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Intuitionistic Fuzzy Ideals in (m, n)-Near Rings

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Abstract

In this article, first we review some basic definitions and results about fuzzy sets and intuitionistic fuzzy sets; then we state the definitions of intuitionistic fuzzy (m, n)-sub near rings and intuitionistic fuzzy ideals of (m, n)-near rings, which are generalizations of intuitionistics subrings and intuitionistic fuzzy ideals of rings and near-rings, respectively. We provide several examples for the definitions and discuss and investigate some results in this respect. Finally, we investigate the direct product of intuitionistic fuzzy (m, n)-sub near rings of two (m, n)-near rings and state and prove some results on these topics.

Keywords: Fuzzy set, Characteristic function, Fuzzy (m, n)-sub near ring, Prime ideal.

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1. Introduction

The theory of fuzzy sets proposed by Zadeh [1] has achieved great success in various fields. Out of several higher-order fuzzy sets, intuitionistic fuzzy sets introduced by Atanassov [2–4] have been found to be highly useful to deal with vagueness.

A function $\mu : R \longrightarrow [0, 1]$ in a set R is called a fuzzy set [1]. Let $\mathrm{Im}(\mu)$ denote the image set of μ. Set $\mu_\alpha = \{x \in R \mid \mu(x) \geq \alpha\}$ and $\mu_\alpha^{\leq} = \{x \in R \mid \mu(x) \leq \alpha\}$ where $\alpha \in [0, 1]$.

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Definition 1.1. Assume that X is a set and D is a subset of X. In this case

$$\mu_D(x) = \begin{cases} 1, & \text{if } x \in D, \\ 0, & \text{if } x \notin D, \end{cases}$$

is said characteristic function of the set D in X.

Example 1.2. Assume that $B = \{z \in \mathbb{N} \mid \frac{z}{3} \in \mathbb{N}\}$ and

$$\mu_B(z) = \begin{cases} 1, & \text{if } z \in B, \\ 0, & \text{if } z \notin B. \end{cases}$$

In this case μ_B is the characteristic function of the set B in $\mathbb{N}$.

Definition 1.3. ([3]). An intuitionistic fuzzy set Q in W is given by:

$$Q = \{\langle z, \mu(z), \eta(z)\rangle \mid z \in W\},$$

where $\mu : W \longrightarrow [0,1]$ and $\eta : W \longrightarrow [0,1]$, with the condition $0 \leq \mu(z)+\eta(z) \leq 1$ for all $z \in W$. The numbers $\mu(z)$ and $\eta(z)$ denote, respectively, the degrees of membership and non-membership of the element $z \in W$ to the set Q.

Every fuzzy set corresponds to the following intuitionistic fuzzy set:

$$FS : \{\langle z, \mu(z), 1 - \mu(z)\rangle \mid z \in W\}.$$

Example 1.4. Assume that $\mu(z) = \frac{z}{2z+2}$, $\eta(z) = \frac{1}{2z+2}$ and $A = \{\langle \mathbb{N}, \mu(z), \eta(z)\rangle \mid z \in X\}$. In this case, A is an intuitionistic fuzzy set.

Definition 1.5. ([3]). For each intuitionistic fuzzy set A, there is another parameter $\pi(x)$, called the degree of non-determinacy of the membership of x of the set A,

$$\pi(z) = 1 - \mu(z) - \eta(z).$$

Example 1.6. In Example 1.4, $\pi(z) = 1 - \mu(z) - \eta(z) = 1 - \frac{z}{2z+2} - \frac{1}{2z+2} = 1 - (\frac{z+1}{2z+2}) = 1 - \frac{1}{2} = \frac{1}{2}$ so $\pi(z) = \frac{1}{2}$.

Definition 1.7. ([3]). Let $D = \{\langle w, \mu(w), \eta(w)\rangle \mid w \in X\}$ and $E = \{\langle w, \mu'(w), \eta'(w)\rangle \mid w \in X\}$ be any two intuitionistic fuzzy sets of X, then

(1) $D \subseteq E$ if and only if $\mu(w) \leq \mu'(w)$ and $\eta(w) \geq \eta'(w)$ for all $w \in X$,

(2) $D = E$ if and only if $\mu(w) = \mu'(w)$ and $\eta(w) = \eta'(w)$ for all $w \in X$,

(3) $D \cap E = \{\langle w, (\mu \cap \mu')(w), (\eta \cap \eta')(w)\rangle \mid w \in X\}$, where $(\mu \cap \mu')(w) = \min\{\mu(w), \mu'(w)\} = \mu(w) \wedge \mu'(w)$ and $(\eta \cap \eta')(w) = \max\{\eta(w), \eta'(w)\} = \eta(w) \vee \eta'(w)$,

(4) $D \cup E = \{\langle w, (\mu \cup \mu')(w), (\eta \cup \eta')(w)\rangle \mid w \in X\}$, where $(\mu \cup \mu')(w) = \max\{\mu(w), \mu'(w)\} = \mu(w) \vee \mu'(w)$ and $(\eta \cup \eta')(w) = \min\{\eta(w), \eta'(w)\} = \eta(w) \wedge \eta'(w)$.

