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Existence Solution for a Non-smooth System

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Abstract

A variational method for a mixed boundary value problem in mathematical physics is considered. Using two-field Lagrange multipliers, we would investigate a variational formulation containing a mixed variational problem which is equivalent with a problem of saddle point type. With a keen focus on two-field Lagrange multipliers, we introduce a variational formulation comprising a mixed variational problem that seamlessly aligns with a saddle point problem (Problem 3). Consequently, the distinctive solvability of the weak formulation we propose is intricately governed by the principles of saddle point theory, marking a significant advancement in our understanding of mathematical physics under challenging conditions.

Keywords: Non-smooth boundary value problems, Weak solutions, Saddle points.

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1. Introduction

This paper is devoted to a unified and general frame work for a class of non-smooth nonlinear problems arising from contact mechanics. May be many of the practical problems in various sciences, including physics and natural sciences have perturbed at the boundary. Nowadays, solving the turbulent boundary value problems for nonlinear elastic materials are important. The solutions of there

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differential equations usually will be done in several ways, the inequality method is in fact a variation which is widely used today. Among others, Joachim's works are notable for their use of the KKM (Knaster-Kuratowski-Mazurkiewicz) method [1]. In fact, by changing the differential equation to a variational inequality and use of the KKM principle, which is an extension of the known topological fixed point theorems, and theorem 2-2 in [1], he investigated the existence of the solution to his problem.

In [2] based on the semi coercive operators and pseudomonotone functions in the variational inequality they looked for the numerical solution of the problem with a sequence of approximate solutions and using Moscow convergence, they showed that the problem $VI(A, \Phi, g, K)$ has a solution. In [3], authors proved the existence and uniqueness of the solution to a class of the history-dependent subdifferential inclusions. They described the model of contact between the elastic beam and the reactive obstacle history-dependent hemivariational inequalities.

Migorski studied a class of quasi variational-hemivariational inequalities which contain a convex potential, a locally Lipschitz superpotential and an implicit obstacle set of constraints in reflexive Banach spaces, to show the existence and uniqueness of the solution. He established the compactness of the solution set in the strong topology [4]. We note that the variational principle of Ekeland, as well as fixed point theorems and optimal algorithms are used to solve these problems which are given in [5–7].

The following variational problem:

$$h(z, v - z) + \jmath(v) - \jmath(z) \geq (\ell, v - z)_X, \quad \text{for all } v \in X, \tag{P}$$

has been studied by many authors, as variational inequality of the second kind, where X is a Hilbert space, $\ell \in X$, $h : X \times X \to \mathbb{R}$ and $\jmath : X \to \mathbb{R}$ are maps under some condition [8–10]. In fact, in third chapter of [9] the following conditions are given:

$$\left.\begin{array}{l} h \text{ is a symmetric bilinear map such that:} \\ \exists\, M_h > 0 \ :\ |h(z, v)| \leq M_h \|z\|_X \|v\|_X \text{ for all } z, v \in X, \\ \exists\, m_h > 0 \ :\ h(v, v) \geq m_h \|v\|_X^2 \text{ for all } v \in X. \end{array}\right\} \tag{I}$$

$$\jmath \text{ is a proper, convex and l.s.c. functional.} \tag{II}$$

Then it is proved that for each $\ell \in X$, there exists $z \in X$ that is a unique solution of (P).

The origin of the disturbance (or perturbation) may have come from physical factors surrounding the system, for instance: passing a heavy car on the road and causing a vibration in the building causes a disturbance, however small, in a system being tested or a small amount of energy entering the system from outside. Sometimes they make a disturbance in system to get the desired result. To design and build toy models, it seems very good reasonable if the irregularities, heterogeneities, impurities,... are ignored. But it is at least as reasonable that the stability of the results obtained for homogeneous systems is questionable in the presence of disorder. This concern exists and many examples of the impact of impurities and small disorders have been revealed [11–13].

Our aim is to show that the following perturbation Problem 1 has a solution by the saddle point method.

Problem 1. *There exists* $z : \bar{\Omega} \longrightarrow \mathbb{R}$ *and* $\boldsymbol{\sigma} : \bar{\Omega} \longrightarrow \mathbb{R}^N$ *such that*

