}\mathscr{I}^{\rho}\left[\mathfrak{h}\right](\wp) &= \int_0^{\wp}\left(\ln\tfrac{\wp}{\tilde{p}}\right)^{\rho-1}\tfrac{\mathfrak{h}(\tilde{p})}{\tilde{p}}\,\tfrac{\mathrm{d}\tilde{p}}{\Gamma(\rho)}, \qquad \rho>0,
\end{aligned}
$$

respectively [3, 4, 35].

Lemma 2.3 ([3]). *Let $\rho>0$. Then for $\mathfrak{h}\in C(0,T)\cap L^1(0,T)$ and ${}_{\mathrm{R.L}}\mathscr{D}^{\rho}\mathfrak{h}\in C(0,T)\cap L^1(0,T)$, we have*

$$
{}_{\mathrm{R.L}}\mathscr{I}^{\rho}\left[{}_{\mathrm{R.L}}\mathscr{D}^{\rho}\left[\mathfrak{h}\right]\right](\wp)=\mathfrak{h}(\wp)+\sum_{i=1}^{\grave{\imath}}e_i\wp^{\rho-i},
$$

for $e_i\in\mathbb{R}$, $i=1,2,\ldots,\grave{\imath}$, $\grave{\imath}=[\rho]+1$.

Lemma 2.4 ([36]). *Let $\mathfrak{h}\in C_{\delta}^{\grave{\imath}}(\Lambda)$. Then*

$$
{}_{\mathrm{H}}\mathscr{I}^{\rho}\left[{}_{\mathrm{C.H}}\mathscr{D}^{\rho}\left[\mathfrak{h}\right]\right](\wp)=\mathfrak{h}(\wp)+\sum_{i=0}^{\grave{\imath}-1}e_i(\ln\wp)^i, \qquad e_i\in\mathbb{R},\ i=1,2,\ldots,\grave{\imath}-1,
$$

where $C_{\delta}^{\grave{\imath}}(\Lambda)=\left\{\mathfrak{h}:\Lambda\to\mathbb{R}\,:\,\delta^{\grave{\imath}-1}\mathfrak{h}\in C(\Lambda)\right\}$.

Lemma 2.5 ([3]). *For $\rho>0$ and $\lambda>0$, we have ${}_{\mathrm{H}}\mathscr{I}^{\rho}\left[\wp^{\lambda}\right]=\lambda^{-\rho}\wp^{\lambda}$.*

Lemma 2.6 ([13]). *Let $\eta,\vartheta>0$ and $\lambda>0$ be given constants. Then, we have*

$$
\begin{aligned}
{}_{\mathrm{H}}\mathscr{I}^{\vartheta}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta}\left[1\right]\right](\wp) &= \tfrac{\eta^{-\vartheta}}{\Gamma(\eta+1)}\wp^{\eta},\\
{}_{\mathrm{R.L}}\mathscr{I}^{\lambda}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta}\left[1\right]\right]\right](\wp) &= \tfrac{\eta^{-\vartheta}}{\Gamma(\eta+\lambda+1)}\wp^{\eta+\lambda}.
\end{aligned}
$$

3. Existence results for proposed system

In this section, we investigate the existence and uniqueness of solution for coupled system R.L and C.H sequential $\mathbb{FPE}$s (1). First, we give the following auxiliary result.

Lemma 3.1. *Let $\Gamma(\eta_k)\,\theta_k^{\eta_k+\alpha_k-1}\neq T^{\eta_k-1}\Gamma(\eta_k+\alpha_k)$ and suppose that $g_k\in C(\Lambda)$, $k=1,2$. Then the unique solution of the fractional problem*

$$
\begin{cases}
{}_{\mathrm{R.L}}\mathscr{D}^{\eta_k}\left[{}_{\mathrm{C.H}}\mathscr{D}^{\vartheta_k}+\beta_k\right]\mathfrak{h}_k(\wp)=g_k(\wp),\\
\left[{}_{\mathrm{C.H}}\mathscr{D}^{\vartheta_k}+\beta_k\right]\mathfrak{h}_k(0)=0, \quad \mathfrak{h}_k(T)={}_{\mathrm{R.L}}\mathscr{I}^{\alpha_1}\mathfrak{h}_k(\theta_k), \quad s_k(0)=0,\\
\wp\in\Lambda,\ 1\le\eta_k\le 2,\ 0\le\vartheta_k\le 1,\ \alpha_k>0,\ 0<\theta_k<T, \quad k=1,2,
\end{cases}
\tag{3}
$$

is given by

$$\begin{aligned}\mathfrak{h}_k(\wp) &= {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}[g_k]\right](\wp) - \beta_k\, {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}[\mathfrak{h}_k](\wp)\\ &+ \tfrac{(\eta_k-1)^{-\vartheta_k}\wp^{\eta_k-1}}{\Pi_k}\Big({}_{\mathrm{R.L}}\mathscr{I}^{\alpha_k}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}[g_k]\right]\right](\theta_k)\\ &- {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}[g_k]\right](T) + \beta_k {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}[\mathfrak{h}_k](T)\\ &- \beta_k\, {}_{\mathrm{R.L}}\mathscr{I}^{\alpha_k}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}[\mathfrak{h}_k]\right](\theta_k)\Big), \end{aligned} \tag{4}$$

where

$$\Pi_k := (\eta_k-1)^{-\vartheta_k}\left(\tfrac{\Gamma(\eta_k)\theta_k^{\eta_k+\alpha_k-1}}{\Gamma(\eta_k+\alpha_k)} - T^{\eta_k-1}\right), \qquad k=1,2. \tag{5}$$

Proof. Applying the operator ${}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}$, $k=1,2$ to both sides of equation in (3) and using Lemma 2.3, we get:

$$\left[{}_{\mathrm{C.H}}\mathscr{D}^{\vartheta_k} + \beta_k\right]\mathfrak{h}_k(\wp) = {}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}[g_k](\wp) + e_{1k}\wp^{\eta_k-1} + e_{2k}\wp^{\eta_k-2}, \tag{6}$$

where $e_{1k}, e_{2k} \in \mathbb{R}$, $k=1,2$. Next, using the operator ${}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}$, $k=1,2$ to both sides of equation in (6) and applying Lemma 2.4, we obtain:

$$\begin{aligned}\mathfrak{h}_k(\wp) &= {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}[g_k]\right](\wp) - \beta_k {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}[\mathfrak{h}_k](\wp)\\ &+ e_{1k}\, {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-1}\right] + e_{2k}\, {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-2}\right] + e_{0k}, \end{aligned} \tag{7}$$

where $e_0 \in \mathbb{R}$. Since $\left[{}_{\mathrm{C.H}}\mathscr{D}^{\vartheta_k} + \beta_k\right]\mathfrak{h}_k(0) = 0$ and $\mathfrak{h}_k(0) = 0$, we get:

$$e_{2k} = 0, \quad e_{0k} = 0, \qquad k=1,2. \tag{8}$$

Now, applying the fractional integral ${}_{\mathrm{R.L}}\mathscr{I}^{\alpha_k}$, $k=1,2$ to both sides of equation in (7), we get,

$$\begin{aligned}{}_{\mathrm{R.L}}\mathscr{I}^{\alpha_k}[\mathfrak{h}_k](\wp) &= {}_{\mathrm{R.L}}\mathscr{I}^{\alpha_k}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}[g_k]\right]\right](\wp)\\ &- \beta_k\, {}_{\mathrm{R.L}}\mathscr{I}^{\alpha_k}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}[\mathfrak{h}_k]\right](\wp)\\ &+ e_{1k}\tfrac{(\eta_k-1)^{-\vartheta_k}\Gamma(\eta_k)}{\Gamma(\eta_k+\alpha_k)}\wp^{\eta_k+\alpha_k-1}. \end{aligned}$$

Since $\mathfrak{h}_k(T) = {}_{\mathrm{R.L}}\mathscr{I}^{\alpha_k}\mathfrak{h}_k(\theta_k)$, $k=1,2$ and from Equation (8), we obtain:

$$\begin{aligned}e_{1k} &= \left[(\eta_k-1)^{-\vartheta_k}\left(T^{\eta_k-1} - \tfrac{\Gamma(\eta_k)}{\Gamma(\eta_k+\alpha_k)}\beta_k^{\eta_k+\alpha_k-1}\right)\right]^{-1}\\ &\cdot\Big({}_{\mathrm{R.L}}\mathscr{I}^{\alpha_k}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}g_k\right]\right](\theta_k) - {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}g_k\right](T)\\ &- \beta_k\, {}_{\mathrm{R.L}}\mathscr{I}^{\alpha_k}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}[\mathfrak{h}_k]\right](\theta_k) + \beta_k\, {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}[\mathfrak{h}_k](T)\Big)\\ &= \tfrac{1}{\Pi_k}\Big({}_{\mathrm{R.L}}\mathscr{I}^{\alpha_k}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}g_k\right]\right](\theta_k) - {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}g_k\right](T)\\ &- \beta_k\, {}_{\mathrm{R.L}}\mathscr{I}^{\alpha_k}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}[\mathfrak{h}_k]\right](\theta_k) + \beta_k\, {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}[\mathfrak{h}_k](T)\Big), \end{aligned}$$

inserting the values of e_0, e_{1k} and e_{2k}, $k = 1, 2$, in Equation (7) yields the solution (4). □

Let us now define the space $\mathcal{Z} = \{\mathfrak{h}_k : \mathfrak{h}_k \in C(\Lambda)\}$, $k = 1, 2$ endowed with the norm $\|(\mathfrak{h}_1, \mathfrak{h}_2)\|_{\mathcal{Z}^2} = \|\mathfrak{h}_1\| + \|\mathfrak{h}_2\|$, where $\|\mathfrak{h}_k\| = \sup\{|\mathfrak{h}_k(\wp)| : \wp \in \Lambda\}$. It is clear that $\left(\mathcal{Z}^2, \|\cdot\|_{\mathcal{Z}^2}\right)$ is a Banach space. In view of Lemma 3.1, we define an operator $\varphi : \mathcal{Z}^2 \to \mathcal{Z}^2$ by

$$\varphi(\mathfrak{h}_1, \mathfrak{h}_2)(\wp) = (\varphi_1(\mathfrak{h}_1, \mathfrak{h}_2)(\wp), \varphi_2(\mathfrak{h}_1, \mathfrak{h}_2)(\wp)), \qquad \wp \in \Lambda,$$

where for all $k = 1, 2$ and $\wp \in \Lambda$, we have:

$$\begin{aligned}
\varphi_1(\mathfrak{h}_1, \mathfrak{h}_2)(\wp) &= {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](\wp) - \beta_1\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}[\mathfrak{h}_1](\wp) \\
&+ \tfrac{(\eta_1-1)^{-\vartheta_1}\wp^{\eta_1-1}}{\Pi_1}\Big({}_{\mathrm{R.L}}\mathscr{I}^{\alpha_1}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right]\right](\theta_1) \\
&- {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](T) + \beta_1\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}[\mathfrak{h}_1](T) \\
&- \beta_1\ {}_{\mathrm{R.L}}\mathscr{I}^{\alpha_1}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}[\mathfrak{h}_1]\right](\theta_1)\Big),
\end{aligned}$$

and

$$\begin{aligned}
\varphi_2(\mathfrak{h}_1, \mathfrak{h}_2)(\wp) &= {}_{\mathrm{H}}\mathscr{I}^{\vartheta_2}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_2}\left[\mathfrak{z}^*_{2,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](\wp) - \beta_{2\mathrm{H}}\mathscr{I}^{\vartheta_2}[\mathfrak{h}_2](\wp) \\
&+ \tfrac{(\eta_2-1)^{-\vartheta_2}\wp^{\eta_2-1}}{\Pi_2}\Big({}_{\mathrm{R.L}}\mathscr{I}^{\alpha_2}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_2}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_2}\left[\mathfrak{z}^*_{2,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right]\right](\theta_2) \\
&- {}_{\mathrm{H}}\mathscr{I}^{\vartheta_2}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_2}\left[\mathfrak{z}^*_{2,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](T) + \beta_2\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_2}[\mathfrak{h}_2](T) \\
&- \beta_2\ {}_{\mathrm{R.L}}\mathscr{I}^{\alpha_2}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_2}[\mathfrak{h}_2]\right](\theta_2)\Big),
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}(\wp) &= \mathfrak{z}_1(\wp, \mathfrak{h}_1(\wp), \mathfrak{h}_1(\omega\wp), \mathfrak{h}_2(\wp)), \\
\mathfrak{z}^*_{2,(\mathfrak{h}_1,\mathfrak{h}_2)}(\wp) &= \mathfrak{z}_2(\wp, \mathfrak{h}_1(\wp), \mathfrak{h}_2(\wp), \mathfrak{h}_2(\omega\wp)),
\end{aligned}$$

and for $k = 1, 2$,

$$\begin{aligned}
{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}\left[\mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}\right](\wp) &= \int_0^{\wp}(\wp - \tilde{p})^{\eta_k - 1}\,\mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}(\tilde{p})\ \tfrac{\mathrm{d}\tilde{p}}{\Gamma(\eta_k)}, \\
{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}[\mathfrak{h}_k](\wp) &= \int_0^{\varsigma}\left(\ln\tfrac{\varsigma}{\tilde{p}}\right)^{\eta_k-1}\mathfrak{h}_k(\tilde{p})\ \tfrac{\mathrm{d}\tilde{p}}{\Gamma(\eta_k)}.
\end{aligned}$$

For convenience, we introduce the quantities:

$$\begin{aligned}
\varDelta_k &:= \tfrac{\eta_k^{-\vartheta_k}T^{\eta_k}}{\Gamma(\eta_k+1)} + \tfrac{(\eta_k-1)^{-\vartheta_k}T^{\eta_k-1}}{|\Pi_k|}\left[\tfrac{\eta_k^{-\vartheta_k}\theta_k^{\eta_k+\alpha_k}}{\Gamma(\eta_k+\alpha_k+1)} + \tfrac{\eta_k^{-\vartheta_k}T^{\eta_k}}{\Gamma(\eta_k+1)}\right], \\
\nabla_k &:= |\beta_k|T + \tfrac{(\eta_k-1)^{-\vartheta_k}T^{\eta_k-1}}{|\Pi_k|}\left[|\beta_k|T + |\beta_k|\,\tfrac{\theta_k^{\eta_k+1}}{\Gamma(\eta_k+2)}\right],
\end{aligned} \tag{9}$$