Definition 1.8. Let (G, f) be an m-ary group. An intuitionistic fuzzy set $A = \{\langle w, \mu(w), \eta(w)\rangle \mid w \in G\}$ of G is said to be an intuitionistic fuzzy subgroup of G if

(1) for all $w_1, w_2, ..., w_m \in R$, $\mu\big(f(w_1, w_2, ..., w_m)\big) \geq \min\{\mu(w_1), \mu(w_2), ..., \mu(w_m)\}$,

(2) for all $a_1^m, b \in R$ and $1 \leq i \leq n$ there is $x_i \in R$ so that $f(a_1^{i-1}, x_i, a_{i+1}^m) = b$ and $\mu(x_i) \geq \min\{\mu(a_1), \mu(a_2), ..., \mu(a_{i-1}), \mu(a_{i+1}), ..., \mu(a_n), \mu(b)\}$,

(3) for all $w_1, w_2, ..., w_m \in R$, $\eta\big(f(w_1, w_2, ..., w_m)\big) \leq \max\{\eta(w_1), \eta(w_2), ..., \eta(w_m)\}$,

(4) for all $w_1^m, b \in R$ and $1 \leq i \leq n$ there is $x_i \in R$ so that $f(w_1^{i-1}, x_i, w_{i+1}^m) = b$ and $\eta(x_i) \leq \max\{\eta(w_1), \eta(w_2), ..., \eta(w_{i-1}), \eta(w_{i+1}), ..., \eta(w_n), \eta(b)\}$.

Definition 1.9. Let (G, f) be an m-ary group. An intuitionistic fuzzy subgroup

$$A = \{\langle w, \mu(w), \eta(w)\rangle \mid w \in G\},$$

of G is said to be an intuitionistic fuzzy normal subgroup of G if for every permutation γ of $\{1, 2, ..., m\}$ and $w_1, w_2, ..., w_m \in G$,

(1) $\mu\big(f(w_1, w_2, ..., w_m)\big) = \mu\big(f(w_{\gamma_1}, w_{\gamma_2}, ..., w_{\gamma_m})\big)$,

(2) $\eta\big(f(w_1, w_2, ..., w_m)\big) = \eta\big(f(w_{\gamma_1}, w_{\gamma_2}, ..., w_{\gamma_m})\big)$.

For more details about the intutionistic fuzzy sets, we refer to [5–8].

A near ring is a non-empty set with two binary operations, addition and multiplication, satisfying all the ring axioms except possibly one of the distributive laws and commutativity of addition.

For details about near ring theory and applications, we refer to [9, 10]. For fuzzy sets and fuzzy groups, we refer to [1, 11, 12]. For fuzzy ideals of near rings, we refer to [13–17]. In [18], Mohammadi and Davvaz characterized a new class of n-ary algebras that we call (m, n)-near rings. They investigated the notions of i-R-groups, i-(m, n)-near field, prime ideals, primary ideals and subtractive ideals of (m, n)-near rings. Then, in [19], they studied fuzzy ideals. We recall the definition.

Definition 1.10. Let U be a non-empty set and h, k be m-ary and n-ary operations on U, respectively. Then (U, h, k) is called an i-(m, n)-near ring, if

(1) (U, h) is an m-ary group (not necessarily abelian),

(2) (U, k) is an n-ary semigroup,

(3) the n-ary operation k is i-distributive with respect to the m-ary operation h.

We define i-distributive low, for every $c_1^n, d_1^m \in U$ as follows:
If $i = n$, then

$$k\big(c_1^{n-1}, h(d_1, d_2, ..., d_m)\big) = h\big(k(c_1^{n-1}, d_1), k(c_1^{n-1}, d_2), ..., k(c_1^{n-1}, d_m)\big).$$

If $i = 1$, then

$$k\big(h(d_1, d_2, ..., d_m), c_2^n\big) = h\big(k(d_1, c_2^n), k(d_2, c_2^n), ..., k(d_m, c_2^n)\big).$$

If $1 < i < n$, then

$$\begin{gathered} k\big(c_1^{i-1}, h(d_1, d_2, ..., d_m), c_{i+1}^n\big) \\ = h\big(k(c_1^{i-1}, d_1, c_{i+1}^n), k(c_1^{i-1}, d_2, c_{i+1}^n), ..., k(c_1^{i-1}, d_m, c_{i+1}^n)\big). \end{gathered}$$

Example 1.11. We know $(\mathbb{R}, +, \cdot)$ is an (m, n)-near ring with two binary operations m-addition and n-multiplication. The element $1 \in \mathbb{R}$ is an identity element in $(\mathbb{R}, +, \cdot)$.