$$\begin{aligned} div\boldsymbol{\sigma}(x) &= f_0(x) - \alpha z(x), && in\ \Omega, && (1)\\ -\boldsymbol{\sigma}(x) &\in \partial\zeta(\nabla z(x)) + \beta\nabla z(x), && in\ \Omega, && (2)\\ z(x) &= 0, && on\ \Gamma_1, && \\ \boldsymbol{\sigma}(x)\cdot\boldsymbol{\nu}(x) &= f_2(x), && on\ \Gamma_2, && (3)\\ \boldsymbol{\sigma}(x)\cdot\boldsymbol{\nu}(x) &\in \partial\xi(z(x)), && on\ \Gamma_3, && (4) \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary Γ which is composed of three parts $\Gamma_1, \Gamma_2, \Gamma_3$, in which their Lebesgue measures are positive. Also, $\boldsymbol{\nu}$ denotes, as usual, the outward normal unit vector which is defined almost sure on Γ. Moreover "$\cdot$" denotes the inner product on $\mathbb{R}^N$. Suppose that $f_0 \in L^2(\Omega)$, $f_2 \in L^2(\Gamma_2)$, β and α are positive parameters, $\zeta : \mathbb{R}^N \to \mathbb{R}$ and $\xi : \mathbb{R} \to \mathbb{R}$ are bounded seminorms such that

$$\exists M_\zeta > 0 \ :\ \ \zeta(\boldsymbol{s}) \leq M_\zeta\|\boldsymbol{s}\| \text{ for all } \boldsymbol{s} \in \mathbb{R}^N,$$

$$\exists M_\xi > 0 \ :\ \ \xi(t) \leq M_\xi|t| \text{ for all } t \in \mathbb{R}.$$

$\|\cdot\|$ as usual represents the Euclidean norm on $\mathbb{R}^N$ and $|\cdot|$ for the real number's absolute magnitude.

Assuming $\alpha = 0$, this problem reduces to a boundary value problem which is studied in [14] with a single-field Lagrange coefficient and in [15] with two-field Lagrange coefficients. Actually $\alpha z(x)$ plays the role of perturbation that we introduced in Problem 1.

Setting $\alpha = 0$, $\zeta \equiv 0$ and $\Gamma_3 \equiv 0$, then we have a classical boundary value problem that in the theory of electricity, there is a physical significance associated with that; see Chapter 8 in [16].

The case $\alpha = 0$, $N = 2$, $\zeta \equiv 0$ and $\xi : \mathbb{R} \longrightarrow [0, \infty)$, $\xi(t) = g|t|$, where g is a positive constant in which

$$\left|\beta\frac{\partial z}{\partial\nu}(x)\right| \leq g, \ \ \beta\frac{\partial z}{\partial\nu}(x) = -g\frac{z(x)}{|z(x)|} \text{ if } z(x) \neq 0 \text{ on } \Gamma_3,$$

which is the well known Tresca's law, Problem 1 reduces to an antiplane frictional contact model for elastic materials. Friction is described with the Tresca's law in Chapter 9 of [9].

In Mechanics, Problem 1 usually uses two-field Lagrange multiplier $\bar{w} = (w_\Omega, w_{\Gamma_3})$, where w_Ω is relevant to $\boldsymbol{\sigma}$ in Ω and w_{Γ_3} is relevant to $\boldsymbol{\sigma}$ on Γ_3. Thus, by a variational formula we may compute not only z in Ω but also $\boldsymbol{\sigma}$ in Ω and $\boldsymbol{\sigma}$ on Γ_3 as well.

Here, our focus is for weak solvability of Problem 1 through two Lagrange coefficient fields. Our proposed approach guides us into addressing saddle point challenges. Employing Lagrange multiplier techniques in mathematical physics problems enables the application of modern methods for efficiently approximating weak solutions, see [17] in which primary-dual active strategy is applied.

2. Notation and preliminaries

The symbols we use here are fairly standard. We list some main concepts and symbols:
$X \coloneqq \{v : \ v \in H^1(\Omega),\ \gamma v = 0 \text{ a.e. on } \Gamma_1\}$, where $\gamma : H^1(\Omega) \to L^2(\Gamma)$ is the trace operator. γ is a continuous, linear and compact operator; to read more details on its properties, we refer to [18]. X is a closed subspace of the Hilbert space $H^1(\Omega)$. To see this, consider a sequence $(v_n)_n \subset X$ such that $v_n \to v$ in $H^1(\Omega)$ as $n \to \infty$. If we show $\|\gamma v\|_{L^2(\Gamma_1)} = 0$ then $v \in X$,

$$0 \le \|\gamma v\|_{L^2(\Gamma_1)} = \|\gamma v - \gamma v_n + \gamma v_n\|_{L^2(\Gamma_1)} \le \|\gamma v - \gamma v_n\|_{L^2(\Gamma_1)} + \|\gamma v_n\|_{L^2(\Gamma_1)}.$$

Then

$$0 \le \|\gamma v\|_{L^2(\Gamma_1)} \le \|\gamma v - \gamma v_n\|_{L^2(\Gamma)} \le c_E \|v - v_n\|_{H^1(\Omega)},$$

where $c_E > 0$ is a constant in the trace theorem. We will get the desired result when we pass to the limit as $n \to \infty$.