$k=1,2,$, and for all $(\mathfrak{h}_1,\mathfrak{h}_2)\in B_\epsilon$,

$$\begin{aligned}\left|h^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}(\wp)\right| &= |\mathfrak{z}_1(\wp,\mathfrak{h}_1(\wp),\mathfrak{h}_1(\omega\wp),\mathfrak{h}_2(\wp))| \\ &\le |\mathfrak{z}_1(\wp,\mathfrak{h}_1(\wp),\mathfrak{h}_1(\omega\wp),\mathfrak{h}_2(\wp)) \\ &\quad -\mathfrak{z}_1(\wp,0,0,0)| + |\mathfrak{z}_1(\wp,0,0,0)| \\ &\le |\mathfrak{h}_1(\wp)| + |\mathfrak{h}_1(\omega\wp)| + |\mathfrak{h}_2(\wp)| \\ &\le \gamma_1(2\|\mathfrak{h}_1\| + \|\mathfrak{h}_2\|) + \Omega_1 \\ &\le 3\gamma_1\|(\mathfrak{h}_1,\mathfrak{h}_2)\|_{\mathcal{Z}^2} + \Omega_1 \le 3\gamma_1\epsilon + \Omega_1, \end{aligned} \tag{10}$$

$$\begin{aligned}\left|\mathfrak{z}^*_{2,(\mathfrak{h}_1,\mathfrak{h}_2)}(\wp)\right| &= |\mathfrak{z}_2(\wp,\mathfrak{h}_1(\wp),\mathfrak{h}_2(\omega\wp),\mathfrak{h}_2(\wp))| \\ &\le |\mathfrak{z}_2(\wp,\mathfrak{h}_1(\wp),\mathfrak{h}_1(\omega\wp),\mathfrak{h}_2(\wp)) \\ &\quad -\mathfrak{z}_2(\wp,0,0,0)| + |\mathfrak{z}_2(\wp,0,0,0)| \\ &\le |\mathfrak{h}_1(\wp)| + |\mathfrak{h}_2(\omega\wp)| + |\mathfrak{h}_2(\wp)| \\ &\le \gamma_2(\|\mathfrak{h}_1\| + 2\|\mathfrak{h}_2\|) + \Omega_2 \\ &\le 3\gamma_2\|(\mathfrak{h}_1,\mathfrak{h}_2)\|_{\mathcal{Z}^2} + \Omega_1 \le 3\gamma_2\epsilon + \Omega_2. \end{aligned} \tag{11}$$

In the sequel, base on the Banach's contraction principle, we give the existence and uniqueness of solutions of the system (1).

Theorem 3.2. *Let $\mathfrak{z}_k\in C(\Lambda\times\mathbb{R}^3)$, $k=1,2$ satisfying the following hypothesis* $(\mathbf{H}_1)$. *There exist a constants $\gamma_k>0$, $k=1,2$ such that for all $\wp\in\Lambda$ and $\mathfrak{h}_j,\acute{\mathfrak{h}}_j\in\mathbb{R}$, $j=1,2,3$, we have:*

$$\left|\mathfrak{z}_k\big(\wp,\mathfrak{h}_1,\mathfrak{h}_2,\mathfrak{h}_3\big)-\mathfrak{z}_k\big(\wp,\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2,\acute{\mathfrak{h}}_3\big)\right| \le \gamma_k\sum_{j=1}^{3}\left|\mathfrak{h}_j-\acute{\mathfrak{h}}_j\right| .$$

In addition, we suppose that

$$3\varDelta_k\gamma_k+\nabla_k<\tfrac{1}{2}, \qquad k=1,2, \tag{12}$$

where $\varDelta_k$ and ∇_k, $k=1,2$ are given by (9). Then system (1) has a unique solution on Λ.

Proof. We first show that $\varphi B_\epsilon\subset B_\epsilon$, where $B_\epsilon=\left\{(\mathfrak{h}_1,\mathfrak{h}_2)\in\mathcal{Z}^2 : \|(\mathfrak{h}_1,\mathfrak{h}_2)\|\le\epsilon\right\}$ such that

$$\epsilon\ge\max\left\{\varDelta_k\Omega_k\left[\tfrac{1}{2}-(3\varDelta_k\gamma_k+\nabla_k)\right]^{-1},\, k=1,2\right\},$$

and $\Omega_k = \sup_{\wp\in\Lambda} |\mathfrak{z}_k(\wp, 0, 0)| < \infty$, $k = 1, 2$. From (10), we get:

$$\begin{aligned}\left|\varphi_1(\mathfrak{h}_1,\mathfrak{h}_2)(\wp)\right| &\le \sup_{\wp\in\Lambda}\Big\{ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](\wp) + |\beta_1|\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}[\mathfrak{h}_1](\wp) \\ &\quad + \tfrac{(\eta_1-1)^{-\vartheta_1}t^{\eta_1-1}}{|\Pi_1|}\Big({}_{\mathrm{R.L}}\mathscr{I}^{\alpha_1}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right]\right](\theta_1) \\ &\quad + {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](T) \\ &\quad + |\beta_1|\,{}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}[\mathfrak{h}_1](T) + |\beta_1|\ {}_{\mathrm{R.L}}\mathscr{I}^{\alpha_1}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}[\mathfrak{h}_1]\right](\theta_1)\Big)\Big\} \\ &\le {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\left|\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}\right|\right]\right](T) + |\beta_1|\,{}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}[|\mathfrak{h}_1|](T) \\ &\quad + \tfrac{(\eta_1-1)^{-\vartheta_1}T^{\eta_1-1}}{|\Pi_1|}\Big({}_{\mathrm{R.L}}\mathscr{I}^{\alpha_1}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\left|\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}\right|\right]\right]\right](\theta_1) \\ &\quad + {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\left|\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}\right|\right]\right](T) \\ &\quad + |\beta_1|\,{}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}[|\mathfrak{h}_1|](T) + |\beta_1|\,{}_{\mathrm{R.L}}\mathscr{I}^{\alpha_1}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}[|\mathfrak{h}_1|]\right](\theta_1)\Big),\end{aligned}$$

which implies that

$$\begin{aligned}\|\varphi_1(\mathfrak{h}_1,\mathfrak{h}_2)\| &\le {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}[1]\right](T)(3\gamma_1\epsilon+\Omega_1) + |\beta_1|\,\epsilon\,{}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}[|1|](T) \\ &\quad + \tfrac{(\eta_1-1)^{-\vartheta_1}T^{\eta_1-1}}{|\Pi_1|}\Big({}_{\mathrm{R.L}}\mathscr{I}^{\alpha_1}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}[1]\right]\right](\theta_1)(3\gamma_1\epsilon+\Omega_1) \\ &\quad + {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}[1]\right](T)(3\gamma_1\epsilon+\Omega_1) + |\beta_1|\,\epsilon\,{}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}[1](T) \\ &\quad + |\beta_1|\,\epsilon\,{}_{\mathrm{R.L}}\mathscr{I}^{\alpha_1}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}[1]\right](\theta_1)\Big).\end{aligned}$$

Thanks to Lemma 2.6, we get:

$$\begin{aligned}\|\varphi_1(\mathfrak{h}_1,\mathfrak{h}_2)\| &\le \left[\tfrac{\eta_1^{-\vartheta_1}T^{\eta_1}}{\Gamma(\eta_1+1)} + \tfrac{(\eta_1-1)^{-\vartheta_1}T^{\eta_1-1}}{|\Pi_1|}\left(\tfrac{\eta_1^{-\vartheta_1}\theta_1^{\eta_1+\alpha_1}}{\Gamma(\eta_1+\alpha_1+1)}\right.\right. \\ &\quad \left.\left. + \tfrac{\eta_1^{-\vartheta_1}T^{\eta_1}}{\Gamma(\eta_1+1)}\right)\right](3\gamma_1\epsilon+\Omega_1) + \Big[|\beta_1|\,T \\ &\quad + \tfrac{(\eta_1-1)^{-\vartheta_1}T^{\eta_1-1}}{|\Pi_1|}\left(|\beta_1|\,T + |\beta_1|\tfrac{\theta_1^{\eta_1+1}}{\Gamma(\eta_1+2)}\right)\Big]\epsilon \\ &= \mathit{\Delta}_1(3\gamma_1\epsilon+\Omega_1) + \nabla_1\epsilon = (\mathit{\Delta}_1 3\gamma_1 + \nabla_1)\epsilon + \mathit{\Delta}_1\Omega_1.\end{aligned}$$

Also, one can observe that $\|\varphi_1(\mathfrak{h}_1,\mathfrak{h}_2)\| \le (\mathit{\Delta}_1 3\gamma_1 + \nabla_1)\epsilon + \mathit{\Delta}_1\Omega_1 \le \frac{\epsilon}{2}$. Similarly, we have:

$$\|\varphi_2(\mathfrak{h}_1,\mathfrak{h}_2)\| \le (\mathit{\Delta}_2 3\gamma_2 + \nabla_2)\epsilon + \mathit{\Delta}_2\Omega_2 \le \tfrac{\epsilon}{2}.$$

From the definition of $\|(\cdot)\|$, we have:

$$\begin{aligned}\|\varphi(\mathfrak{h}_1,\mathfrak{h}_2)\|_{\mathcal{Z}^2} &= \|\varphi_1(\mathfrak{h}_1,\mathfrak{h}_2)\|_{\mathcal{Z}^2} + \|\varphi_2(\mathfrak{h}_1,\mathfrak{h}_2)\|_{\mathcal{Z}^2} \\ &\le [3(\mathit{\Delta}_1\gamma_1 + \mathit{\Delta}_2\gamma_2) + \nabla_1 + \nabla_2]\epsilon + \mathit{\Delta}_1\Omega_1 + \mathit{\Delta}_2\Omega_2 \le \epsilon,\end{aligned}$$

which implies that $\varphi B_\epsilon \subset B_\epsilon$. For $(\mathfrak{h}_1, \mathfrak{h}_2)$, $\left(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2\right) \in B_\epsilon$ and for each $\wp \in \Lambda$, we have:

$$
\begin{aligned}
&\left|\varphi_1(\mathfrak{h}_1, \mathfrak{h}_2)(\wp) - \varphi_1(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2)(\wp)\right| \\
&\qquad \le \sup_{\wp\in\Lambda} \Bigg\{ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\left|\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)} - \mathfrak{z}^*_{1,\left(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2\right)}\right|\right]\right](\wp) \\
&\qquad + |\beta_1|\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[|\mathfrak{h}_1 - t_1|\right](\wp) + \frac{(\eta_1-1)^{-\vartheta_1}\acute{\mathfrak{h}}^{\eta_1-1}}{|\Pi_1|} \\
&\qquad \times \Bigg({}_{\mathrm{R.L}}\mathscr{I}^{\alpha_1}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\left|\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}\right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left. - \mathfrak{z}^*_{1,\left(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2\right)}\right|\right]\right]\right](\theta_1) + {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\left|\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}\right.\right.\right. \\
&\qquad \left.\left.\left. - \mathfrak{z}^*_{1,\left(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2\right)}\right|\right]\right](T) + |\beta_1|\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[\left|\mathfrak{h}_1 - \acute{\mathfrak{h}}_1\right|\right](T) \\
&\qquad + |\beta_1|\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\left|\mathfrak{h}_1 - \acute{\mathfrak{h}}_1\right|\right]\right](\theta_1)\Bigg)\Bigg\}.
\end{aligned}
$$

By $(\mathbf{H}_1)$ and Lemma 2.6, we can write:

$$
\begin{aligned}
&\left\|\varphi_1(\mathfrak{h}_1, \mathfrak{h}_2) - \varphi_1(t_1, t_2)\right\| \\
&\qquad \le \left[\frac{\eta_1^{-\vartheta_1}T^{\eta_1}}{\Gamma(\eta_1+1)} + \frac{(\eta_1-1)^{-\vartheta_1}T^{\eta_1-1}}{|\Pi_1|}\right. \\
&\qquad\quad \left.\times\left(\frac{\eta_1^{-\vartheta_1}\theta_1^{\eta_1+\alpha_1}}{\Gamma(\eta_1+\alpha_1+1)} + \frac{\eta_1^{-\vartheta_1}T^{\eta_1}}{\Gamma(\eta_1+1)}\right)\right]3\gamma_1 \\
&\qquad\quad + \left[|\beta_1|T + \frac{(\eta_1-1)^{-\vartheta_1}T^{\eta_1-1}}{|\Pi_1|}\left(|\beta_1|T\right.\right. \\
&\qquad\quad \left.\left. + |\beta_1|\frac{\eta_1^{-\vartheta_1}\theta_1^{\eta_1}}{\Gamma(\eta_1+1)}\right)\right]\left(\left\|\mathfrak{h}_1 - \acute{\mathfrak{h}}_1\right\| + \left\|\mathfrak{h}_2 - \acute{\mathfrak{h}}_2\right\|\right) \\
&\qquad = (\varDelta_1 3\gamma_1 + \nabla_1)\left(\left\|\mathfrak{h}_1 - \acute{\mathfrak{h}}_1\right\| + \left\|\mathfrak{h}_2 - \acute{\mathfrak{h}}_2\right\|\right).
\end{aligned}
$$

Hence,

$$
\left\|\varphi_1(\mathfrak{h}_1, \mathfrak{h}_2) - \varphi_1\left(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2\right)\right\|_{\mathcal{Z}^2} \le (3\varDelta_1\gamma_1 + \nabla_1)\left\|(\mathfrak{h}_1, \mathfrak{h}_2) - \left(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2\right)\right\|_{\mathcal{Z}^2}.
$$

Similarly, by (11) and Lemma 2.6, we get:

$$
\left\|\varphi_2(\mathfrak{h}_1, \mathfrak{h}_2) - \varphi_2\left(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2\right)\right\|_{\mathcal{Z}^2} \le (3\varDelta_2\gamma_2 + \nabla_2)\left\|(\mathfrak{h}_1, \mathfrak{h}_2) - \left(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2\right)\right\|_{\mathcal{Z}^2}.
$$

Consequently, we obtain:

$$
\begin{aligned}
\left\|\varphi(\mathfrak{h}_1, \mathfrak{h}_2) - \varphi\left(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2\right)\right\|_{\mathcal{Z}^2} &= \left\|\varphi_1(\mathfrak{h}_1, \mathfrak{h}_2) - \varphi_1\left(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2\right)\right\|_{\mathcal{Z}^2} \\
&\quad + \left\|\varphi_2(\mathfrak{h}_1, \mathfrak{h}_2) - \varphi_2\left(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2\right)\right\|_{\mathcal{Z}^2} \\
&\le \left[3(\varDelta_1\gamma_1 + \varDelta_2\gamma_2) + \nabla_1 + \nabla_2\right]\left\|(\mathfrak{h}_1, \mathfrak{h}_2) - \left(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2\right)\right\|_{\mathcal{Z}^2}.
\end{aligned}
$$

So, from (12), φ is contractive. Hence, the Banach contraction principlec implies there exists a unique fixed point which is a solution of system (1)-(2). □

In what sequel, by employing the Leray-Schauder nonlinear alternative, we present the existence of solutions of fractional problem (1)-(2).