Example 1.12. Assume that $\mathbb{Z}$ is a set of integer numbers and h, k are m-ary and n-ary operations on $\mathbb{Z}$, respectively, which are defined below; in this case $(\mathbb{Z}, h, k)$ is an (m, n)-near ring. For all $o_1, o_2 ..., o_m, f_1, f_2, ..., f_n \in \mathbb{Z}$

$$\begin{gathered} h(o_1, o_2, ..., o_m) = o_1 + o_2 + ... + o_m, \\ k(f_1, f_2, ..., f_n) = f_n. \end{gathered}$$

It is clear that $(\mathbb{Z}, h)$ is an m-ary group. We prove that $(\mathbb{Z}, k)$ is an n-ary semigroup. If $1 \leq i < n$, then

$$k(f_1^{i-1}, k(f_i, f_{i+1}, ..., f_{n+i-1}), f_{n+i}, ..., f_{2n-1}) = f_{2n-1}.$$

If $i = n$, then

$$k(f_1^{n-1}, f_{n-1}, k(f_n, f_{n+1}, ..., f_{2n-1})) = k(f_n, f_{n+1}, ..., f_{2n-1}) = f_{2n-1}.$$

We prove that the n-ary operation k is n-distributive with respect to the m-ary operation h. We have

$$\begin{gathered} k(f_1, f_2, ..., f_{n-1}, h(o_1, o_2, ..., o_m)) = h(o_1, o_2, ..., o_m) \\ h(k(f_1, f_2, ..., f_{n-1}, o_1), k(f_1, f_2, ..., f_{n-1}, o_2), ..., k(f_1, f_2, ..., f_{n-1}, o_m)) = \\ h(o_1, o_2, ..., o_m)). \end{gathered}$$

Therefore, we obtain

$$\begin{gathered} k(f_1, f_2, ..., f_{n-1}, h(o_1, o_2, ..., o_m)) = \\ h(k(f_1, f_2, ..., f_{n-1}, o_1), k(f_1, f_2, ..., f_{n-1}, o_2), ..., k(f_1, f_2, ..., f_{n-1}, o_m)). \end{gathered}$$

2 Intuitionistic fuzzy (m,n)-subnear ring

Definition 2.1. Let (R,f,g) be an (m,n)-near ring and $A=\{\langle w,\mu(w),\eta(w)\rangle \mid w\in X\}$ be an intuitionistic fuzzy set of R. In this case, A is called an intuitionistic fuzzy (m,n)-subnear ring of R, if

(1) for all $w_1,w_2,...,w_m\in R$,
$$\mu\big(f(w_1,w_2,...,w_m)\big)\geq \min\{\mu(w_1),\mu(w_2),...,\mu(w_m)\},$$

(2) for all $d_1^m,b\in R$ and $1\leq i\leq n$ there is $x_i\in R$ so that
$$f(d_1^{i-1},x_i,d_{i+1}^m)=b,$$
$$\mu(x_i)\geq \min\{\mu(d_1),\mu(d_2),...,\mu(d_{i-1}),\mu(d_{i+1}),...,\mu(d_m),\mu(b)\},$$

(3) for all $w_1,w_2,...,w_n\in R$,
$$\mu\big(g(w_1,w_2,...,w_n)\big)\geq \min\{\mu(w_1),\mu(w_2),...,\mu(w_n)\},$$

(4) for all $w_1,w_2,...,w_m\in R$,
$$\eta\big(f(w_1,w_2,...,w_m)\big)\leq \max\{\eta(w_1),\eta(w_2),...,\eta(w_m)\},$$

(5) for all $a_1^m,b\in R$ and $1\leq i\leq n$ there is $x_i\in R$ so that
$$f(d_1^{i-1},x_i,d_{i+1}^m)=b,$$
$$\eta(x_i)\leq \max\{\eta(d_1),\eta(d_2),...,\eta(d_{i-1}),\eta(d_{i+1}),...,\eta(d_m),\eta(b)\},$$

(6) for all $w_1,w_2,...,w_n\in R$,
$$\eta\big(g(w_1,w_2,...,w_n)\big)\leq \max\{\eta(w_1),\eta(w_2),...,\eta(w_n)\}.$$

Example 2.2. In Example 1.12, let
$$\mu(w)=\begin{cases}0.9, & \text{if } w\in 2\mathbb{Z},\\ 0.3, & \text{if } w\notin 2\mathbb{Z},\end{cases}$$
$$\eta(w)=\begin{cases}0.4, & \text{if } w\in 2\mathbb{Z},\\ 0.6, & \text{if } w\notin 2\mathbb{Z}.\end{cases}$$

In this case, we have $\mu_{0.9}=2\mathbb{Z}$ and $\mu_{0.3}=\mathbb{Z}$ are both subnear-rings and $\eta_{0.4}^{\leq}=2\mathbb{Z}$ and $\eta_{0.6}^{\leq}=\mathbb{Z}$ are both subnear-rings, hence μ and η are fuzzy subnear-rings of $(\mathbb{Z},h,k)$. So according to Theorem 2.6, $A=\{\langle w,\mu(w),\eta(w)\rangle \mid w\in\mathbb{Z}\}$ is an intuitionistic fuzzy subnear-rings of the (m,n)-near ring $(\mathbb{Z},h,k)$.