$(X, (\ ,\ )_{H^1(\Omega)}, \|\ \|_{H^1(\Omega)})$ is a Hilbert space because it is a closed subspace of a suitable Hilbert space, see, e.g., [19]. X can be equipped with the following inner product:

$$(z, v)_X = \int_\Omega \nabla z(x) \cdot \nabla v(x) dx \text{ for all } z, v \in X,$$

and the resulting induced norm is

$$\|v\|_X \coloneqq \|\nabla v\|_{L^2(\Omega)^N} \text{ for all } v \in X.$$

It is straightforward to see that $(X, (\ ,\ )_X, \|\ \|_X)$ is also a Hilbert space [20]. Using Poincaré's inequality, we have

$$\|z\|_{L^2(\Omega)} \le c_G \|\nabla z\|_{L^2(\Omega)^N} \text{ for all } z \in X,$$

where $c_G = c_G(\Omega, \Gamma_1) > 0$, see [21]. This shows the equivalence of the norms $\|\ \|_X$ and $\|\ \|_{H^1(\Omega)}$.

Consider the following Hilbert space:

$$Y \coloneqq \{\widetilde{v} : \ \widetilde{v} \in H^{\frac{1}{2}}(\Gamma),\ \exists\ v \in X \text{ such that } \widetilde{v} = \gamma v \text{ a.e. on } \Gamma\},$$

where $H^{\frac{1}{2}}(\Gamma)$ is the set of functions in $L^2(\Gamma)$ which are traces of functions in $H^1(\Omega)$ [22]. For specifics regarding fractional spaces at the boundary, one may consult [23, 24]. As usual the dual of Y will be denoted by Y'.

We revisit certain tools in saddle point theory that will be utilized in this research. For more information on saddle point theory and its applications one can refer to [25, 26].

Definition 2.1. For non-empty sets A_0 and B_0, a pair $(z, w) \in A_0 \times B_0$ is called a saddle point of a bifunctional $\mathscr{J} : A_0 \times B_0 \to \mathbb{R}$ if

$$\mathscr{J}(z, \kappa) \leq \mathscr{J}(z, w) \leq \mathscr{J}(v, w) \text{ for all } v \in A_0,\ \kappa \in B_0.$$

Theorem 2.2. *Suppose that V_1, V_2 are two Hilbert spaces and $A_1 \subseteq V_1$, $A_2 \subseteq V_2$ are non-empty, closed, convex subsets. If the bifunctional $\mathscr{J} : A_1 \times A_2 \to \mathbb{R}$ satisfies:*

($\mathfrak{a}$) $v \to \mathscr{J}(v, \kappa)$ *is lower semi-continuous and convex with respect to all* $\kappa \in A_2$,

($\mathfrak{b}$) $\kappa \to \mathscr{J}(v, \kappa)$ *is upper semi-continuous and concave with respect to all* $v \in A_1$,

($\mathfrak{c}$) A_1 *is bounded or* $\lim\limits_{\|v\|_{V_1} \to \infty, v \in A_1} \mathscr{J}(v, \kappa_0) = \infty$ *for some* $\kappa_0 \in A_2$,

($\mathfrak{d}$) A_2 *is bounded or* $\lim\limits_{\|\kappa\|_{V_2} \to \infty, \kappa \in A_2} \inf\limits_{v \in A_1} \mathscr{J}(v, \kappa) = -\infty$,

Then $\mathscr{J}$ has at least one saddle point.

Proposition 2.3. *For non-empty, closed, convex subsets $A_1 \subseteq V_1$, $A_2 \subseteq V_2$, if the bifunctional $\mathscr{J} : A_1 \times A_2 \to \mathbb{R}$ satisfies the hypotheses ($\mathfrak{a}$), ($\mathfrak{b}$), ($\mathfrak{c}$) and ($\mathfrak{d}$), then $A_0 \times B_0$ of saddle points of $\mathscr{J}$ is convex, where $A_0 \subset A_1$ and $B_0 \subset A_2$. Moreover,*
($\mathfrak{e}$) *If $v \to \mathscr{J}(v, \kappa)$ is strictly convex for all $\kappa \in A_2$, then A_0 is at most a set with one element.*
($\mathfrak{f}$) *If $\kappa \to \mathscr{J}(v, \kappa)$ is strictly concave for all $v \in A_1$, then B_0 is at most a set with one element.*

One can see the proof of Theorem 2.2 and Proposition 2.3 in [25].