Lemma 3.3 (Leray-Schauder alternative [37]). *Assume that* $\varphi : \mathbb{W} \to \mathbb{W}$ *be a completely continuous operator, and let* $\mathbb{E}_\varphi = \{\mathfrak{h} \in \mathbb{W} : \mathfrak{h} = \xi\varphi(\mathfrak{h})$ *for some* $0 < \xi < 1\}$. *Then either the set* $\mathbb{E}_\varphi$ *is unbounded, or* φ *has at least one fixed point.*

Theorem 3.4. *Let* $\mathfrak{z}_k \in C(\Lambda \times \mathbb{R}^3)$, $k = 1, 2$. *In addition we assume that:*
$(\mathbf{H}_2)$ *There exist real constants* μ_i, $\varpi_i \geq 0$, $i = 1, 3$ *and* $\mu_0 > 0$, $\varpi_0 > 0$ *such that for any* $\mathfrak{h}_k \in \mathbb{R}, k = 1, 2,$ *we have:*

$$|\mathfrak{z}_1(\wp, \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)| \leq \mu_0 + \mu_1 |\mathfrak{h}_1| + \mu_2 |\mathfrak{h}_2| + \mu_3 |\mathfrak{h}_3|,$$
$$|\mathfrak{z}_2(\wp, \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)| \leq \varpi_0 + \varpi_1 |\mathfrak{h}_1| + \varpi_2 |\mathfrak{h}_2| + \varpi_3 |\mathfrak{h}_3|.$$

If

$$\begin{aligned} \varDelta_1\mu_1 + \varDelta_1\mu_2 &< 1 - (\nabla_1 + \varDelta_2\varpi_1), \\ \varDelta_2\varpi_2 + \varDelta_2\varpi_3 &< 1 - (\nabla_2 + \varDelta_1\mu_3), \end{aligned} \tag{13}$$

where $\varDelta_k$, ∇_k, $k = 1, 2$ *are given by* (9). *Then the problem* (1)-(2) *has at least one solution on* Λ.

Proof. We begin by showing the operator $\varphi : \mathcal{Z}^2 \to \mathcal{Z}^2$ is completely continuous. By continuity of the functions $\mathfrak{z}_k$, $k = 1, 2$, it follows that the operator φ is continuous. Let $\Theta \subset \mathcal{Z}^2$ be bounded. Then there exist positive constants Φ_k, $k = 1, 2$ such that

$$\left|\mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}(\wp)\right| \leq \Phi_k, \qquad k = 1, 2, \ \forall\, (\mathfrak{h}_1, \mathfrak{h}_2) \in \Theta.$$

Then for all $(\mathfrak{h}_1, \mathfrak{h}_2) \in \Theta$, we have:

$$\begin{aligned} |\varphi_1(\mathfrak{h}_1, \mathfrak{h}_2)(\wp)| \leq{} & {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\left|\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}\right|\right]\right](T) + |\beta_1|\, {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}[|\mathfrak{h}_1|](T) \\ & + \tfrac{(\eta_1-1)^{-\vartheta_1}T^{\eta_1-1}}{|\Pi_1|}\Big({}_{\mathrm{R.L}}\mathscr{I}^{\alpha_1}\left[{}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\left|\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}\right|\right]\right]\right](\theta_1) \\ & + {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\left|\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}\right|\right]\right](T) \\ & + |\beta_1|\, {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}[|\mathfrak{h}_1|](T) + |\beta_1|\, {}_{\mathrm{H}}\mathscr{I}^{\vartheta_1}[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}[|\mathfrak{h}_1|]](\theta_1)\Big), \end{aligned}$$

which implies that,

$$\begin{aligned} \|\varphi_1(\mathfrak{h}_1, \mathfrak{h}_2)\| \leq{} & \left[\tfrac{\eta_1^{-\vartheta_1}T^{\eta_1}}{\Gamma(\eta_1+1)} + \tfrac{(\eta_1-1)^{-\vartheta_1}T^{\eta_1-1}}{|\Pi_1|}\left(\tfrac{\eta_1^{-\vartheta_1}\theta_1^{\eta_1+\alpha_1}}{\Gamma(\eta_1+\alpha_1+1)} + \tfrac{\eta_1^{-\vartheta_1}T^{\eta_1}}{\Gamma(\eta_1+1)}\right)\right]\Phi_1 \\ & + \left[|\beta_1| T + \tfrac{(\eta_1-1)^{-\vartheta_1}T^{\eta_1-1}}{|\Pi_1|}\left(|\beta_1| T + |\beta_1| \tfrac{\theta_1^{\eta_1+1}}{\Gamma(\eta_1+2)}\right)\right]\epsilon. \end{aligned}$$

Indeed $\|\varphi_1(\mathfrak{h}_1,\mathfrak{h}_2)\| \leq \varDelta_1\Phi_1 + \nabla_1\epsilon < \infty$. We have also, $\|\varphi_2(\mathfrak{h}_1,\mathfrak{h}_2)\| \leq \varDelta_2\Phi_2 + \nabla_2\epsilon < \infty$. Thus, these follows that φ is uniformly bounded. Next, we show that φ is equicontinuous . For all $0 \leq \wp_1 < \wp_2 \leq T$, we have:

$$\begin{aligned}
&\left\|\varphi_1(\mathfrak{h}_1,\mathfrak{h}_2)(\wp_2) - \varphi_1(\mathfrak{h}_1,\mathfrak{h}_2)(\wp_1)\right\| \\
&\leq \frac{\Phi_1\eta_1^{-\vartheta_1}}{\Gamma(\eta_1+1)}\left|\wp_2^{\eta_1} - \wp_1^{\eta_1}\right| + \epsilon\left|\beta_1\right|\left|\wp_2 - \wp_1\right| \\
&+ \frac{\Phi_1(\eta_1-1)^{-\vartheta_1}}{|\Pi_1|}\left(\frac{\eta_1^{-\vartheta_1}\theta_1^{\eta_1+\alpha_1}}{\Gamma(\eta_1+\alpha_1+1)} + \frac{\eta_1^{-\vartheta_1}T^{\eta_1}}{\Gamma(\eta_1+1)}\right)\left|\wp_2^{\eta_1-1} - \wp_1^{\eta_1-1}\right| \\
&+ \frac{\epsilon|\beta_1|(\eta_1-1)^{-\vartheta_1}}{|\Pi_1|}\left(T + \frac{\theta_1^{\eta_1+1}}{\Gamma(\eta_1+2)}\right)\left|\wp_2^{\eta_1-1} - \wp_1^{\eta_1-1}\right|,
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\varphi_2(\mathfrak{h}_1,\mathfrak{h}_2)(\wp_2) - \varphi_1(\mathfrak{h}_1,\mathfrak{h}_2)(\wp_1)\right\| \\
&\leq \frac{\Phi_2\eta_2^{-\vartheta_2}}{\Gamma(\eta_2+1)}\left|\wp_2^{\eta_2} - \wp_1^{\eta_2}\right| + \epsilon\left|\beta_2\right|\left|\wp_2 - \wp_1\right| \\
&+ \frac{\Phi_2(\eta_2-1)^{-\vartheta_2}}{|\Pi_2|}\left(\frac{\eta_2^{-\vartheta_2}\theta_2^{\eta_2+\alpha_2}}{\Gamma(\eta_2+\alpha_2+1)} + \frac{\eta_2^{-\vartheta_2}T^{\eta_2}}{\Gamma(\eta_2+1)}\right)\left|\wp_2^{\eta_2-1} - \wp_1^{\eta_2-1}\right| \\
&+ \frac{\epsilon|\beta_2|(\eta_2-1)^{-\vartheta_2}}{|\Pi_2|}\left(T + \frac{\theta_2^{\eta_2+1}}{\Gamma(\eta_2+2)}\right)\left|\wp_2^{\eta_2-1} - \wp_1^{\eta_2-1}\right|.
\end{aligned}$$

Obviously, these inequalities tend to zero independently of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ as $\wp_2 \to \wp_1$. Therefore, the operator $\varphi(\mathfrak{h}_1,\mathfrak{h}_2)$ is equicontinuous and thus it is completely continuous. Finally, it will be verified that the set

$$\Psi = \Big\{(\mathfrak{h}_1,\mathfrak{h}_2) \in \mathcal{Z} : (\mathfrak{h}_1,\mathfrak{h}_2) = \upsilon\varphi(\mathfrak{h}_1,\mathfrak{h}_2),\, 0 \leq \upsilon \leq 1\Big\},$$

is bounded. Let $(\mathfrak{h}_1,\mathfrak{h}_2) \in \Psi$. Then, for each $\wp \in \Lambda$, we have $\mathfrak{h}_1(\wp) = \upsilon\varphi_1(\mathfrak{h}_1,\mathfrak{h}_2)(\wp)$ and $\mathfrak{h}_2(\wp) = \upsilon\varphi_2(\mathfrak{h}_1,\mathfrak{h}_2)(\wp)$. Then, $(\mathbf{H}_2)$ implies

$$\begin{aligned}
\left|\mathfrak{h}_1(\wp)\right| \leq &\left[\frac{\eta_1^{-\vartheta_1}T^{\eta_1}}{\Gamma(\eta_1+1)} + \frac{(\eta_1-1)^{-\vartheta_1}T^{\eta_1-1}}{|\Pi_1|}\left(\frac{\eta_1^{-\vartheta_1}\theta_1^{\eta_1+\alpha_1}}{\Gamma(\eta_1+\alpha_1+1)}\right.\right. \\
&\left.\left.+ \frac{\eta_1^{-\vartheta_1}T^{\eta_1}}{\Gamma(\eta_1+1)}\right)\right](\mu_0 + (\mu_1+\mu_2)\|\mathfrak{h}_1\| + \mu_3\|\mathfrak{h}_2\|) \\
&+ \left[|\beta_1|T + \frac{(\eta_1-1)^{-\vartheta_1}T^{\eta_1-1}}{|\Pi_1|}\left(|\beta_1|T + |\beta_1|\frac{\theta_1^{\eta_1+1}}{\Gamma(\eta_1+2)}\right)\right]\|\mathfrak{h}_1\|,
\end{aligned}$$

and

$$\begin{aligned}
\left|\mathfrak{h}_2(\wp)\right| \leq &\left[\frac{\eta_2^{-\vartheta_2}T^{\eta_2}}{\Gamma(\eta_2+1)} + \frac{(\eta_2-1)^{-\vartheta_2}T^{\eta_2-1}}{|\Pi_2|}\left(\frac{\eta_2^{-\vartheta_2}\theta_2^{\eta_2+\alpha_2}}{\Gamma(\eta_2+\alpha_2+1)}\right.\right. \\
&\left.\left.+ \frac{\eta_2^{-\vartheta_2}T^{\eta_2}}{\Gamma(\eta_2+1)}\right)\right](\varpi_0 + \varpi_1\|\mathfrak{h}_1\| + (\varpi_2+\varpi_3)\|\mathfrak{h}_2\|) \\
&+ \left[|\beta_2|T + \frac{(\eta_2-1)^{-\vartheta_2}T^{\eta_2-1}}{|\Pi_2|}\left(|\beta_2|T + |\beta_2|\frac{\theta_2^{\eta_2+1}}{\Gamma(\eta_2+2)}\right)\right]\|\mathfrak{h}_2\|,
\end{aligned}$$

which imply that

$$\|\mathfrak{h}_1\| \leq \varDelta_1\mu_0 + [\varDelta_1(\mu_1+\mu_2)+\nabla_1]\|\mathfrak{h}_1\| + \varDelta_1\mu_3\|\mathfrak{h}_2\|,$$
$$\|\mathfrak{h}_2\| \leq \varDelta_2\varpi_0 + \varDelta_2\varpi_1\|\mathfrak{h}_1\| + [\varDelta_2(\varpi_2+\varpi_3)+\nabla_2]\|\mathfrak{h}_2\|.$$

In view of the above estimates, we get:

$$\|\mathfrak{h}_1\| + \|\mathfrak{h}_2\| = \varDelta_1\mu_0 + \varDelta_2\varpi_0 + (\varDelta_1(\mu_1+\mu_2)+\nabla_1+\varDelta_2\varpi_1)\|\mathfrak{h}_1\| + (\nabla_2+\varDelta_1\mu_3+\varDelta_2(\varpi_2+\varpi_3))\|\mathfrak{h}_2\|.$$

Consequently,

$$\|(\mathfrak{h}_1,\mathfrak{h}_2)\|_{\mathcal{Z}^2} \leq \frac{1}{\min\{\Upsilon_1,\Upsilon_2\}}(\varDelta_1\mu_0+\varDelta_2\varpi_0),$$

here

$$\Upsilon_1 := 1 - [\varDelta_1(\mu_1+\mu_2)+\nabla_1+\varDelta_2\varpi_1],$$
$$\Upsilon_2 := 1 - [\nabla_2+\varDelta_1\mu_3+\varDelta_2(\varpi_2+\varpi_3)],$$

where $\varDelta_k$, ∇_k, $k=1,2$ are given by (9) and μ_i, $\varpi_i \geq 0$, $i=1,2,3$. This shows that Ψ is bounded. Hence, by Lemma 3.3, the operator φ has at least one fixed point. Hence, the system! (1)-(2) has at least one solution on Λ. The proof is complete. □

4. Ulam stability of coupled system (1)

We will define and study the Ulam stability of the system (1). For $\wp \in \Lambda$ and $k=1,2$, we give the following inequalities

$$\left|{}_{\mathrm{R.L}}\mathscr{D}^{\eta_k}\left[{}_{\mathrm{C.H}}\mathscr{D}^{\vartheta_k}+\beta_k\right]\mathfrak{h}_k(\wp) - \mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}(\wp)\right| \leq \delta_k, \tag{14}$$

and

$$\left|{}_{\mathrm{R.L}}\mathscr{D}^{\eta_k}\left[{}_{\mathrm{C.H}}\mathscr{D}^{\vartheta_k}+\beta_k\right]\mathfrak{h}_k(\wp) - \mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}(\wp)\right| \leq \delta_k\,\psi_k(\wp), \tag{15}$$

where δ_k are positive real numbers and $\psi_k : \Lambda \to \mathbb{R}_{\geq 0}$, $k=1,2$ are continuous functions.