Theorem 2.3. *Assume that $A = \{\langle z, \mu(z), \eta(z)\rangle \mid z \in X\}$ is an intuitionistic fuzzy subnear-ring of an (m,n)-near ring (R, f, g), then $A' = \{\langle z, \mu(z), \mu^c(z)\rangle \mid z \in X\}$ is an intuitionistic fuzzy subnear-ring of R.*

Proof. All the conditions of the definition for μ are satisfied, it is enough to check the conditions stated for μ^c in the definition

$$\begin{aligned}\mu^c\big(f(z_1, z_2, ..., z_m)\big) &= 1 - \mu\big(f(z_1, z_2, ..., z_m)\big)\\ &\leq 1 - \min\{\mu(z_1), \mu(z_2), ..., \mu(z_m)\}\\ &= 1 - \min\{1 - \mu^c(z_1), 1 - \mu^c(z_2), ..., 1 - \mu^c(z_m)\}\\ &= \max\{\mu^c(z_1), \mu^c(z_2), ..., \mu^c(z_m)\},\end{aligned}$$

for all $d_1^m, b \in R$ and $1 \leq i \leq n$ there is $x_i \in R$ so that

$$f(d_1^{i-1}, x_i, d_{i+1}^m) = b,$$
$$\mu(x_i) \geq \min\{\mu(d_1), \mu(d_2), ..., \mu(d_{i-1}), \mu(d_{i+1}), ..., \mu(d_m), \mu(b)\},$$

so

$$\begin{aligned}&\mu^c(x_i) = 1 - \mu(x_i)\\ &\leq 1 - \min\{\mu(d_1), \mu(d_2), ..., \mu(d_{i-1}), \mu(d_{i+1}), ..., \mu(d_m), \mu(b)\}\\ &= 1 - \min\{1 - \mu^c(d_1), ..., 1 - \mu^c(d_{i-1}), 1 - \mu^c(d_{i+1}), ..., 1 - \mu^c(d_m), 1 - \mu^c(b)\}\\ &= \max\{\mu^c(d_1), \mu^c(d_2), ..., \mu^c(d_{i-1}), \mu^c(d_{i+1}), ..., \mu^c(d_m), \mu^c(b)\},\end{aligned}$$

$$\begin{aligned}\mu^c\big(g(v_1, v_2, ..., v_n)\big) &= 1 - \mu\big(g(v_1, v_2, ..., v_n)\big)\\ &\leq 1 - \min\{\mu(v_1), \mu(v_2), ..., \mu(v_n)\}\\ &= 1 - \min\{1 - \mu^c(v_1), 1 - \mu^c(v_2), ..., 1 - \mu^c(v_n)\}\\ &= \max\{\mu^c(v_1), \mu^c(v_2), ..., \mu^c(v_n)\}.\end{aligned}$$

□

Theorem 2.4. *Let $A = \{\langle z, \mu(z), \eta(z)\rangle \mid z \in X\}$ be an intuitionistic fuzzy subnear-ring of an (m,n)-near ring (R, f, g). Then $A'' = \{\langle z, \eta^c(z), \eta(z)\rangle \mid z \in X\}$ is an intuitionistic fuzzy subnear-ring of R.*

Proof. Assume that $A = \{\langle z, \mu(z), \eta(z)\rangle \mid z \in X\}$ is an intuitionistic fuzzy subnear-ring of an (m,n)-near ring (R, f, g). All the conditions of the definition for η are satisfied, it is enough to check the conditions stated for η^c in the definition.
For all $z_1^{i-1}, z_{i+1}^m, z \in R$ and $1 \leq j \leq m$, there is $a \in R$ that $f(z_1^{j-1}, z, z_{j+1}^m) = f(z_1^{i-1}, a, z_{i+1}^m)$, $\eta(a) \leq \eta(z)$, so $\eta^c(a) = 1 - \eta(a) \geq 1 - \eta(z) = \eta^c(z)$.
For all $z_1^n, z \in R$ and $1 \leq i \leq n$, $\eta\big(g(z_1^{i-1}, z, z_{i+1}^n)\big) \leq \eta(z)$, then

$$\eta^c\big(g(z_1^{i-1}, z, z_{i+1}^n)\big) = 1 - \eta\big(g(z_1^{i-1}, z, z_{i+1}^n)\big) \geq 1 - \eta(z) = \eta^c(z).$$

For all $k \neq i$, $1 \leq k \leq m$, $d_1^{i-1}, d_{i+1}^n, z_1^m \in R$ and $z_k \in R$ there is $h_k \in R$ so that $g\big(d_1^{i-1}, f(z_1, z_2, ..., z_m), d_{i+1}^n\big) = f\big(g(d_1^{i-1}, z_1, d_{i+1}^n), g(d_1^{i-1}, z_2, d_{i+1}^n), ..., g(d_1^{i-1}, z_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, z_{k+1}, d_{i+1}^n), ..., g(d_1^{i-1}, z_m, d_{i+1}^n)\big)$ and $\eta(h_k) \leq \eta(z_k)$.
Thus $\eta^c(h_k) = 1 - \eta(h_k) \geq 1 - \eta(z_k) = \eta^c(z_k)$. □