3. Main result

Suppose z and $\boldsymbol{\sigma}$ are functions that satisfy the assumptions of Problem 1. We multiply relation (1) by a function $v \in X$ and after integration we obtain:

$$\overbrace{\int_\Omega \nabla \cdot \boldsymbol{\sigma}(x) v(x)\, dx}^{①} = \overbrace{\int_\Omega f_0(x) v(x)\, dx}^{②} - \overbrace{\int_\Omega \alpha z(x) v(x)\, dx}^{③}. \tag{5}$$

Using Green's identities for Sobolev spaces, [10], we get:

$$\textcircled{1} = \overbrace{-\int_{\Omega} \boldsymbol{\sigma}(x) \cdot \nabla v(x)\, dx}^{\textcircled{4}} + \overbrace{\int_{\Gamma} \boldsymbol{\sigma}(x) \cdot \boldsymbol{\nu}(x) \gamma v(x)\, d\Gamma}^{\textcircled{5}}. \tag{6}$$

We can write

$$\begin{aligned} \textcircled{4} &= -\int_{\Omega} (\boldsymbol{\sigma}(x) + \beta \nabla z(x) - \beta \nabla z(x)) \cdot \nabla v(x)\, dx \\ &= \overbrace{-\int_{\Omega} (\boldsymbol{\sigma}(x) + \beta \nabla z(x)) \cdot \nabla v(x)\, dx}^{\textcircled{6}} + \overbrace{\int_{\Omega} \beta \nabla z(x) \cdot \nabla v(x)\, dx}^{\textcircled{7}}, \end{aligned} \tag{7}$$

from relations (5), (6) and (7) we have:

$$\textcircled{2} - \textcircled{3} = \textcircled{1} = \textcircled{4} + \textcircled{5} = \textcircled{6} + \textcircled{7} + \textcircled{5}. \tag{8}$$

But

$$\begin{aligned} \textcircled{5} = & \overbrace{\int_{\Gamma_1} \boldsymbol{\sigma}(x) \cdot \boldsymbol{\nu}(x) \gamma v(x)\, d\Gamma}^{\textcircled{8}} + \overbrace{\int_{\Gamma_2} \boldsymbol{\sigma}(x) \cdot \boldsymbol{\nu}(x) \gamma v(x)\, d\Gamma}^{\textcircled{9}} \\ & + \overbrace{\int_{\Gamma_3} \boldsymbol{\sigma}(x) \cdot \boldsymbol{\nu}(x) \gamma v(x)\, d\Gamma}^{\textcircled{10}}. \end{aligned}$$

Since $v \in X$, we get $\textcircled{8} = 0$, this means relation (8) becomes

$$\textcircled{2} - \textcircled{3} = \textcircled{6} + \textcircled{7} + \textcircled{9} + \textcircled{10},$$

and then

$$\textcircled{2} - \textcircled{9} = \textcircled{7} + \textcircled{3} + \textcircled{6} + \textcircled{10}. \tag{9}$$

But from (3) we get:

$$\textcircled{9} = \int_{\Gamma_2} f_2(x) \gamma v(x)\, d\Gamma.$$

Here, $\langle\ ,\ \rangle_{Y \times Y'}$ will denote the duality pairing. Using Riesz representation theorem, there is a unique element $\ell \in X$;

$$\textcircled{2} - \textcircled{9} = (\ell, v)_X. \tag{10}$$

For simplicity, consider the following three bilinear maps:

$$\phi : X \times X \to \mathbb{R}, \quad \phi(z, v) := ⑦ + ③, \tag{11}$$

$$\varpi_1 : X \times X \to \mathbb{R}, \quad \varpi_1(v, \kappa) := (\kappa, v)_X, \tag{12}$$

$$\varpi_2 : X \times Y' \to \mathbb{R}, \quad \varpi_2(v, \eta) := \langle \eta, \gamma v \rangle. \tag{13}$$

Consider $\bar{w} = (w_\Omega, w_{\Gamma_3}) \in X \times Y'$ as two-field Lagrange multiplier, where w_Ω and w_{Γ_3} are defined as follows:

$$(w_\Omega, v)_X = ⑥ \quad \text{for all } v \in X, \tag{14}$$

$$\langle w_{\Gamma_3}, \tilde{k} \rangle = \int_{\Gamma_3} \boldsymbol{\sigma}(x) \cdot \boldsymbol{\nu}(x) \tilde{k}(x)\, d\Gamma = ⑩, \quad \text{for all } \tilde{k} \in Y. \tag{15}$$

From (9)-(15), we have:

$$\phi(z, v) + \varpi_1(v, w_\Omega) + \varpi_2(v, w_{\Gamma_3}) = (\ell, v)_X, \quad \text{for all } v \in X. \tag{16}$$

We define a form $\varpi : X \times (X \times Y') \longrightarrow \mathbb{R}$ by

$$\varpi(v, \bar{w}) = \varpi_1(v, w_\Omega) + \varpi_2(v, w_{\Gamma_3}), \tag{17}$$

where $\bar{w} = (w_\Omega, w_{\Gamma_3}) \in X \times Y'$. Then we can rewrite (16) as follows:

$$\phi(z, v) + \varpi(v, \bar{w}) = (\ell, v)_X, \quad \text{for all } v \in X. \tag{18}$$

The following lemma motivates us to replace Problem 1 with Problem 2.