Definition 4.1. System (1)-(2) is

- Ulam-Hyers stable if there exist a real number $\chi_{\mathfrak{z}_1,\mathfrak{z}_2} > 0$, such that for each $\delta = \max\{\delta_k : k=1,2\} > 0$ and for each solution $(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2) \in \mathcal{Z}^2$ of the inequality (14), there exists a solution $(\mathfrak{h}_1,\mathfrak{h}_2) \in \mathcal{Z}^2$ of the system (1)-(2) with
$$\left\|\left(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2\right) - (\mathfrak{h}_1,\mathfrak{h}_2)\right\| \leq \chi_{\mathfrak{z}_1,\mathfrak{z}_2}\delta.$$

- Ulam-Hyers-Rassias stable with respect to $\psi_k \in C(\Lambda)$, $k = 1, 2$ if there exists a real number $\epsilon_{\psi_1,\psi_2} > 0$ such that for each $\delta = \max\{\delta_k : k = 1, 2\} > 0$ and for each solution $(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2) \in \mathcal{Z}^2$ of the inequality (15), there exists a solution $(\mathfrak{h}_1, \mathfrak{h}_2) \in \mathcal{Z}^2$ of system (1)-(2) with

$$\left\|\left(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2\right) - (\mathfrak{h}_1, \mathfrak{h}_2)\right\| \leq \delta\epsilon_{\psi_1,\psi_2}\psi(\wp), \qquad \wp \in \Lambda,$$

where $\psi = \max\{\psi_k : k = 1, 2\}$.

Theorem 4.2. *Let $h_k \in C(\Lambda \times \mathbb{R}^3)$, $k = 1, 2$, satisfying $(\mathbf{H}_1)$. If the inequality*

$$\gamma_k \tfrac{\eta_k^{-\vartheta_k} T^{\eta_k}}{\Gamma(\eta_k+1)} < \tfrac{1}{3}(1 - |\beta_k| T), \qquad k = 1, 2, \tag{16}$$

is valid, then the system (1)-(2) is stable in Ulam-Hyers sens.

Proof. Let $\mathfrak{h}_k \in \mathcal{Z}$, $k = 1, 2$ be the unique solution of the system

$$\begin{cases} {}_{\text{R.L}}\mathscr{D}^{\eta_k}\left[{}_{\text{C.H}}\mathscr{D}^{\vartheta_k} + \beta_k\right]\mathfrak{h}_k(\wp) = \mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}(\wp), \quad \wp \in \Lambda, \\ \left[{}_{\text{C.H}}\mathscr{D}^{\vartheta_k} + \beta_k\right]\mathfrak{h}_k(0) = \left[{}_{\text{C.H}}\mathscr{D}^{\vartheta_k} + \beta_k\right]\acute{\mathfrak{h}}_k(0), \\ \mathfrak{h}_k(T) = \acute{\mathfrak{h}}_k(T), \ \mathfrak{h}_k(0) = \acute{\mathfrak{h}}_k(0), \end{cases}$$

where $\acute{\mathfrak{h}}_k \in \mathcal{Z}$, $k = 1, 2$ is a solution of the inequality (14). Thanks to Lemma 3.1, we have:

$$\begin{aligned} \mathfrak{h}_k(\wp) = {}& {}_{\text{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\text{R.L}}\mathscr{I}^{\eta_k}\left[\mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](\wp) \\ & - \beta_k \ {}_{\text{H}}\mathscr{I}^{\vartheta_k}[\mathfrak{h}_k](\wp) + e_{1k} \ {}_{\text{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-1}\right] \\ & + e_{2k} \ {}_{\text{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-2}\right] + e_{0k}, \qquad k = 1, 2, \end{aligned}$$

where e_{0k}, e_{1k}, $e_{2k} \in \mathbb{R}$, $k = 1, 2$. By integration of (14), we can obtain:

$$\begin{aligned} &\Big|\acute{\mathfrak{h}}_k(\wp) - {}_{\text{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\text{R.L}}\mathscr{I}^{\eta_k}\left[\mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](\wp) - \beta_k \ {}_{\text{H}}\mathscr{I}^{\vartheta_k}\left[\acute{\mathfrak{h}}_k\right](\wp) \\ &\qquad - c_{1k} \ {}_{\text{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-1}\right] - c_{2k} \ {}_{\text{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-2}\right] - c_{0k}\Big| \\ &\leq \delta_k \ {}_{\text{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\text{R.L}}\mathscr{I}^{\eta_k}[1]\right](\wp) \leq \tfrac{\delta_k \eta_k^{-\vartheta_k} T^{\eta_k}}{\Gamma(\eta_k+1)}, \qquad k = 1, 2. \end{aligned}$$

Using $(\mathbf{H}_1)$, we get:

$$\begin{aligned}
\left|\acute{\mathfrak{h}}_k(\wp) - \mathfrak{h}_k(\wp)\right| &= \Big| \acute{\mathfrak{h}}_k(\wp) - {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}\left[\mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](\wp) \\
&- \beta_k\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\acute{\mathfrak{h}}_k\right](\wp) - c_{1k}\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-1}\right] \\
&- c_{2k}\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-2}\right] - c_{0k} + \beta_k\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\acute{\mathfrak{h}}_k - \mathfrak{h}_k\right](\wp) \\
&+ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}\left[\mathfrak{z}^*_{k,(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2)} - \mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](\wp)\Big| \\
&\leq \Big| \acute{\mathfrak{h}}_k(\wp) - {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}\left[\mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](\wp) \\
&- \beta_k\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\acute{\mathfrak{h}}_k\right](\wp) - c_{1k}\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-1}\right] \\
&- c_{2k}{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-2}\right] - c_{0k}\Big| + |\beta_k|\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\left|\acute{\mathfrak{h}}_k - \mathfrak{h}_k\right|\right](\wp) \\
&+ \Big|{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}\left[\mathfrak{z}^*_{k,(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2)} - \mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](\wp)\Big| \\
&\leq \frac{\delta_k \eta_k^{-\vartheta_k} T^{\eta_k}}{\Gamma(\eta_k+1)} + \left(|\beta_k| T + 3\frac{\gamma_k \eta_k^{-\vartheta_k} T^{\eta_k}}{\Gamma(\eta_k+1)}\right)\left\|(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2) - (\mathfrak{h}_1,\mathfrak{h}_2)\right\|_{\mathcal{Z}^2}.
\end{aligned}$$

This implies that

$$\left\|\acute{\mathfrak{h}}_k - \mathfrak{h}_k\right\| \leq \frac{\delta_k \eta_k^{-\vartheta_k} T^{\eta_k}}{\Gamma(\eta_k+1)} + \left(|\beta_k| T + 3\frac{\gamma_k \eta_k^{-\vartheta_k} T^{\eta_k}}{\Gamma(\eta_k+1)}\right)\left\|(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2) - (\mathfrak{h}_1,\mathfrak{h}_2)\right\|_{\mathcal{Z}^2}.$$

Consequently for $k = 1, 2$, we obtain:

$$\left\|(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2) - (\mathfrak{h}_1,\mathfrak{h}_2)\right\|_{\mathcal{Z}^2} \leq \frac{1}{\min\{A_1, A_2\}}\left(\frac{\eta_1^{-\vartheta_1} T^{\eta_1}}{\Gamma(\eta_1+1)} + \frac{\eta_2^{-\vartheta_2} T^{\eta_2}}{\Gamma(\eta_2+1)}\right)\delta,$$

if we put

$$\chi_{\mathfrak{z}_1,\mathfrak{z}_2} := \frac{1}{\min\{A_1, A_2\}}\left[\frac{\eta_1^{-\vartheta_1} T^{\eta_1}}{\Gamma(\eta_1+1)} + \frac{\eta_2^{-\vartheta_2} T^{\eta_2}}{\Gamma(\eta_2+1)}\right], \tag{17}$$

where

$$A_k := 1 - \left(|\beta_k| T + 3\frac{\gamma_k \eta_k^{-\vartheta_k} T^{\eta_k}}{\Gamma(\eta_k+1)}\right), \qquad k = 1, 2, \tag{18}$$

then $\left\|(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2) - (\mathfrak{h}_1,\mathfrak{h}_2)\right\|_{\mathcal{Z}^2} \leq \chi_{\mathfrak{z}_1,\mathfrak{z}_2}\delta$. Hence, the system (1)-(2) is stable in Ulam-Hyers sens. □

Theorem 4.3. *Assume that $\mathfrak{z}_k \in C(\Lambda \times \mathbb{R}^3)$, $k = 1, 2$, satisfying $(\mathbf{H}_1)$ and (16) holds. If there exists $\epsilon_{\psi_k} > 0$, $k = 1, 2$ such that*

$${}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}[\psi_k]\right](\wp) \leq \epsilon_{\psi_k}\psi_k(\wp), \qquad \wp \in \Lambda,\ k = 1, 2, \tag{19}$$

where $\psi_k \in C(\Lambda, \mathbb{R}_{\geq 0})$ are nondecreasing. Then the our system (1)-(2) is stable in Ulam-Hyers-Rassias sens with respect to ψ_k, $k = 1, 2$.

Proof. From (15), we see

$$\Big|\acute{\mathfrak{h}}_k(\wp) - {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}\left[\mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](\wp) - \beta_k\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\acute{\mathfrak{h}}_k\right](\wp)$$
$$- c_{1k}{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-1}\right] - c_{2k}{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-2}\right] - c_{0k}\Big|$$
$$\leq \delta_k{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}\left[\psi_k\right]\right](\wp) \leq \delta_k\epsilon_{\psi_k}\psi_k(\wp), \qquad k=1,2,$$

where $\acute{\mathfrak{h}}_k \in \mathcal{Z}$, $k=1,2$ is a solution of inequality (15). So, for $k=1,2$, we have:

$$\left|\acute{\mathfrak{h}}_k(\wp) - \mathfrak{h}_k(\wp)\right| = \Big|\acute{\mathfrak{h}}_k(\wp) - {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}\left[\mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](\wp)$$
$$- \beta_k{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[t_k\right](\wp) - c_{1k}{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-1}\right]$$
$$- c_{2k}{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-2}\right] - c_{0k} + \beta_k\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\acute{\mathfrak{h}}_k - \mathfrak{h}_k\right](\wp)$$
$$+ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}\left[\mathfrak{z}^*_{k,(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2)} - \mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](\wp)\Big|$$
$$\leq \Big|\acute{\mathfrak{h}}_k(\wp) - {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}\left[\mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](\wp)$$
$$- \beta_k{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[t_k\right](\wp) - c_{1k}{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-1}\right]$$
$$- c_{2k}\ {}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\wp^{\eta_k-2}\right] - c_{0k}\Big| + |\beta_k|{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[\left|\acute{\mathfrak{h}}_k - \acute{\mathfrak{h}}_k\right|\right](\wp)$$
$$+ \left|{}_{\mathrm{H}}\mathscr{I}^{\vartheta_k}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_k}\left[\mathfrak{z}^*_{k,(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2)} - \mathfrak{z}^*_{k,(\mathfrak{h}_1,\mathfrak{h}_2)}\right]\right](\wp)\right|,$$

such that $\mathfrak{h}_k \in \mathcal{Z}$, $k=1,2$ is a unique solution of system (1)-(2). Then, by ($\mathbf{H}_1$) and (19), we have:

$$\left\|\acute{\mathfrak{h}}_k - \mathfrak{h}_k\right\| \leq \delta_k\epsilon_{\psi_k}\psi_k(\wp) + \left(|\beta_k|T + 3\frac{\gamma_k\eta_k^{-\vartheta_k}T^{\eta_k}}{\Gamma(\eta_k+1)}\right)\left\|\left(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2\right) - (\mathfrak{h}_1,\mathfrak{h}_2)\right\|_{\mathcal{Z}^2}.$$

For $k=1,2$, we obtain:

$$\left\|\left(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2\right) - (\mathfrak{h}_1,\mathfrak{h}_2)\right\|_{\mathcal{Z}^2} \leq \tfrac{1}{\min(A_1,A_2)}\left(\epsilon_{\psi_1}+\epsilon_{\psi_2}\right)\delta\psi(\wp) := \delta\epsilon_{\psi_1,\psi_2}\psi(\wp),$$

where $\psi = \max\{\psi_k : k=1,2\}$ and $\epsilon_{\psi_1,\psi_2} = \frac{1}{\min(A_1,A_2)}\left(\epsilon_{\psi_1}+\epsilon_{\psi_2}\right)$. Hence, (1) is stable in Ulam-Hyers-Rassias sense. □

5. Examples with numerical study

In Example 5.1, we check the results for system (1) for different values of R.L derivative order η_1 in first mixed equation of the system.