Lemma 2.5. *The set* $A = \{\langle w, \mu(w), \eta(w)\rangle \mid w \in X\}$ *is an intuitionistic fuzzy subnear-ring of an* (m, n)*-near ring* (R, f, g) *if and only if* μ *and* η^c *are fuzzy subnear-rings of* R*.*

Proof. Assume that $A = \{\langle w, \mu(w), \eta(w)\rangle \mid w \in X\}$ is an intuitionistic fuzzy set of an (m, n)-near ring (R, f, g). Clearly μ is a fuzzy subnear-ring. Moreover, we have

$$\begin{aligned} \eta^c\big(f(w_1, w_2, ..., w_m)\big) &= 1 - \eta\big(f(w_1, w_2, ..., w_m)\big) \\ &\geq 1 - \max\{\eta(w_1), \eta(w_2), ..., \eta(w_m)\} \\ &= \min\{1 - \eta(w_1), 1 - \eta(w_2), ..., 1 - \eta(w_m)\} \\ &= \min\{\eta^c(w_1), \eta^c(w_2), ..., \eta^c(w_m)\}. \end{aligned}$$

For all $d_1^m, b \in R$ and $1 \leq i \leq n$ there is $x_i \in R$ so that

$$f(d_1^{i-1}, x_i, d_{i+1}^m) = b,$$
$$\eta(x_i) \leq \max\{\eta(d_1), \eta(d_2), ..., \eta(d_{i-1}), \eta(d_{i+1}), ..., \eta(d_m), \eta(b)\},$$

and so

$$\begin{aligned} &\eta^c(x_i) = 1 - \eta(x_i) \\ &\geq 1 - \max\{\eta(d_1), \eta(d_2), ..., \eta(d_{i-1}), \eta(d_{i+1}), ..., \eta(d_m), \eta(b)\} \\ &= \min\{1 - \eta(d_1), 1 - \eta(d_2), ..., 1 - \eta(d_{i-1}), 1 - \eta(d_{i+1}), ..., 1 - \eta(d_m), 1 - \eta(b)\} \\ &= \min\{\eta^c(d_1), \eta^c(d_2), ..., \eta^c(d_{i-1}), \eta^c(d_{i+1}), ..., \eta^c(d_m), \eta^c(b)\}. \end{aligned}$$

Also, we have

$$\begin{aligned} \eta^c\big(g(q_1, q_2, ..., q_n)\big) &= 1 - \eta\big(g(q_1, q_2, ..., q_n)\big) \\ &\geq 1 - \max\{\eta(q_1), \eta(q_2), ..., \eta(q_n)\} \\ &= \min\{1 - \eta(q_1), 1 - \eta(q_2), ..., 1 - \eta(q_n)\} \\ &= \min\{\eta^c(q_1), \eta^c(q_2), ..., \eta^c(q_n)\}. \end{aligned}$$

Therefore η^c is a fuzzy subnear-ring of R.

Conversely, assume that μ and η^c are fuzzy subnear-rings of R. Then, we have

$$\begin{aligned} \eta\big(f(w_1, w_2, ..., w_m)\big) &= 1 - \eta^c\big(f(w_1, w_2, ..., w_m)\big) \\ &\leq 1 - \min\{\eta^c(w_1), \eta^c(w_2), ..., \eta^c(w_m)\} \\ &= 1 - \min\{1 - \eta(w_1), 1 - \eta(w_2), ..., 1 - \eta(w_m)\} \\ &= 1 - (1 - \max\{\eta(w_1), \eta(w_2), ..., \eta(w_m)\}) \\ &= \max\{\eta(w_1), \eta(w_2), ..., \eta(w_m)\}. \end{aligned}$$

For all $d_1^m, b \in R$ and $1 \leq i \leq n$ there is $x_i \in R$ so that

$$f(d_1^{i-1}, x_i, d_{i+1}^m) = b,$$
$$\eta^c(x_i) \geq \min\{\eta^c(d_1), \eta^c(d_2), ..., \eta^c(d_{i-1}), \eta^c(d_{i+1}), ..., \eta^c(d_m), \eta^c(b)\},$$

and so

$$\begin{aligned}\eta(x_i) &= 1-\eta^c(x_i)\\ &\leq 1-\min\{\eta^c(d_1),\eta^c(d_2),...,\eta^c(d_{i-1}),\eta^c(d_{i+1}),...,\eta^c(d_m),\eta^c(b)\}\\ &= 1-\min\{1-\eta(d_1),...,1-\eta(d_{i-1}),1-\eta(d_{i+1}),...,1-\eta(d_m),1-\eta(b)\}\\ &= \max\{\eta(d_1),\eta(d_2),...,\eta(d_{i-1}),\eta(d_{i+1}),...,\eta(d_m),\eta(b)\}.\end{aligned}$$

Also, we have

$$\begin{aligned}\eta\big(g(q_1,q_2,...,q_n)\big) &= 1-\eta^c\big(g(q_1,q_2,...,q_n)\big)\\ &\leq 1-\min\{\eta^c(q_1),\eta^c(q_2),...,\eta^c(q_n)\}\\ &= 1-\min\{1-\eta(q_1),1-\eta(q_2),...,1-\eta(q_n)\}\\ &= \min\{\eta(q_1),\eta(q_2),...,\eta(q_n)\}.\end{aligned}$$