Lemma 3.1. *For z and $\boldsymbol{\sigma}$ which verify Problem 1, the following relations are true*

$$-\int_\Omega (\boldsymbol{\sigma}(x) + \beta \nabla z(x)) \cdot \nabla v(x)\, dx \le \int_\Omega \zeta(\nabla v(x))\, dx, \text{ for all } v \in X, \tag{19}$$

$$-\int_\Omega (\boldsymbol{\sigma}(x) + \beta \nabla z(x)) \cdot \nabla z(x)\, dx = \int_\Omega \zeta(\nabla z(x))\, dx. \tag{20}$$

Proof. Let $x \in \Omega$. From (2),

$$\zeta(\boldsymbol{\theta}) - \zeta(\nabla z(x)) \ge -(\boldsymbol{\sigma}(x) + \beta \nabla z(x)) \cdot (\boldsymbol{\theta} - \nabla z(x)), \text{ for all } \boldsymbol{\theta} \in \mathbb{R}^N, \tag{21}$$

Setting $\boldsymbol{\theta} = 0$ in (21), ζ is a seminorm and then

$$-\zeta(\nabla z(x)) \ge (\boldsymbol{\sigma}(x) + \beta \nabla z(x)) \cdot \nabla z(x).$$

After integration on Ω,

$$-\int_\Omega (\boldsymbol{\sigma}(x) + \beta \nabla z(x)) \cdot \nabla z(x)\, dx \ge \int_\Omega \zeta(\nabla z(x))\, dx. \tag{22}$$

Since ζ is a seminorm, considering $\boldsymbol{\theta} = 2\nabla z(x)$ in (21),

$$\zeta(\nabla z(x)) \geq -(\boldsymbol{\sigma}(x) + \beta\nabla z(x)) \cdot \nabla z(x).$$

Then

$$-\int_{\Omega} (\boldsymbol{\sigma}(x) + \beta\nabla z(x)) \cdot \nabla z(x)\, dx \leq \int_{\Omega} \zeta(\nabla z(x))\, dx. \tag{23}$$

From (22) and (23), relation (20) is obtained.
ζ is a seminorm, if we set $\boldsymbol{\theta} = \nabla v(x) + \nabla z(x)$ in (21) then

$$\zeta(\nabla v(x)) \geq -(\boldsymbol{\sigma}(x) + \beta\nabla z(x)) \cdot \nabla v(x).$$

By integrating on Ω the last relation (19) is obtained. □

In a similar way to the proof of Lemma 3.1, it can be shown that for z and $\boldsymbol{\sigma}$ satisfying the conditions of Problem 1 the following relations are valid:

$$\int_{\Gamma_3} \boldsymbol{\sigma}(x) \cdot \boldsymbol{\nu}(x) \gamma v(x)\, d\Gamma \leq \int_{\Gamma_3} \xi(\gamma v(x))\, d\Gamma, \quad \text{for all } v \in X, \tag{24}$$

$$\int_{\Gamma_3} \boldsymbol{\sigma}(x) \cdot \boldsymbol{\nu}(x) \gamma z(x)\, d\Gamma = \int_{\Gamma_3} \xi(\gamma z(x))\, d\Gamma. \tag{25}$$

For next problem, at this moment we consider $\bar{\Lambda}$ as set of Lagrange multipliers by

$$\bar{\Lambda} := \Lambda_{\zeta} \times \Lambda_{\xi}, \tag{26}$$

where

$$\Lambda_{\zeta} := \big\{\kappa_{\Omega} \in X : (\kappa_{\Omega}, v)_X \leq \int_{\Omega} \zeta(\nabla v(x))\, dx \ \text{ for all } \ v \in X\big\}, \tag{27}$$

$$\Lambda_{\xi} := \big\{\kappa_{\Gamma_3} \in Y' : \langle \kappa_{\Gamma_3}, \tilde{w} \rangle \leq \int_{\Gamma_3} \xi(\tilde{w}(x))\, d\Gamma \ \text{ for all } \ \tilde{w} \in Y\big\}. \tag{28}$$