Example 5.1. According to system (1)-(2), consider the following sequential $\mathbb{FPE}$

$$\begin{cases} {}_{\mathrm{R.L}}\mathscr{D}^{\eta_1}\left[{}_{\mathrm{C.H}}\mathscr{D}^{\sqrt{5}/7}+\frac{e}{13}\right]\mathfrak{h}_1(\wp) \\ \quad =\frac{10^{-2}\cos(\wp)}{\sqrt{90^2+\wp^2}}\left(\cos^2\mathfrak{h}_1(\wp)+\mathfrak{h}_1\left(\frac{2\wp}{3}\right)+\frac{|\mathfrak{h}_2(\wp)|}{|\mathfrak{h}_2(\wp)|+2}+\ln(\wp+2)\right), \\ \left[{}_{\mathrm{C.H}}\mathscr{D}^{\sqrt{5}/7}+\frac{e}{13}\right]\mathfrak{h}_1(0)=0, \quad \mathfrak{h}_1(1)={}_{\mathrm{R.L}}\mathscr{I}^{3/2}\mathfrak{h}_1\left(\frac{2}{5}\right), \quad \mathfrak{h}_1(0)=0, \\ {}_{\mathrm{R.L}}\mathscr{D}^{4/3}\left[{}_{\mathrm{C.H}}\mathscr{D}^{6/7}+\frac{\ln 5}{11}\right]\mathfrak{h}_2(\wp) \\ \quad =\frac{e^{-\wp}}{37^2+\wp^2}\left(\frac{|\mathfrak{h}_1(\wp)|}{|\mathfrak{h}_1(\wp)|+3}+\sin\mathfrak{h}_2(\wp)+\mathfrak{h}_2\left(\frac{2\wp}{3}\right)+\arctan(\wp+1)\right), \\ \left[{}_{\mathrm{C.H}}\mathscr{D}^{6/7}+\frac{\ln 5}{11}\right]\mathfrak{h}_2(0)=0, \quad \mathfrak{h}_2(1)={}_{\mathrm{R.L}}\mathscr{I}^{7/6}\mathfrak{h}_2\left(\frac{3}{7}\right), \quad \mathfrak{h}_2(0)=0, \end{cases} \tag{20}$$

for $\wp\in\Lambda=[0,1]$, $T=1$ and three values of $\eta_1=\left\{\frac{5}{4},\frac{5}{3},\frac{19}{10}\right\}\subseteq[1,2]$. Clearly, $\vartheta_1=\frac{\sqrt{5}}{7}\in\Lambda$, $\eta_2=\frac{4}{3}\in[1,2]$, $\vartheta_2=\frac{\sqrt{6}}{7}\in\Lambda$, $\omega=\frac{2}{3}\in(0,1)$, $\beta_1=\frac{1}{13}e$, $\beta_2=\frac{1}{11}\ln 5$, $\alpha_1=\frac{3}{2}$, $\alpha_2=\frac{7}{6}$ and $\theta_1=\frac{2}{5}\in(0,1)$, $\theta_2=\frac{3}{7}\in(0,1)$. Furthermore, assume that,

$$\begin{aligned} \left|{}_{\mathrm{R.L}}\mathscr{D}^{5/3}\left[{}_{\mathrm{C.H}}\mathscr{D}^{\sqrt{5}/7}+\frac{e}{13^2}\right]\mathfrak{h}_1(\wp)-\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}(\wp)\right| &\leq\delta_1, \\ \left|{}_{\mathrm{R.L}}\mathscr{D}^{4/3}\left[{}_{\mathrm{C.H}}\mathscr{D}^{6/7}+\frac{\ln 5}{15^2}\right]\mathfrak{h}_2(\wp)-\mathfrak{z}^*_{2,(\mathfrak{h}_1,\mathfrak{h}_2)}(\wp)\right| &\leq\delta_2, \end{aligned} \tag{21}$$

and

$$\begin{aligned} \left|{}_{\mathrm{R.L}}\mathscr{D}^{5/3}\left[{}_{\mathrm{C.H}}\mathscr{D}^{\sqrt{5}/7}+\frac{e}{13^2}\right]\mathfrak{h}_1(\wp)-\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}(\wp)\right| &\leq\delta_1\psi_1(\wp), \\ \left|{}_{\mathrm{R.L}}\mathscr{D}^{4/3}\left[{}_{\mathrm{C.H}}\mathscr{D}^{6/7}+\frac{\ln 5}{15^2}\right]\mathfrak{h}_2(\wp)-\mathfrak{z}^*_{2,(\mathfrak{h}_1,\mathfrak{h}_2)}(\wp)\right| &\leq\delta_2\psi_2(\wp). \end{aligned} \tag{22}$$

By the given data and definitions,

$$\begin{aligned} \mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2)}(\wp) &=\mathfrak{z}_1(\wp,\mathfrak{h}_1(\wp),\mathfrak{h}_1(\omega\wp),\mathfrak{h}_2(\wp)) \\ &=\frac{10^{-2}\cos(\wp)}{\sqrt{90^2+\wp^2}}\left(\cos^2\mathfrak{h}_1(\wp)+\mathfrak{h}_1\left(\frac{2\wp}{3}\right)+\frac{|\mathfrak{h}_2(\wp)|}{|\mathfrak{h}_2(\wp)|+2}+\ln(\wp+2)\right), \\ \mathfrak{z}^*_{2,(\mathfrak{h}_1,\mathfrak{h}_2)}(\wp) &=\mathfrak{z}_2(\wp,\mathfrak{h}_1(\wp),\mathfrak{h}_1(\omega\wp),\mathfrak{h}_2(\wp)) \\ &=\frac{e^{-\wp}}{37^2+\wp^2}\left(\frac{|\mathfrak{h}_1(\wp)|}{|\mathfrak{h}_1(\wp)|+3}+\sin\mathfrak{h}_2(\wp)+\mathfrak{h}_2\left(\frac{2\wp}{3}\right)+\arctan(\wp+1)\right), \end{aligned}$$

for all $\wp \in \Lambda$ and $\mathfrak{h}_j$, $\acute{\mathfrak{h}}_j \in \mathbb{R}$, $j = 1, 2, 3$, we have:

$$\begin{aligned}
&\left|\mathfrak{z}^*_{1,(\mathfrak{h}_1,\mathfrak{h}_2,\mathfrak{h}_3)}(\wp) - \mathfrak{z}^*_{1,\left(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2,\acute{\mathfrak{h}}_3\right)}(\wp)\right| \\
&= \left|\tfrac{10^{-2}\cos(\wp)}{\sqrt{90^2+\wp^2}}\left(\cos^2\mathfrak{h}_1(\wp) + \mathfrak{h}_2\left(\tfrac{2\wp}{3}\right) + \tfrac{|\mathfrak{h}_3(\wp)|}{|\mathfrak{h}_3(\wp)|+2}\right.\right. \\
&\left. + \ln(\wp+2)\right) - \left[\tfrac{10^{-2}\cos(\wp)}{\sqrt{90^2+\wp^2}}\left(\cos^2\acute{\mathfrak{h}}_1(\wp) + \acute{\mathfrak{h}}_2\left(\tfrac{2\wp}{3}\right)\right.\right. \\
&\left.\left.\left. + \tfrac{\left|\acute{\mathfrak{h}}_3(\wp)\right|}{\left|\acute{\mathfrak{h}}_3(\wp)\right|+2} + \ln(\wp+2)\right)\right]\right| \\
&\leq \left|\tfrac{10^{-2}\cos(\wp)}{\sqrt{90^2+\wp^2}}\right| \left|\cos^2\mathfrak{h}_1(\wp) - \cos^2\acute{\mathfrak{h}}_1(\wp)\right| \quad + \left|\tfrac{10^{-2}\cos(\wp)}{\sqrt{90^2+\wp^2}}\right| \left|\mathfrak{h}_2\left(\tfrac{2\wp}{3}\right) - \acute{\mathfrak{h}}_2\left(\tfrac{2\wp}{3}\right)\right| \\
&+ \left|\tfrac{10^{-2}\cos(\wp)}{\sqrt{90^2+\wp^2}}\right| \left|\tfrac{|\mathfrak{h}_3(\wp)|}{|\mathfrak{h}_3(\wp)|+2} - \tfrac{\left|\acute{\mathfrak{h}}_3(\wp)\right|}{\left|\acute{\mathfrak{h}}_3(\wp)\right|+2}\right| \\
&\leq \tfrac{1}{9000}\left|\sin^2\acute{\mathfrak{h}}_1(\wp) - \sin^2\mathfrak{h}_1(\wp)\right| + \tfrac{1}{9000}\left\|\mathfrak{h}_2 - \acute{\mathfrak{h}}_2\right\| \quad + \tfrac{1}{9000}\left|\tfrac{2\left(|\mathfrak{h}_3(\wp)| - \left|\acute{\mathfrak{h}}_3(\wp)\right|\right)}{(|\mathfrak{h}_3(\wp)|+2)\left(\left|\acute{\mathfrak{h}}_3(\wp)\right|+2\right)}\right| \\
&\leq \tfrac{2}{9000}\left|\acute{\mathfrak{h}}_1(\wp) - \mathfrak{h}_1(\wp)\right| + \tfrac{1}{9000}\left\|\mathfrak{h}_2 - \acute{\mathfrak{h}}_2\right\| \quad + \tfrac{1}{18000}\left|\mathfrak{h}_3(\wp) - \acute{\mathfrak{h}}_3(\wp)\right| \\
&\leq \tfrac{1}{4500}\sum_{j=1}^{3}\left|\mathfrak{h}_j - \acute{\mathfrak{h}}_j\right|,
\end{aligned}$$

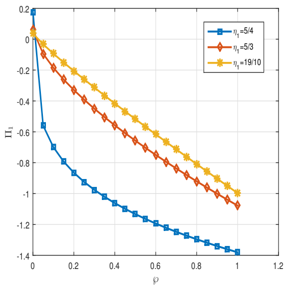
and

$$\begin{aligned}
&\left|\mathfrak{z}^*_{2,(\mathfrak{h}_1,\mathfrak{h}_2,\mathfrak{h}_3)}(\wp) - \mathfrak{z}^*_{2,\left(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2,\acute{\mathfrak{h}}_3\right)}(\wp)\right| \\
&\qquad = \left|\tfrac{e^{-\wp}}{37^2+\wp^2}\left(\tfrac{|\mathfrak{h}_1(\wp)|}{|\mathfrak{h}_1(\wp)|+3} + \sin\mathfrak{h}_2(\wp) + \mathfrak{h}_3\left(\tfrac{2\wp}{3}\right)\right.\right. \\
&\qquad \left. + \arctan(\wp+1)\right) - \left[\tfrac{e^{-\wp}}{37^2+\wp^2}\left(\tfrac{\left|\acute{\mathfrak{h}}_1(\wp)\right|}{\left|\acute{\mathfrak{h}}_1(\wp)\right|+3}\right.\right. \\
&\qquad \left.\left.\left. + \sin\acute{\mathfrak{h}}_2(\wp) + \acute{\mathfrak{h}}_3\left(\tfrac{2\wp}{3}\right) + \arctan(\wp+1)\right)\right]\right| \\
&\qquad \leq \left|\tfrac{e^{-\wp}}{37^2+\wp^2}\right| \left[\left|\tfrac{|\mathfrak{h}_1(\wp)|}{|\mathfrak{h}_1(\wp)|+3} - \tfrac{\left|\acute{\mathfrak{h}}_1(\wp)\right|}{\left|\acute{\mathfrak{h}}_1(\wp)\right|+3}\right|\right. \\
&\qquad \left. + \left|\sin\mathfrak{h}_2(\wp) - \sin\acute{\mathfrak{h}}_2(\wp)\right| + \left|\mathfrak{h}_3\left(\tfrac{2\wp}{3}\right) - \acute{\mathfrak{h}}_3\left(\tfrac{2\wp}{3}\right)\right|\right] \\
&\qquad \leq \tfrac{1}{37^2}\left|\left[\left|\tfrac{3\left|\mathfrak{h}_1(\wp) - \acute{\mathfrak{h}}_1(\wp)\right|}{(|\mathfrak{h}_1(\wp)|+3)\left(\left|\acute{\mathfrak{h}}_1(\wp)\right|+3\right)}\right|\right.\right. \\
&\qquad \left. + \left|\mathfrak{h}_2(\wp) - \acute{\mathfrak{h}}_2(\wp)\right| + \left\|\mathfrak{h}_3 - \acute{\mathfrak{h}}_3\right\|\right] \leq \tfrac{1}{37^2}\sum_{j=1}^{3}\left|\mathfrak{h}_j - \acute{\mathfrak{h}}_j\right|.
\end{aligned}$$

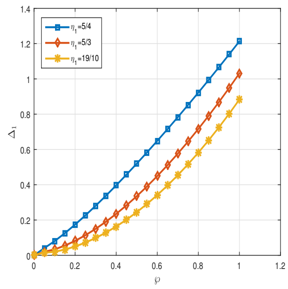
Hence condition $(\mathbf{H}_1)$ holds with $\gamma_1 = \frac{1}{4500}$ and $\gamma_2 = \frac{1}{1369}$. With the given data, and employing Equations (5) and (9), it is found that

$$\begin{aligned}\Pi_1 &= (\eta_1 - 1)^{-\vartheta_1}\left(\frac{\Gamma(\eta_1)\theta_1^{\eta_1+\alpha_1-1}}{\Gamma(\eta_1+\alpha_1)} - T^{\eta_1-1}\right)\\ &= (\eta_1 - 1)^{-\sqrt{5}/7}\left(\frac{\Gamma(\eta_1)\left(\frac{2}{5}\right)^{\eta_1+\frac{3}{2}-1}}{\Gamma\left(\eta_1+\frac{3}{2}\right)}\right) = \begin{cases} -1.3806, & \eta_1 = \frac{5}{4},\\ -1.0781, & \eta_1 = \frac{5}{3},\\ -0.9972, & \eta_1 = \frac{19}{10},\end{cases}\end{aligned}$$

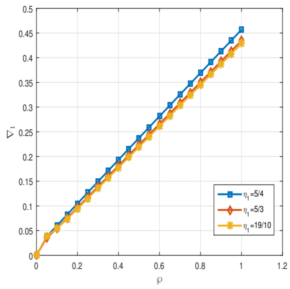
$$\begin{aligned}\Pi_2 &= (\eta_2 - 1)^{-\vartheta_2}\left(\frac{\Gamma(\eta_2)\theta_2^{\eta_2+\alpha_2-1}}{\Gamma(\eta_2+\alpha_2)} - T^{\eta_2-1}\right)\\ &= \left(\frac{4}{3} - 1\right)^{-\sqrt{6}/7}\left(\frac{\Gamma\left(\frac{4}{3}\right)\left(\frac{\sqrt{6}}{7}\right)^{\frac{4}{3}+\frac{7}{6}-1}}{\Gamma\left(\frac{4}{3}+\frac{7}{6}\right)}\right) = -1.1920,\end{aligned}$$



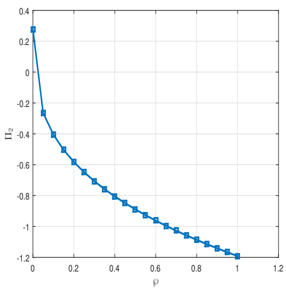
(a) Π_1

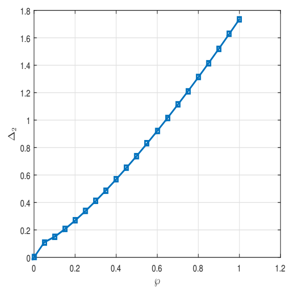


(b) $\mathit{\Delta}_1$



(c) ∇_1


(d) Π_2


(e) $\mathit{\Delta}_2$

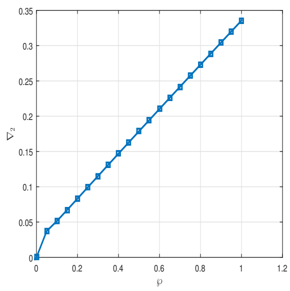

(f) ∇_2

Figure 1: 2D plot of Π_i, $\mathit{\Delta}_i$ and ∇_i, $i = 1, 2$ for three different values of η_1 in the system (20) in Example 5.1.