This yields that A is an intuitionistic fuzzy subnear-ring of R. □

Theorem 2.6. *The set $A=\{\langle x,\mu(x),\eta(x)\rangle \mid x\in X\}$ is an intuitionistic fuzzy subnear-ring of an (m,n)-near ring (R,f,g) if and only if for all $s,t\in[0,1]$, the non-empty sets μ_t and $\eta_s^{\leq}$ are subnear-rings of R.*

Proof. Suppose that μ is a fuzzy subnear ring and $\mu_t\neq 0$, then for all $q_1,q_2,...,q_m\in\mu_t$,

$$\mu\big(f(q_1,q_2,...,q_m)\big)\geq\min\{\mu(q_1),\mu(q_2),...,\mu(q_m)\}\geq t.$$

Then $\mu\big(f(q_1,q_2,...,q_m)\big)\geq t$ so $f(q_1,q_2,...,q_m)\in\mu_t$. Hence (μ_t,f) is a semigroup. For all $d_1^{i-1},d_{i+1}^m,b\in\mu_t$ there is $x_i\in R$ that $f(d_1^{i-1},x_i,d_{i+1}^m)=b$ and

$$\mu(x_i)\geq\min\{\mu(d_1),\mu(d_2),...,\mu(d_{i-1}),\mu(d_{i+1}),...,\mu(d_m)\}\geq t.$$

Hence $x_i\in\mu_t$, and so (μ_t,f) is an m-group. For all $w_1,w_2,...,w_n\in\mu_t$ we have

$$\mu\big(g(w_1,w_2,...,w_n)\big)\geq\min\{\mu(w_1),\mu(w_2),...,\mu(w_n)\}\geq t.$$

Thus $\mu\big(g(w_1,w_2,...,w_n)\big)\geq t$, this gives that $g(w_1,w_2,...,w_n)\in\mu_t$. Therefore (μ_t,g) is an n-semigroup. Then the level subset μ_t, $t\in(0,1]$, is a subnear-ring of R.

Assume that $\eta_s^{\leq}\neq\emptyset$. Then for all $d_1,d_2,...,d_m\in\eta_s^{\leq}$, we have

$$\eta\big(f(d_1,d_2,...,d_m)\big)\leq\max\{\eta(d_1),\eta(d_2),...,\eta(d_m)\}\leq s.$$

Then $\eta\big(f(d_1,d_2,...,d_m)\big)\leq s$, and so $f(d_1,d_2,...,d_m)\in\eta_s^{\leq}$, hence (η_s,f) is a semigroup.

For all $d_1^{i-1},d_{i+1}^m,b\in\eta_s^{\leq}$ there is $x_i\in R$ such that $f(d_1^{i-1},x_i,d_{i+1}^m)=b$ and

$$\eta(x_i)\leq\max\{\eta(d_1),\eta(d_2),...,\eta(d_{i-1}),\eta(d_{i+1}),...,\eta(d_m)\}\leq s.$$

Thus, we have $x_i\in\eta_s$, which implies that (η_s,f) is an m-group. For all $w_1,...,w_n\in\eta_s$ we have

$$\eta\big(g(w_1,w_2,...,w_n)\big)\leq\max\{\eta(w_1),\eta(w_2),...,\eta(w_n)\}\leq s.$$

Thus $\eta\big(g(w_1, w_2, ..., w_n)\big) \leq s$, and so $g(w_1, w_2, ..., w_n) \in \eta_s$. Consequently, (η_s, g) is an n-semigroup. This yields that the level subset η_s, for $s \in (0, 1]$, is a subnear-ring of R.

Now, suppose that the level subset μ_t, $t \in (0, 1]$, is a subnear-ring of R. For all $w_1, w_2, ..., w_m \in R$ let $b = \min\{\mu(w_1), ..., \mu(w_m)\}$ so for all $1 \leq i \leq m$, $\mu(w_i) \geq b$. As a result $w_1, w_2, ..., w_m \in \mu_b$, $f(w_1, w_2, ..., w_m) \in \mu_b$, and so $\mu\big(f(w_1, w_2, ..., w_m)\big) \geq b$, or

$$\mu\big(f(w_1, w_2, ..., w_m)\big) \geq \min\{\mu(w_1), \mu(w_2), ..., \mu(w_m)\}.$$

For all $q_1, q_2, ..., q_n \in R$ let $a = \min\{\mu(q_1), \mu(q_2), ..., \mu(q_n)\}$, then for all $1 \leq i \leq n$, $\mu(q_i) \geq a$. As a result $q_1, q_2, ..., q_n \in \mu_a$, and so $g(q_1, q_2, ..., q_n) \in \mu_a$. Hence $\mu\big(g(q_1, q_2, ..., q_n)\big) \geq a$, or equivalently

$$\mu\big(g(q_1, q_2, ..., q_n)\big) \geq \min\{\mu(q_1), \mu(q_2), ..., \mu(q_n)\}.$$