Equations (14) and (19) imply that $w_{\Omega} \in \Lambda_{\zeta}$. Also $w_{\Gamma_3} \in \Lambda_{\xi}$ is easily obtained from (15) and (24). According to (12), (14), (19), (20) and (27) we deduce that

$$\varpi_1(z, \kappa_{\Omega} - w_{\Omega}) \leq 0, \quad \text{for all } \ \kappa_{\Omega} \in \Lambda_{\zeta}. \tag{29}$$

Also, from (13), (15), (24), (25) and (28),

$$\varpi_2(z, \kappa_{\Gamma_3} - w_{\Gamma_3}) \leq 0, \quad \text{for all } \ \kappa_{\Gamma_3} \in \Lambda_{\xi}. \tag{30}$$

Looking at relations (17), (29) and (30) we get the following result for $\bar{\kappa} = (\kappa_{\Omega}, \kappa_{\Gamma_3})$ and $\bar{w} = (w_{\Omega}, w_{\Gamma_3})$

$$\varpi(z, \bar{\kappa} - \bar{w}) \leq 0, \quad \text{for all } \ \bar{\kappa} \in \bar{\Lambda}. \tag{31}$$

Problem 2. *Find $z \in X$ and $\bar{w} \in \bar{\Lambda} \subseteq X \times Y'$ satisfying* (18) *and* (31).

A solution satisfying Problem 2 is designated as a weak solution for Problem 1. Equivalently we would prove the existence and uniqueness of the solution for the following problem.

Problem 3. *Let $(V_1, (\ ,\ )_{V_1}, \|\ \|_{V_1})$, $(V_2, (\ ,\ )_{V_2}, \|\ \|_{V_2})$ be two Hilbert spaces. Given $\ell \in V_1$, find $z \in V_1$ and $w \in \Lambda \subseteq V_2$ such that*

$$
\begin{aligned}
a_0(z, v) + b_0(v, w) &= (\ell, v)_{V_1}, && \text{for all } v \in V_1, && (32)\\
b_0(v, \kappa - w) &\leq 0, && \text{for all } \kappa \in \Lambda, && (33)
\end{aligned}
$$

where Λ is any bounded, closed and convex subset of V_2 containing 0_{V_2}.

We assume that $a_0 : V_1 \times V_1 \longrightarrow \mathbb{R}$ satisfies (I). Moreover we consider the following conditions are met,

$$
\left.
\begin{aligned}
&b_0 : V_1 \times V_2 \longrightarrow \mathbb{R} \text{ is a bilinear form such that:}\\
&\text{there exists } M_{b_0} > 0 : |b_0(v, \kappa)| \leq M_{b_0} \|v\|_{V_1} \|\kappa\|_{V_2} \text{ for all } v \in V_1,\ \kappa \in V_2.
\end{aligned}
\right\} \tag{III}
$$

$$
\text{there exists } \iota > 0 \text{ such that } \inf_{\kappa \in V_2, \kappa \neq 0_{V_2}} \sup_{v \in V_1, v \neq 0_{V_1}} \frac{b_0(v, \kappa)}{\|v\|_{V_1} \|\kappa\|_{V_2}} \geq \iota. \tag{IV}
$$

We note that a_0 is continuous, bilinear and V_1-elliptic and b_0 is continuous bilinear [20]. The following results are standard, but we will present them here for the sake. For Problem 3 we would show any solution is a saddle point, where $\mathscr{J} : V_1 \times \Lambda \to \mathbb{R}$ is defined as follows:

$$
\mathscr{J}(v, \kappa) := \frac{1}{2} a_0(v, v) + b_0(v, \kappa) - (\ell, v)_{V_1}, \text{ for all } v \in V_1, \kappa \in \Lambda. \tag{34}
$$

According to Proposition 1.30 in [10] since a_0 is a continuous, bilinear, symmetric and V_1-elliptic form (thus, a positive form), so $v \to a_0(v, v)$ is lower semi-continuous and strictly convex. Then we deduce that $\mathscr{J}$ is lower semi-continuous and strictly convex with respect to the first argument. It is well-known that $\mathscr{J}$ displays upper semi-continuity and concavity in its second argument, [15].