Table 1: System (20): Numerical values of Π_i and $\varDelta_i$ in Example 5.1 when $\eta_1 = \frac{5}{4}, \frac{5}{3}, \frac{19}{10}$.

	Π_1			Π_2	$\varDelta_1$			$\varDelta_2$
$\wp$	$\eta_1=\frac{5}{4}$	$\eta_1=\frac{5}{3}$	$\eta_1=\frac{19}{10}$		$\eta_1=\frac{5}{4}$	$\eta_1=\frac{5}{3}$	$\eta_1=\frac{19}{10}$	
0.00	0.1765	0.0602	0.0370	0.2768	0.0000	0.0000	0.0000	0.0000
0.05	−0.5598	−0.0943	−0.0328	−0.2643	0.0406	0.0197	0.0152	0.1096
0.10	−0.6991	−0.1851	−0.0932	−0.4049	0.0802	0.0335	0.0200	0.1497
0.15	−0.7925	−0.2612	−0.1505	−0.5036	0.1253	0.0549	0.0325	0.2050
0.20	−0.8648	−0.3291	−0.2060	−0.5821	0.1742	0.0818	0.0500	0.2682
0.25	−0.9245	−0.3915	−0.2600	−0.6485	0.2263	0.1136	0.0719	0.3370
0.30	−0.9758	−0.4499	−0.3130	−0.7064	0.2809	0.1499	0.0982	0.4103
0.35	−1.0211	−0.5051	−0.3650	−0.7583	0.3377	0.1904	0.1286	0.4877
0.40	−1.0618	−0.5578	−0.4164	−0.8054	0.3966	0.2349	0.1632	0.5685
0.45	−1.0988	−0.6083	−0.4671	−0.8487	0.4573	0.2831	0.2019	0.6526
0.50	−1.1328	−0.6569	−0.5172	−0.8890	0.5196	0.3350	0.2445	0.7395
0.55	−1.1644	−0.7039	−0.5669	−0.9266	0.5835	0.3904	0.2911	0.8292
0.60	−1.1939	−0.7496	−0.6161	−0.9620	0.6487	0.4491	0.3416	0.9214
0.65	−1.2216	−0.7940	−0.6648	−0.9955	0.7153	0.5112	0.3960	1.0159
0.70	−1.2477	−0.8372	−0.7132	−1.0273	0.7832	0.5765	0.4543	1.1128
0.75	−1.2725	−0.8795	−0.7613	−1.0577	0.8522	0.6449	0.5163	1.2117
0.80	−1.2961	−0.9208	−0.8091	−1.0867	0.9224	0.7164	0.5822	1.3127
0.85	−1.3186	−0.9612	−0.8565	−1.1145	0.9936	0.7909	0.6519	1.4157
0.90	−1.3401	−1.0009	−0.9037	−1.1413	1.0659	0.8683	0.7253	1.5206
0.95	−1.3607	−1.0398	−0.9506	−1.1671	1.1391	0.9486	0.8024	1.6272
1.00	−1.3806	−1.0781	−0.9972	−1.1920	1.2133	1.0317	0.8832	1.7356

and

$$
\begin{aligned}
\varDelta_1 &= \frac{\eta_1^{-\vartheta_1} T^{\eta_1}}{\Gamma(\eta_1+1)} + \frac{(\eta_1-1)^{-\vartheta_1} T^{\eta_1-1}}{|\Pi_1|}\left[\frac{\eta_1^{-\vartheta_1}\theta_1^{\eta_1+\alpha_1}}{\Gamma(\eta_1+\alpha_1+1)} + \frac{\eta_1^{-\vartheta_1} T^{\eta_1}}{\Gamma(\eta_1+1)}\right] \\
&= \frac{\eta_1^{-\sqrt{5}/7}}{\Gamma(\eta_1+1)} + \frac{(\eta_1-1)^{-\sqrt{5}/7}}{|\Pi_1|}\left[\frac{\eta_1^{-\sqrt{5}/7}\left(\frac{2}{5}\right)^{\eta_1+\frac{3}{2}}}{\Gamma\left(\eta_1+\frac{3}{2}+1\right)} + \frac{\eta_1^{-\sqrt{5}/7}}{\Gamma(\eta_1+1)}\right] \\
&= \begin{cases} 1.2133, & \eta_1 = \frac{5}{4}, \\ 1.0317, & \eta_1 = \frac{5}{3}, \\ 0.8832, & \eta_1 = \frac{19}{10}, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\varDelta_2 &= \frac{\eta_2^{-\vartheta_2} T^{\eta_2}}{\Gamma(\eta_2+1)} + \frac{(\eta_2-1)^{-\vartheta_2} T^{\eta_2-1}}{|\Pi_2|}\left[\frac{\eta_2^{-\vartheta_2}\theta_2^{\eta_2+\alpha_2}}{\Gamma(\eta_2+\alpha_2+1)} + \frac{\eta_2^{-\vartheta_2} T^{\eta_2}}{\Gamma(\eta_2+1)}\right] \\
&= \frac{\left(\frac{4}{3}\right)^{-\sqrt{6}/7}}{\Gamma\left(\frac{4}{3}+1\right)} + \frac{\left(\frac{4}{3}-1\right)^{-\sqrt{6}/7}}{|\Pi_2|}\left[\frac{\left(\frac{4}{3}\right)^{-\sqrt{6}/7}\left(\frac{3}{7}\right)^{\frac{4}{3}+\frac{7}{6}}}{\Gamma\left(\frac{3}{7}+\frac{7}{6}+1\right)} + \frac{\left(\frac{4}{3}\right)^{-\sqrt{6}/7}}{\Gamma\left(\frac{3}{7}+1\right)}\right] = 1.7356,
\end{aligned}
$$

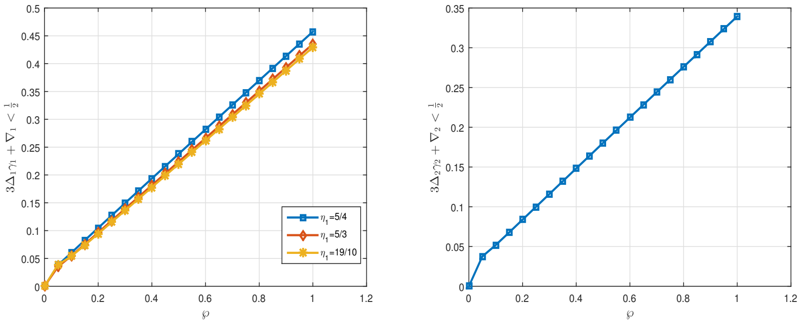
Table 2: System (20): results of ∇_i and inequality (12) in Example 5.1 when $\eta_1 = \frac{5}{4}, \frac{5}{3}, \frac{19}{10}$.

	∇_1			∇_2	Ineq. (12), $k=1$			Ineq. (12)
$\wp$	$\eta_1=\frac{5}{4}$	$\eta_1=\frac{5}{3}$	$\eta_1=\frac{19}{10}$		$\eta_1=\frac{5}{4}$	$\eta_1=\frac{5}{3}$	$\eta_1=\frac{19}{10}$	$k=2$
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.05	0.0379	0.0350	0.0386	0.0372	0.0380	0.0350	0.0386	0.0375
0.10	0.0602	0.0546	0.0540	0.0515	0.0602	0.0546	0.0540	0.0519
0.15	0.0825	0.0755	0.0739	0.0673	0.0826	0.0756	0.0739	0.0677
0.20	0.1047	0.0966	0.0944	0.0832	0.1049	0.0967	0.0945	0.0838
0.25	0.1270	0.1178	0.1152	0.0992	0.1271	0.1179	0.1152	0.0999
0.30	0.1491	0.1390	0.1360	0.1151	0.1493	0.1391	0.1360	0.1160
0.35	0.1713	0.1602	0.1568	0.1311	0.1715	0.1603	0.1569	0.1321
0.40	0.1934	0.1813	0.1777	0.1470	0.1936	0.1815	0.1778	0.1482
0.45	0.2154	0.2025	0.1986	0.1628	0.2157	0.2027	0.1988	0.1643
0.50	0.2375	0.2236	0.2195	0.1787	0.2378	0.2238	0.2197	0.1803
0.55	0.2595	0.2448	0.2405	0.1945	0.2599	0.2450	0.2407	0.1963
0.60	0.2815	0.2659	0.2614	0.2102	0.2819	0.2662	0.2616	0.2122
0.65	0.3034	0.2870	0.2823	0.2260	0.3039	0.2873	0.2826	0.2282
0.70	0.3254	0.3081	0.3032	0.2417	0.3259	0.3085	0.3035	0.2441
0.75	0.3473	0.3292	0.3242	0.2574	0.3479	0.3296	0.3245	0.2600
0.80	0.3692	0.3503	0.3451	0.2731	0.3698	0.3508	0.3455	0.2759
0.85	0.3911	0.3714	0.3660	0.2887	0.3918	0.3719	0.3665	0.2918
0.90	0.4130	0.3925	0.3870	0.3044	0.4137	0.3931	0.3874	0.3077
0.95	0.4349	0.4136	0.4079	0.3200	0.4356	0.4142	0.4084	0.3236
1.00	0.4567	0.4346	0.4288	0.3356	0.4575	0.4353	0.4294	0.3394

$$\begin{aligned}\nabla_1 &= |\beta_1|T + \frac{(\eta_1-1)^{-\vartheta_1}T^{\eta_1-1}}{|\Pi_1|}\left[|\beta_1|T + |\beta_1|\frac{\theta_1^{\eta_1+1}}{\Gamma(\eta_1+2)}\right] \\ &= \left|\frac{e}{13}\right| + \frac{(\eta_1-1)^{-\sqrt{5}/7}}{|\Pi_1|}\left[\left|\frac{e}{13}\right| + \left|\frac{e}{13}\right|\frac{\left(\frac{2}{5}\right)^{\eta_1+1}}{\Gamma(\eta_1+2)}\right] \\ &= \begin{cases} 0.4567, & \eta_1 = \frac{5}{4}, \\ 0.4346, & \eta_1 = \frac{5}{3}, \\ 0.4288, & \eta_1 = \frac{19}{10}, \end{cases}\end{aligned}$$

$$\begin{aligned}\nabla_2 &= |\beta_2|T + \frac{(\eta_2-1)^{-\vartheta_2}T^{\eta_2-1}}{|\Pi_2|}\left[|\beta_2|T + |\beta_2|\frac{\theta_2^{\eta_2+1}}{\Gamma(\eta_2+2)}\right] \\ &= \left|\frac{\ln 5}{11}\right| + \frac{\left(\frac{4}{3}-1\right)^{-\sqrt{6}/7}}{|\Pi_2|}\left[\left|\frac{\ln 5}{11}\right| + \left|\frac{\ln 5}{11}\right|\frac{\left(\frac{3}{7}\right)^{\eta_2+1}}{\Gamma\left(\frac{4}{3}+2\right)}\right] = 0.3356.\end{aligned}$$

Tables 1 to 3, show the calculated values. In Figures 1a to 1f, we have plotted the results of Π_k, Δ_k, ∇_k, $k = 1, 2$ respectively for the system (20). Further, in Figures 2a and 2b, we have plotted $3\Delta_{ii} + \nabla_i, i = 1, 2$ for three different values of



(a) $3\mathit{\Delta}_1\gamma_1+\nabla_1$ (b) $3\mathit{\Delta}_2\gamma_2+\nabla_2$

Figure 2: 2D plot of Inequality (12), $3\mathit{\Delta}_i\gamma_i+\nabla_i$, $i=1,2$ for three different values of η_1 in the system (20) in Example 5.1.