For all $d_1^{i-1}, d_{i+1}^m, b \in R$ let $d = \min\{\mu(d_1), \mu(d_2), ..., \mu(d_{i-1}), \mu(d_{i+1}), ..., \mu(d_m), \mu(b)\}$, then $d_1^{i-1}, d_{i+1}^m, b \in \mu_d$. Hence, there is $x_i \in \mu_d$ so that $b = f(d_1^{i-1}, x_i, d_{i+1}^m)$,

$$\mu(x_i) \geq d = \min\{\mu(d_1), \mu(d_2), ..., \mu(d_{i-1}), \mu(d_{i+1}), ..., \mu(d_m), \mu(b)\}.$$

Therefore, μ is a fuzzy subnear-ring.

Now, suppose that the level subset $\eta_s^{\leq}$, $s \in (0, 1]$, is a subnear-ring of R. For all $w_1, w_2, ..., w_m \in R$ let $b = \max\{\eta(w_1), \eta(w_2), ..., \eta(w_m)\}$ so for all $1 \leq i \leq m$, $\eta(w_i) \leq b$. As a result $w_1, w_2, ..., w_m \in \eta_b$, $f(w_1, w_2, ..., w_m) \in \eta_b$, and so $\eta\big(f(w_1, w_2, ..., w_m)\big) \leq b$

$$\eta\big(f(w_1, w_2, ..., w_m)\big) \leq \max\{\eta(w_1), \eta(w_2), ..., \eta(w_m)\}.$$

For all $q_1, q_2, ..., q_n \in R$ let $a = \max\{\eta(q_1), \eta(q_2), ..., \eta(q_n)\}$, so for all $1 \leq i \leq n$, $\eta(q_i) \leq a$. As a result $q_1, q_2, ..., q_n \in \eta_a$ so $g(q_1, q_2, ..., q_n) \in \eta_a$. Hence $\eta\big(g(q_1, q_2, ..., q_n)\big) \leq a$, which implies that

$$\eta\big(g(q_1, q_2, ..., q_n)\big) \leq \max\{\eta(q_1), \eta(q_2), ..., \eta(q_n)\}.$$

For all $d_1^{i-1}, d_{i+1}^m, b \in R$ let $d = \max\{\eta(d_1), \eta(d_2), ..., \eta(d_{i-1}), \eta(d_{i+1}), ..., \eta(d_m), \eta(b)\}$, so $d_1^{i-1}, d_{i+1}^m, b \in \eta_d$. Hence there is $x_i \in \eta_d$ so that $b = f(d_1^{i-1}, x_i, d_{i+1}^m)$,

$$\eta(x_i) \leq d = \max\{\eta(d_1), \eta(d_2), ..., \eta(d_{i-1}), \eta(d_{i+1}), ..., \eta(d_m), \eta(b)\}.$$

Therefore η is a fuzzy subnear-ring. □

Example 2.7. In Example 1.11, let

$$\mu(w) = \begin{cases} 0.7, & \text{if } w \in \mathbb{Z}, \\ 0.2, & \text{if } w \notin \mathbb{Z}. \end{cases}$$

$$\eta(w) = \begin{cases} 0.2, & \text{if } w \in \mathbb{Z}, \\ 0.9, & \text{if } w \notin \mathbb{Z}. \end{cases}$$

In this case, we have $\mu_{0.7} = \mathbb{Z}$ and $\mu_{0.2} = \mathbb{R}$ are both subnear-rings, and $\eta^{\leq}_{0.2} = \mathbb{Z}$ and $\eta^{\leq}_{0.9} = \mathbb{R}$ are both subnear-rings. Hence μ and η are fuzzy subnear-rings, of $(\mathbb{R}, +, \cdot)$. So, according to Theorem 2.6, $A = \{\langle w, \mu(w), \eta(w)\rangle \mid w \in X\}$ is an intuitionistic fuzzy subnear-ring of (R, f, g)

Definition 2.8. Assume that (R, h, k) is an (m, n)-near ring and W is a non empty subset of R. The intuitionistic characteristic function of W is denoted by $\chi_W = \langle \mu_{\chi_W}, \eta_{\chi_W})\rangle$ and is defined by

$$\mu_{\chi_W} : R \longrightarrow [0,1] \mid w \longrightarrow \mu_{\chi_W} := \begin{cases} 1, & \text{if } w \in W, \\ 0, & \text{if } w \notin W, \end{cases}$$

$$\eta_{\chi_W} : R \longrightarrow [0,1] \mid w \longrightarrow \eta_{\chi_W} := \begin{cases} 0, & \text{if } w \in W, \\ 1, & \text{if } w \notin W. \end{cases}$$

Example 2.9. In Example 1.11, let $A = \{2w \mid w \in \mathbb{N}\}$ and

$$\mu_{\chi_A} = \begin{cases} 1, & \text{if } w \in A, \\ 0, & \text{if } w \notin A, \end{cases}$$

$$\eta_{\chi_A} = \begin{cases} 0, & \text{if } w \in A, \\ 1, & \text{if } w \notin A. \end{cases}$$

Then $\chi_A = \langle \mu_{\chi_A}, \eta_{\chi_A}\rangle$ is intuitionistic characteristic function of A.