Lemma 3.2. *$(z, w) \in V_1 \times \Lambda$ is a solution for Problem 3 if and only if it be a saddle point for $\mathscr{J}$.*

Proof. For a solution $(z, w) \in V_1 \times \Lambda$ of Problem 3, adding $\frac{1}{2} a_0(z, z) - (\ell, z)_{V_1}$ to both sides of (33), then

$$
b_0(v, \kappa - w) + \frac{1}{2} a_0(z, z) - (\ell, z)_{V_1} \leq \frac{1}{2} a_0(z, z) - (\ell, z)_{V_1}, \text{ for all } \kappa \in \Lambda.
$$

Since b_0 is bilinear, so

$$\frac{1}{2}a_0(z,z)+b_0(z,\kappa)-(\ell,z)_{V_1}\leq\frac{1}{2}a_0(z,z)+b_0(z,w)-(\ell,z)_{V_1},\ \text{for all}\ \kappa\in\Lambda,$$

and

$$\mathscr{J}(z,\kappa)\leq\mathscr{J}(z,w),\ \text{for all}\ \kappa\in\Lambda. \tag{35}$$

From (34), we get:

$$\mathscr{J}(z,w)-\mathscr{J}(v,w)=\frac{1}{2}a_0(z,z)+b_0(z,w)-(\ell,z)_{V_1}-\frac{1}{2}a_0(v,v)-b_0(v,w)+(\ell,v)_{V_1}. \tag{36}$$

Since a_0 is a symmetric, bilinear and V_1-elliptic form, so from the right hand of (36) and (32), we obtain:

$$\begin{aligned}&\frac{1}{2}a_0(z,z)+b_0(z,w)-a_0(z,z)-b_0(z,w)-\frac{1}{2}a_0(v,v)-b_0(v,w)+a_0(v,v)+b_0(v,w)\\&=\frac{1}{2}a_0(z,z)-a_0(z,z)-\frac{1}{2}a_0(v,v)+a_0(v,v)=-\frac{1}{2}a_0(z-v,z-v)\leq 0.\end{aligned} \tag{37}$$

From relations (36), (37)

$$\mathscr{J}(z,w)\leq\mathscr{J}(v,w),\ \text{for all}\ v\in V_1.$$

Hence, $(z,w)\in V_1\times\Lambda$ is a saddle point for $\mathscr{J}$. We show the converse implication. Assuming $(z,w)\in V_1\times\Lambda$ is a saddle point for $\mathscr{J}$ and because of (35) and definition of $\mathscr{J}$, so (33) satisfies. From

$$\mathscr{J}(z,w)\leq\mathscr{J}(y,w),\ \text{for all}\ y\in V_1.$$

Then

$$\frac{1}{2}a_0(z,z)-\frac{1}{2}a_0(y,y)+b_0(z-y,w)+(\ell,y-z)_{V_1}\leq 0,\ \text{for all}\ y\in V_1. \tag{38}$$

Setting $y=z+sv$ for $s>0$ in (38), then one has

$$-sa_0(z,v)-\frac{s^2}{2}a_0(v,v)-sb_0(v,w)+s(\ell,v)_{V_1}\leq 0,\ \text{for all}\ v\in V_1.$$

Dividing by $s>0$ and then computing the limit as $s\to 0$,

$$a_0(z,v)+b_0(v,w)\ \geq\ (\ell,v)_{V_1},\ \text{for all}\ v\in V_1. \tag{39}$$

Setting $y=z-sv$ for $s>0$ in (38) and divide both sides of it by $s>0$, we have:

$$a_0(z,z)-\frac{s}{2}a_0(v,v)+b_0(v,w)-(\ell,v)_{V_1}\leq 0,\ \text{for all}\ v\in V_1,$$

taking limits as $s \to 0$,

$$a_0(z, v) + b_0(v, w) \;\leq\; (\ell, v)_{V_1}, \text{ for all } v \in V_1. \tag{40}$$

Adding (39) and (40), we get (32). So, $(z, w) \in V_1 \times \Lambda$ is a solution of Problem 3. □

Theorem 3.3. *Suppose that (I) and (III) are satisfied. Then Problem 3 has a unique solution $(z, w) \in V_1 \times \Lambda$. If (IV) holds too, then w is also unique.*

For the proof of Theorem 3.3 one can refer to [15].

Theorem 3.4. *Problem 1 has a unique weak solution $(z, \bar{w}) \in X \times \bar{\Lambda}$ in first component.*

Proof. Both of Λ_ζ and Λ_ξ dafined in (27) and (28) are nonempty, bounded, closed and convex subsets, so $\bar{\Lambda} := \Lambda_\zeta \times \Lambda_\xi$ is nonempty, bounded, closed and convex subset of $X \times Y'$. We set $a_0 := \phi$, $b_0 := \varpi$, $V_1 := X$, $V_2 := X \times Y'$ and $\Lambda := \bar{\Lambda}$. Therefore, Problem 2 changes to Problem 3. Since the conditions of Theorem 3.3, (I) and (III) are verified, so Problem 2 has a solution $(z, \bar{w}) \in X \times \bar{\Lambda}$ which its first component is unique and that means $(z, \bar{w})$ is a weak solution for Problem 1 which is unique in first component. □

Since condition (IV) of Theorem 3.3 for Problem 2 is not established in [15], so it is not possible to give a definite opinion about the uniqueness of the second component.