η_1 for the system (20). Therefore, conditions

$$3\mathit{\Delta}_1\gamma_1+\nabla_1=\left\{\begin{array}{ll}0.4567, & \eta_1=\frac{5}{4},\\ 0.4346, & \eta_1=\frac{5}{3},\\ 0.4288, & \eta_1=\frac{19}{10},\end{array}\right\}<\frac{1}{2},\qquad 3\mathit{\Delta}_2\gamma_2+\nabla_2=0.04109<\frac{1}{2},$$

and

$$\gamma_1\frac{\eta_1^{-\vartheta_1}T^{\eta_1}}{\Gamma(\eta_1+1)}=\left\{\begin{array}{ll}0.000182, & \eta_1=\frac{5}{4},\\ 0.000125, & \eta_1=\frac{5}{3},\\ 0.000099, & \eta_1=\frac{19}{10},\end{array}\right\}<0.263633=\frac{1}{3}\left(1-|\beta_1|T\right),$$

$$\gamma_2\frac{\eta_2^{-\vartheta_2}T^{\eta_2}}{\Gamma(\eta 2+1)}=0.0005547<0.284562=\frac{1}{3}\left(1-|\beta_2|T\right),$$

are satisfied. It follows from Theorem 3.2, that the problem (20) has a unique solution on Λ, and by considering the Equation (17),

$$\chi_{\mathfrak{z}_1,\mathfrak{z}_2}=\frac{1}{\min\{A_1,A_2\}}\left[\frac{\eta_1^{-\vartheta_1}T^{\eta_1}}{\Gamma(\eta_1+1)}+\frac{\eta_2^{-\vartheta_2}T^{\eta_2}}{\Gamma(\eta_2+1)}\right]=\left\{\begin{array}{ll}2.00079, & \eta_1=\frac{5}{4},\\ 1.67486, & \eta_1=\frac{5}{3},\\ 1.52445, & \eta_1=\frac{19}{10},\end{array}\right.$$

is Ulam-Hyers stable with

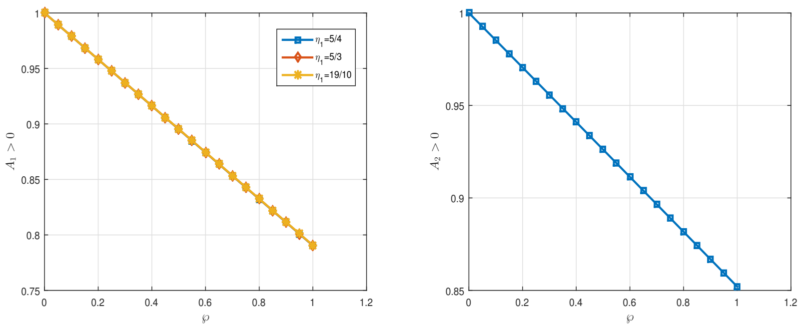
$$\left\|\left(\acute{\mathfrak{h}}_1,\acute{\mathfrak{h}}_2\right)-\left(\mathfrak{h}_1,\mathfrak{h}_2\right)\right\|\leq\left\{\begin{array}{ll}2.00079\,\delta, & \eta_1=\frac{5}{4},\\ 1.67486\,\delta, & \eta_1=\frac{5}{3},\\ 1.52445\,\delta, & \eta_1=\frac{19}{10},\end{array}\right.\qquad \delta>0.$$

Table 3: System (20): Numerical values of A_k, $k = 1, 2$, inequality (18) in Example 5.1 when $\eta_1 = \frac{5}{4}, \frac{5}{3}, \frac{19}{10}$.

	A_1			A_2
$\wp$	$\eta_1 = \frac{5}{4}$	$\eta_1 = \frac{5}{3}$	$\eta_1 = \frac{19}{10}$	
0.00	1.0000	1.0000	1.0000	1.0000
0.05	0.9895	0.9895	0.9895	0.9927
0.10	0.9791	0.9791	0.9791	0.9853
0.15	0.9686	0.9686	0.9686	0.9779
0.20	0.9581	0.9582	0.9582	0.9705
0.25	0.9476	0.9477	0.9477	0.9632
0.30	0.9371	0.9372	0.9372	0.9558
0.35	0.9267	0.9268	0.9268	0.9484
0.40	0.9162	0.9163	0.9163	0.9410
0.45	0.9057	0.9058	0.9058	0.9336
0.50	0.8952	0.8953	0.8954	0.9262
0.55	0.8847	0.8849	0.8849	0.9188
0.60	0.8743	0.8744	0.8744	0.9114
0.65	0.8638	0.8639	0.8640	0.9040
0.70	0.8533	0.8534	0.8535	0.8965
0.75	0.8428	0.8429	0.8430	0.8891
0.80	0.8323	0.8325	0.8325	0.8817
0.85	0.8218	0.8220	0.8220	0.8743
0.90	0.8113	0.8115	0.8116	0.8669
0.95	0.8008	0.8010	0.8011	0.8594
1.00	0.7904	0.7905	0.7906	0.8520

In Figures 3a and 3b, we have plotted the results of A_k, $k = 1, 2$ respectively for the system (20). Because the values of A_1 for $\eta_1 = \frac{5}{4}, \frac{5}{3}, \frac{19}{10}$ are very close to each other, the three curves almost coincide. Let $\psi_1(\wp) = \psi_2(\wp) = \wp$, then

$$\begin{aligned}
{}_{\mathrm{H}}\mathscr{I}^{\sqrt{5}/7}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}[\psi_1]\right](\wp) &= {}_{\mathrm{H}}\mathscr{I}^{\sqrt{5}/7}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}[1]\right](\wp) \\
&\leq \left\{\begin{array}{ll} 0.82188\,\delta, & \eta_1 = \frac{5}{4}, \\ 0.56457\,\delta, & \eta_1 = \frac{5}{3}, \\ 0.44579\,\delta, & \eta_1 = \frac{19}{10}, \end{array}\right\} = \left.\frac{(\eta_1)^{-\sqrt{5}/7}}{\Gamma\left(\frac{5}{3}+1\right)}\wp\right|_{\wp=T} = \epsilon_{\psi_1}\psi_1(\wp), \\
{}_{\mathrm{H}}\mathscr{I}^{6/7}\left[{}_{\mathrm{R.L}}\mathscr{I}^{4/3}[\psi_2]\right](\wp) &= {}_{\mathrm{H}}\mathscr{I}^{6/7}\left[{}_{\mathrm{R.L}}\mathscr{I}^{4/3}[1]\right](\wp) \\
&\leq 0.75945 = \left.\frac{\left(\frac{4}{3}\right)^{-6/7}}{\Gamma\left(\frac{4}{3}+1\right)}\wp\right|_{\wp=T} = \epsilon_{\psi_2}\psi_2(\wp).
\end{aligned}$$



(a) A_1 (b) A_2

Figure 3: 2D plot of Inequality (18), A_k, $k = 1, 2$ for three different values of η_1 in the system (20) in Example 5.1.

Thus condition (19) is satisfied with $\psi_1(\wp) = \psi_2(\wp) = \wp$ and

$$\epsilon_{\psi_1} = \frac{(\eta_1)^{-\sqrt{5}/7}}{\Gamma(\eta_1+1)}, \qquad \epsilon_{\psi_2} = \frac{\left(\frac{4}{3}\right)^{-6/7}}{\Gamma\left(\frac{4}{3}+1\right)}.$$

It follows from Theorem 4.3, problem (20) is Ulam-Hyers-Rassias stable with

$$\left\| (\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2) - (\mathfrak{h}_1, \mathfrak{h}_2) \right\| \leq \begin{cases} 2.00079\, \delta\psi(\wp), & \eta_1 = \frac{5}{4}, \\ 1.67486\, \delta\psi(\wp), & \eta_1 = \frac{5}{3}, \\ 1.52445\, \delta\psi(\wp), & \eta_1 = \frac{19}{10}, \end{cases} \qquad \delta > 0,\ \wp \in \Lambda.$$

In the next Example 5.2, the truth of the issues of existence and stability for system (1) and different values of C.H derivative order ϑ_1 in the second mixed equation of the system are examined.

Example 5.2. We consider the same sequential sustem of $\mathbb{FPE}$ (20) in Example 5.1, as form

$$\begin{cases} {}^{\mathrm{R.L}}\mathscr{D}^{5/3}\left[{}^{\mathrm{C.H}}\mathscr{D}^{\sqrt{5}/7} + \frac{e}{13}\right]\mathfrak{h}_1(\wp) \\ \quad = \frac{10^{-2}\cos(\wp)}{\sqrt{90^2+\wp^2}}\left(\cos^2 \mathfrak{h}_1(\wp) + \mathfrak{h}_1\left(\frac{2\wp}{3}\right) + \frac{|\mathfrak{h}_2(\wp)|}{|\mathfrak{h}_2(\wp)|+2} + \ln(\wp+2)\right), \\ {}^{\mathrm{R.L}}\mathscr{D}^{\eta_2}\left[{}^{\mathrm{C.H}}\mathscr{D}^{6/7} + \frac{\ln 5}{11}\right]\mathfrak{h}_2(\wp) \\ \quad = \frac{e^{-\wp}}{37^2+\wp^2}\left(\frac{|\mathfrak{h}_1(\wp)|}{|\mathfrak{h}_1(\wp)|+3} + \sin \mathfrak{h}_2(\wp) + \mathfrak{h}_2\left(\frac{2\wp}{3}\right) + \arctan(\wp+1)\right), \end{cases} \tag{23}$$

with the difference is that $\eta_1 = \frac{5}{3} \in \Lambda$ is fixed but η_2 changes for the following values $\left\{\frac{2\sqrt{7}}{5}, \frac{3}{2}, \frac{23}{12}\right\} \subseteq [1, 2]$. Furthermore, assume that the relations (21) and (22) be valid. By the given data and definitions $\mathfrak{z}_1(\wp, \mathfrak{h}_1(\wp), \mathfrak{h}_1(\omega\wp), \mathfrak{h}_2(\wp))$ and $\mathfrak{z}_2(\wp, \mathfrak{h}_1(\wp), \mathfrak{h}_1(\omega\wp), \mathfrak{h}_2(\wp))$, We have shown that condition $(\mathbf{H}_1)$ is valid with $\gamma_1 = \frac{1}{4500}$ and $\gamma_2 = \frac{1}{1369}$. Now, Equations (5) and (9) imply that $\Pi_1 = -1.0781$,

$$\Pi_2 = (\eta_2 - 1)^{-\vartheta_2}\left(\frac{\Gamma(\eta_2)\theta_2^{\eta_2+\alpha_2-1}}{\Gamma(\eta_2+\alpha_2)} - T^{\eta_2-1}\right) = \begin{cases} -1.8723, & \eta_2 = \frac{2\sqrt{7}}{5}, \\ -1.0916, & \eta_2 = \frac{3}{2}, \\ -0.9519, & \eta_2 = \frac{23}{12}, \end{cases}$$

$\varDelta_1 = 1.0317$,

$$\begin{aligned} \varDelta_2 &= \frac{\eta_2^{-\vartheta_2}T^{\eta_2}}{\Gamma(\eta_2+1)} + \frac{(\eta_2-1)^{-\vartheta_2}T^{\eta_2-1}}{|\Pi_2|}\left[\frac{\eta_2^{-\vartheta_2}\theta_2^{\eta_2+\alpha_2}}{\Gamma(\eta_2+\alpha_2+1)} + \frac{\eta_2^{-\vartheta_2}T^{\eta_2}}{\Gamma(\eta_2+1)}\right] \\ &= \begin{cases} 2.4214, & \eta_2 = \frac{2\sqrt{7}}{5}, \\ 1.4412, & \eta_2 = \frac{3}{2}, \\ 0.9039, & \eta_2 = \frac{23}{12}, \end{cases} \end{aligned}$$

$\nabla_1 = 0.4346$,

$$\begin{aligned} \nabla_2 &= |\beta_2| T + \frac{(\eta_2-1)^{-\vartheta_2}T^{\eta_2-1}}{|\Pi_2|}\left[|\beta_2| T + |\beta_2| \frac{\theta_2^{\eta_2+1}}{\Gamma(\eta_2+2)}\right] \\ &= \begin{cases} 0.3750, & \eta_2 = \frac{2\sqrt{7}}{5}, \\ 0.3233, & \eta_2 = \frac{3}{2}, \\ 0.3072, & \eta_2 = \frac{23}{12}. \end{cases} \end{aligned}$$

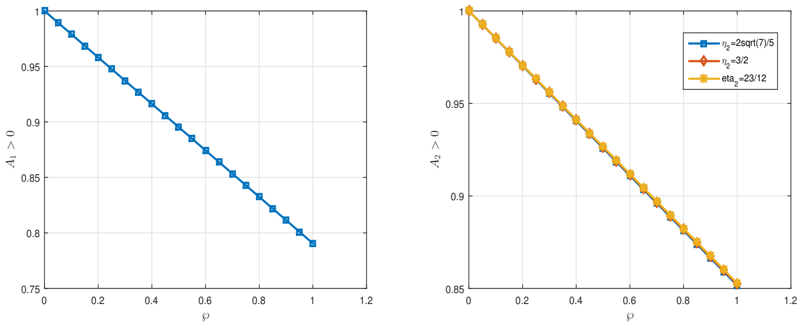
Therefore, conditions

$$3\varDelta_1\gamma_1 + \nabla_1 \simeq 0.4353 < \tfrac{1}{2}, \qquad 3\varDelta_2\gamma_2 + \nabla_2 = \left\{\begin{array}{ll} 0.3803, & \eta_2 = \frac{2\sqrt{7}}{5}, \\ 0.3264, & \eta_2 = \frac{3}{2}, \\ 0.3092, & \eta_2 = \frac{23}{12}. \end{array}\right\} < \tfrac{1}{2},$$

and

$$\begin{aligned} &\gamma_1 \frac{\eta_1^{-\vartheta_1}T^{\eta_1}}{\Gamma(\eta_1+1)} \simeq 0.000125 < 0.263633 \simeq \tfrac{1}{3}\left(1 - |\beta_1| T\right), \\ &\gamma_2 \frac{\eta_2^{-\vartheta_2}T^{\eta_2}}{\Gamma(\eta 2+1)} \simeq \left\{\begin{array}{ll} 0.000697, & \eta_2 = \frac{2\sqrt{7}}{5}, \\ 0.000476, & \eta_2 = \frac{3}{2}, \\ 0.000313, & \eta_2 = \frac{23}{12}. \end{array}\right\} < 0.284562 \simeq \tfrac{1}{3}\left(1 - |\beta_2| T\right), \end{aligned}$$

are satisfied. It follows from Theorem 3.2, that the problem (23) has a unique



(a) A_1 (b) A_2

Figure 4: 2D plot of Inequality (18), A_k, $k = 1, 2$ for three different values of η_2 in the system (23) in Example 5.2.

solution on Λ, and by considering the Equation (17),

$$\chi_{\mathfrak{z}_1,\mathfrak{z}_2} = \frac{1}{\min\{A_1, A_2\}}\left[\frac{\eta_1^{-\vartheta_1} T^{\eta_1}}{\Gamma(\eta_1+1)} + \frac{\eta_2^{-\vartheta_2} T^{\eta_2}}{\Gamma(\eta_2+1)}\right] = \begin{cases} 1.9228, & \eta_2 = \frac{2\sqrt{7}}{5}, \\ 1.5398, & \eta_2 = \frac{3}{2}, \\ 1.2573, & \eta_2 = \frac{23}{12}, \end{cases}$$

is Ulam-Hyers stable with

$$\left\| \left(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2\right) - \left(\mathfrak{h}_1, \mathfrak{h}_2\right) \right\| \leq \begin{cases} 1.9228, & \eta_2 = \frac{2\sqrt{7}}{5}, \\ 1.5398, & \eta_2 = \frac{3}{2}, \\ 1.2573, & \eta_2 = \frac{23}{12}, \end{cases} \qquad \delta > 0.$$

In Figures 4a and 4b, we have plotted the results of A_k, $k = 1, 2$ respectively for the system (23). Because the values of A_2 for $\eta_2 = \frac{2\sqrt{7}}{5}, \frac{3}{2}, \frac{23}{12}$ are very close to each other, the three curves almost coincide. Let $\psi_1(\wp) = \psi_2(\wp) = \wp$, then

$$\begin{aligned} {}_{\mathrm{H}}\mathscr{I}^{\sqrt{5}/7}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\psi_1\right]\right](\wp) &= {}_{\mathrm{H}}\mathscr{I}^{\sqrt{5}/7}\left[{}_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[1\right]\right](\wp) \\ &\leq 0.5645 \simeq \left.\frac{(\eta_1)^{-\sqrt{5}/7}}{\Gamma\left(\frac{5}{3}+1\right)}\wp\right|_{\wp=T} = \epsilon_{\psi_1}\psi_1(\wp), \\ {}_{\mathrm{H}}\mathscr{I}^{6/7}\left[{}_{\mathrm{R.L}}\mathscr{I}^{4/3}\left[\psi_2\right]\right](\wp) &= {}_{\mathrm{H}}\mathscr{I}^{6/7}\left[{}_{\mathrm{R.L}}\mathscr{I}^{4/3}\left[1\right]\right](\wp) \\ &\leq \left\{\begin{array}{ll} 0.9554, & \eta_2 = \frac{2\sqrt{7}}{5}, \\ 0.6527, & \eta_2 = \frac{3}{2}, \\ 0.4294, & \eta_2 = \frac{23}{12}, \end{array}\right\} \simeq \left.\frac{\left(\frac{4}{3}\right)^{-6/7}}{\Gamma\left(\frac{4}{3}+1\right)}\wp\right|_{\wp=T} \\ &= \epsilon_{\psi_2}\psi_2(\wp). \end{aligned}$$

Thus condition (19) is satisfied with $\psi_1(\wp) = \psi_2(\wp) = \wp$ and

$$\epsilon_{\psi_1} = \frac{\left(\frac{5}{3}\right)^{-\sqrt{5}/7}}{\Gamma\left(\frac{5}{3}+1\right)}, \qquad \epsilon_{\psi_2} = \frac{(\eta_2)^{-6/7}}{\Gamma(\eta_2+1)}. \tag{24}$$

It follows from Theorem 4.3, problem (23) is Ulam-Hyers-Rassias stable with

$$\left\|\left(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2\right) - (\mathfrak{h}_1, \mathfrak{h}_2)\right\| \leq \begin{cases} 1.9228\delta\psi(\wp), & \eta_2 = \frac{2\sqrt{7}}{5}, \\ 1.5398\delta\psi(\wp), & \eta_2 = \frac{3}{2}, \\ 1.2573\delta\psi(\wp), & \eta_2 = \frac{23}{12}, \end{cases} \qquad \delta > 0,\ \wp \in \Lambda.$$

In the next Example 5.3, we present significant results for system (1) for changes in the order of the C.H derivative.

Example 5.3. We consider the same sequential sustem of $\mathbb{FPE}$ (20) in Example 5.1, as form

$$\begin{cases} {}_{\mathrm{R.L}}\mathscr{D}^{5/3}\left[{}_{\mathrm{C.H}}\mathscr{D}^{\vartheta_1} + \frac{e}{13}\right]\mathfrak{h}_1(\wp) \\ \quad = \frac{10^{-2}\cos(\wp)}{\sqrt{90^2+\wp^2}}\left(\cos^2\mathfrak{h}_1(\wp) + \mathfrak{h}_1\left(\frac{2\wp}{3}\right) + \frac{|\mathfrak{h}_2(\wp)|}{|\mathfrak{h}_2(\wp)|+2} + \ln(\wp+2)\right), \\ {}_{\mathrm{R.L}}\mathscr{D}^{11/6}\left[{}_{\mathrm{C.H}}\mathscr{D}^{6/7} + \frac{\ln 5}{11}\right]\mathfrak{h}_2(\wp) \\ \quad = \frac{e^{-\wp}}{37^2+\wp^2}\left(\frac{|\mathfrak{h}_1(\wp)|}{|\mathfrak{h}_1(\wp)|+3} + \sin\mathfrak{h}_2(\wp) + \mathfrak{h}_2\left(\frac{2\wp}{3}\right) + \arctan(\wp+1)\right), \end{cases} \tag{25}$$

with the difference is that $\eta_1 = \frac{5}{3} \in \Lambda$, $\eta_2 = \frac{11}{6} \in \Lambda$ are fixed but ϑ_1 changes for the following values $\left\{\frac{\sqrt{5}}{7}, \frac{1}{2}, \frac{9}{10}\right\} \subseteq \Lambda$. Furthermore, assume that the relations (21) and (22) be valid. We have shown that condition $(\mathbf{H}_1)$ is valid with $\gamma_1 = \frac{1}{4500}$ and $\gamma_2 = \frac{1}{1369}$. Now, Equations (5) and (9) imply that

$$\Pi_1 = (\eta_1 - 1)^{-\vartheta_1}\left(\frac{\Gamma(\eta_1)\theta_1^{\eta_1+\alpha_1-1}}{\Gamma(\eta_1+\alpha_1)} - T^{\eta_1-1}\right)$$
$$\simeq \begin{cases} -1.0781, & \vartheta_1 = \frac{\sqrt{5}}{7}, \\ -1.1599, & \vartheta_1 = \frac{1}{2}, \\ -1.3642, & \vartheta_1 = \frac{9}{10}, \end{cases} \qquad \Pi_2 \simeq -0.9738,$$

$$\varDelta_1 = \frac{\eta_1^{-\vartheta_1}T^{\eta_1}}{\Gamma(\eta_1+1)} + \frac{(\eta_1-1)^{-\vartheta_1}T^{\eta_1-1}}{|\Pi_1|}\left[\frac{\eta_1^{-\vartheta_1}\theta_1^{\eta_1+\alpha_1}}{\Gamma(\eta_1+\alpha_1+1)} + \frac{\eta_1^{-\vartheta_1}T^{\eta_1}}{\Gamma(\eta_1+1)}\right]$$
$$\simeq \begin{cases} 1.0317, & \vartheta_1 = \frac{\sqrt{5}}{7}, \\ 0.8839, & \vartheta_1 = \frac{1}{2}, \\ 0.6392, & \vartheta_1 = \frac{9}{10}, \end{cases} \qquad \varDelta_2 = 0.99405,$$

$$\nabla_1 = |\beta_1| T + \frac{(\eta_1-1)^{-\vartheta_1} T^{\eta_1-1}}{|\Pi_1|}\left[|\beta_1| T + |\beta_1| \frac{\theta_1^{\eta_1+1}}{\Gamma(\eta_1+2)}\right]$$
$$\simeq \begin{cases} 0.4346, & \vartheta_1 = \frac{\sqrt{5}}{7}, \\ 0.4346, & \vartheta_1 = \frac{1}{2}, \\ 0.4346, & \vartheta_1 = \frac{9}{10}, \end{cases} \qquad \nabla_2 \simeq 0.3094.$$

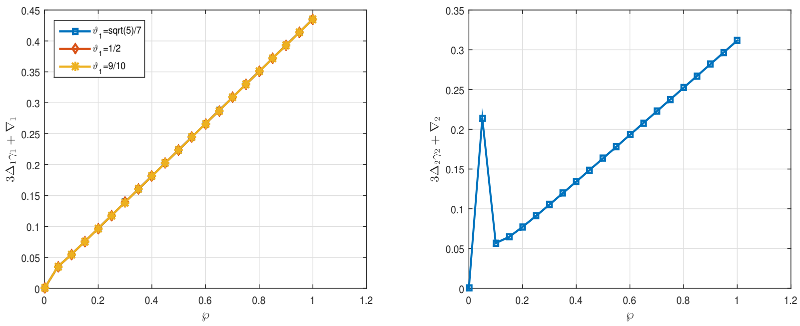
Therefore, conditions

$$3\Delta_1\gamma_1 + \nabla_1 \simeq \left\{\begin{array}{ll} 0.4353, & \vartheta_1 = \frac{\sqrt{5}}{7}, \\ 0.4352, & \vartheta_1 = \frac{1}{2}, \\ 0.4350, & \vartheta_1 = \frac{9}{10}, \end{array}\right\} < \frac{1}{2}, \qquad 3\Delta_2\gamma_2 + \nabla_2 \simeq 0.3116 < \frac{1}{2},$$

and

$$\gamma_1 \frac{\eta_1^{-\vartheta_1} T^{\eta_1}}{\Gamma(\eta_1+1)} \simeq \left\{\begin{array}{ll} 0.000125, & \vartheta_1 = \frac{\sqrt{5}}{7}, \\ 0.000114, & \vartheta_1 = \frac{1}{2}, \\ 0.000093, & \vartheta_1 = \frac{9}{10}, \end{array}\right\} < 0.263633 \simeq \frac{1}{3}\left(1 - |\beta_1| T\right),$$
$$\gamma_2 \frac{\eta_2^{-\vartheta_2} T^{\eta_2}}{\Gamma(\eta 2+1)} \simeq 0.000342 < 0.284562 \simeq \frac{1}{3}\left(1 - |\beta_2| T\right),$$

are satisfied. It follows from Theorem 3.2, that the problem (23) has a unique



(a) $3\Delta_1\gamma_1 + \nabla_1$ (b) $3\Delta_2\gamma_2 + \nabla_2$

Figure 5: 2D plot of $3\Delta_k\gamma_k + \nabla_k$, $k = 1, 2$ for three different values of ϑ_1 in the system (25) in Example 5.3.

solution on Λ, and by considering the Equation (17),

$$\chi_{\mathfrak{z}_1,\mathfrak{z}_2} = \frac{1}{\min\{A_1, A_2\}}\left[\frac{\eta_1^{-\vartheta_1} T^{\eta_1}}{\Gamma(\eta_1+1)} + \frac{\eta_2^{-\vartheta_2} T^{\eta_2}}{\Gamma(\eta_2+1)}\right] \simeq \begin{cases} 1.3075, & \vartheta_1 = \frac{\sqrt{5}}{7}, \\ 1.2445, & \vartheta_1 = \frac{1}{2}, \\ 1.1240, & \vartheta_1 = \frac{9}{10}, \end{cases}$$

is Ulam-Hyers stable with

$$\left\|\left(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2\right)-\left(\mathfrak{h}_1, \mathfrak{h}_2\right)\right\| \leq \begin{cases} 1.3075, & \vartheta_1=\frac{\sqrt{5}}{7}, \\ 1.2445, & \vartheta_1=\frac{1}{2}, \\ 1.1240, & \vartheta_1=\frac{9}{10}, \end{cases} \qquad \delta>0.$$

In Figures 5a and 5b, we have plotted the results of $3\varDelta_k\gamma_k+\nabla_k$, $k=1,2$ respectively for the system (25). Because the values of $3\varDelta_1\gamma_1+\nabla_1$, for $\vartheta_1=\frac{\sqrt{5}}{7}, \frac{1}{2}, \frac{9}{10}$ are very close to each other, the three curves almost coincide. Let $\psi_1(\wp)=\psi_2(\wp)=\wp$, then

$$_{\mathrm{H}}\mathscr{I}^{\sqrt{5}/7}\left[_{\mathrm{R.L}}\mathscr{I}^{\eta_1}\left[\psi_1\right]\right](\wp) \leq \left\{\begin{array}{ll} 0.5645, & \vartheta_1=\frac{\sqrt{5}}{7}, \\ 0.5148, & \vartheta_1=\frac{1}{2}, \\ 0.4196, & \vartheta_1=\frac{9}{10}, \end{array}\right\} \simeq \left.\frac{\left(\frac{5}{3}\right)^{-\vartheta_1}}{\Gamma\left(\frac{5}{3}+1\right)} \wp\right|_{\wp=T}=\epsilon_{\psi_1} \psi_1(\wp),$$

$$_{\mathrm{H}}\mathscr{I}^{6/7}\left[_{\mathrm{R.L}}\mathscr{I}^{4/3}\left[\psi_2\right]\right](\wp) \leq 0.5645 \simeq \left.\frac{\left(\frac{11}{6}\right)^{-6/7}}{\Gamma\left(\frac{11}{6}+1\right)} \wp\right|_{\wp=T}=\epsilon_{\psi_2} \psi_2(\wp).$$

Thus condition (19) is satisfied with $\psi_1(\wp)=\psi_2(\wp)=\wp$ and

$$\epsilon_{\psi_1}=\frac{\left(\frac{5}{3}\right)^{-\vartheta_1}}{\Gamma\left(\frac{5}{3}+1\right)}, \qquad \epsilon_{\psi_2}=\frac{\left(\frac{11}{6}\right)^{-6/7}}{\Gamma\left(\frac{11}{6}+1\right)}.$$

It follows from Theorem 4.3, problem (25) is Ulam-Hyers-Rassias stable with

$$\left\|\left(\acute{\mathfrak{h}}_1, \acute{\mathfrak{h}}_2\right)-\left(\mathfrak{h}_1, \mathfrak{h}_2\right)\right\| \leq \begin{cases} 1.3075\,\delta\psi(\wp), & \vartheta_1=\frac{\sqrt{5}}{7}, \\ 1.2445\,\delta\psi(\wp), & \vartheta_1=\frac{1}{2}, \\ 1.1240\,\delta\psi(\wp), & \vartheta_1=\frac{9}{10}, \end{cases} \qquad \delta>0,\ \wp\in\Lambda.$$

6. Conclusion

The fractional pantograph system involving both Riemann-Liouville and Caputo-Hadamard fractional derivatives has been explored in this work in points of interest. We investigated the existence and stability criteria of solutions for teh system by applying the Leray-Schauder nonlinear alternative. Finally, some illustrative examples are provided to support the theoretical findings.

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