Example 3.5. Assume that in Problem 1, $\Omega = B_1(0) \subset \mathbb{R}^3$, $\Gamma = \{x \in \Omega : \|x\| = 1\}, \Gamma_1 = \{(0,0,1)\}, \Gamma_2 = \{x = (x_1, x_2, x_3) \in \Omega : \|x\| = 1, 0 \leq x_3 < 1\}, \Gamma_3 = \{x = (x_1, x_2, x_3) \in \Omega : \|x\| = 1, x_3 < 0\}$,
$\zeta : \mathbb{R}^3 \to \mathbb{R}$ by $\zeta(s) = \|s\|$, $\xi : \mathbb{R} \to \mathbb{R}$ by $\xi(t) = |t|$,
$f_2 : \mathbb{R}^3 \to \mathbb{R}$ by $f_2(x) = \|x\|^2$ and $f_0 : \mathbb{R}^3 \to \mathbb{R}$ by $f_0(x) = \|x\|$.

Then $\partial\zeta(s) = \{v \in \mathbb{R}^3 : \langle v, s\rangle = \|s\|, \|v\|_* \leq 1\}$ where $\|s\|_*$ is the dual norm of $\|.\|$, defined as $\|s\|_* := \sup_{\|v\| \leq 1} \langle s, v\rangle$, [27]. Also by referring to [28] we will find

$$\partial\xi(t) = \begin{cases} [-1, 1], & \xi(t) = 0, \\ \{1\}, & \xi(t) > 0, \\ \{-1\}, & \xi(t) < 0. \end{cases}$$

By rewriting Problem 1, it becomes an attempt to find $z : \bar{\Omega} \longrightarrow \mathbb{R}$ and $\boldsymbol{\sigma} : \bar{\Omega} \longrightarrow$

$\mathbb{R}^3$ such that

$$
\begin{aligned}
&\operatorname{div} \boldsymbol{\sigma}(x) = \|x\| - \alpha z(x), && \text{in } \Omega,\\
&\beta \nabla z(x) - \boldsymbol{\sigma}(x) \in \left\{ v \in \mathbb{R}^3 : \langle v, \nabla z(x) \rangle = \|\nabla z(x)\|, \|v\|_* \le 1 \right\}, && \text{in } \Omega,\\
&z(x) = 0, && \text{on } \Gamma_1,\\
&\boldsymbol{\sigma}(x) \cdot \boldsymbol{\nu}(x) = \|x\|^3, && \text{on } \Gamma_2,\\
&\boldsymbol{\sigma}(x), \cdot \boldsymbol{\nu}(x) \in \begin{cases} [-1, 1], & \xi(z(x)) = 0,\\ \{1\}, & \xi(z(x)) > 0,\\ \{-1\}, & \xi(z(x)) < 0. \end{cases} && \text{on } \Gamma_3.
\end{aligned}
$$

If we assume ϕ is the same function in (11), ℓ in (10) and

$$
J : X \to \mathbb{R}, \quad J(v) = \int_\Omega \zeta(\nabla v(x))\, dx + \int_{\Gamma_3} \xi(\gamma v(x))\, d\Gamma, \tag{41}
$$

then Problem 1 leads us to (P). It is immediate that ϕ is a continuous, bilinear, symmetric, X-elliptic form and J is a proper, convex and l.s.c. functional. Then conditions (I) and (II) are valid, so the existence and uniqueness is guaranteed.

Proposition 3.6. *If $z_0 \in X$ is the unique solution for (P), then*

$$
\phi(z_0, v) + J(v) \ge (\ell, v)_X, \quad \text{for all } v \in X. \tag{42}
$$

Proof. We set $v = 0_X$ in (P), then

$$
-\phi(z_0, z_0) - J(z_0) \ge -(\ell, z_0)_X. \tag{43}
$$

Set $v = 2z_0$ in (P), then

$$
\phi(z_0, z_0) + J(z_0) \ge (\ell, z_0)_X. \tag{44}
$$

From (43) and (44) we have:

$$
\phi(z_0, z_0) + J(z_0) = (\ell, z_0)_X.
$$

By the last relation and (P), we get (42). □

Proposition 3.7. *If $(z, \bar{w}) \in X \times \bar{\Lambda}$ is the unique solution of Problem 2 and ϕ, ℓ and J are defined as in (10), (11) and (41), then*

$$
\phi(z, v) + J(v) \ge (\ell, v)_X, \quad \text{for all } v \in X. \tag{45}
$$

Proof. From (12), (13), (16), (17), (26), (27) and (28), the inequality (45) will be conclude. □

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