Lemma 2.10. *Let I be a subset of an (m, n)-near ring (R, f, g). Then I is a subnear-ring of R if and only if the intuitionistic characteristic function $\chi_I = \langle \mu_{\chi_I}, \eta_{\chi_I}\rangle$ of I is an intuitionistic fuzzy subnear-ring of R.*

Proof. Assume that the intuitionistic characteristic function $\chi_I = \langle \mu_{\chi_I}, \eta_{\chi_I}\rangle$ of I is an intuitionistic fuzzy subnear-ring of R.

(1) Let $w_1, w_2, ..., w_m \in I$, then $\mu_{\chi_I}(w_1) = \mu_{\chi_I}(w_2) = ... = \mu_{\chi_I}(w_m) = 1$, and so

$$\mu_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) \geq \min\{\mu_{\chi_I}(w_1), \mu_{\chi_I}(w_2), ..., \mu_{\chi_I}(w_m)\} = 1.$$

Let $w_1, w_2, ..., w_m \in I$, then $\eta_{\chi_I}(w_1) = \eta_{\chi_I}(w_2) = ... = \eta_{\chi_I}(w_m) = 0$, which implies that

$$\eta_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) \leq \max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_m)\} = 0.$$

Therefore $f(w_1, w_2, ..., w_m) \in I$.

(2) Let $d_1^n, b \in I$, then there is $x_i \in R$, $1 \leq i \leq n$, so that $f(d_1^{i-1}, x_i, d_{i+1}^m) = b$ and $\mu_{\chi_I}(x_i) \geq \min\{\mu_{\chi_I}(d_1), \mu_{\chi_I}(d_2), ..., \mu_{\chi_I}(d_{i-1}), \mu_{\chi_I}(d_{i+1}), ..., \mu_{\chi_I}(d_n), \mu_{\chi_I}(b)\} = 1$. Let $d_1^n, b \in I$, then there is $x_i \in R$, $1 \leq i \leq n$, so that $f(d_1^{i-1}, x_i, d_{i+1}^m) = b$ and

$\eta_{\chi_I}(x_i) \leq \max\{\eta_{\chi_I}(d_1), \eta_{\chi_I}(d_2), ..., \eta_{\chi_I}(d_{i-1}), \eta_{\chi_I}(d_{i+1}), ..., \eta_{\chi_I}(d_n), \eta_{\chi_I}(b)\} = 0$, and so $x_i \in I$.

(3) Let $w_1, w_2, ..., w_n \in I$, then $\mu_{\chi_I}(w_1) = \mu\chi_I(w_2) = ... = \mu_{\chi_I}(w_m) = 1$, which implies that

$\mu_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) \geq \min\{\mu_{\chi_I}(w_1), \mu_{\chi_I}(w_2), ..., \mu_{\chi_I}(w_n)\} = 1$.

Let $w_1, w_2, ..., w_n \in I$, then $\eta_{\chi_I}(w_1) = \eta_{\chi_I}(w_2) = ... = \eta_{\chi_I}(w_m) = 0$, and hence $\eta_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) \leq \max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_n)\} = 0$. This implies that $g(w_1, w_2, ..., w_n) \in I$.

Conversely, assume that I is a subnear-ring of R.

(1) Let $w_1, w_2, ..., w_m \in R$. If there is $1 \leq i \leq m$ that $w_i \notin I$, then

$$\min\{\mu_{\chi_I}(w_1), \mu_{\chi_I}(w_2), ..., \mu_{\chi_I}(w_m)\} = 0,$$
$$\mu_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) \geq \min\{\mu_{\chi_I}(w_1), \mu_{\chi_I}(w_2), ..., \mu_{\chi_I}(w_m)\}.$$

Otherwise, $w_1, w_2, ..., w_m \in I$ and I is an ideal of R so $f(w_1, w_2, ..., w_m) \in I$. Hence, we get

$$\mu_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) = 1,$$
$$\mu_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) \geq \min\{\mu_{\chi_I}(w_1), \mu_{\chi_I}(w_2), ..., \mu_{\chi_I}(w_m)\}.$$

(2) Let $d_1^n, b \in R$. Since R is an (m, n)-near ring, it follows that for all $1 \leq i \leq n$ there is $x_i \in R$, $f(d_1^{i-1}, x_i, d_{i+1}^m) = b$. If there is $1 \leq i \leq m$ such that $d_i \notin I$ or $b \notin I$, then

$$\min\{\mu_{\chi_I}(d_1), \mu_{\chi_I}(d_2), ..., \mu_{\chi_I}(d_{i-1}), \mu_{\chi_I}(d_{i+1}), ..., \mu_{\chi_I}(d_n), \mu_{\chi_I}(b)\} = 0,$$
$$\mu_{\chi_I}(x_i) \geq \min\{\mu_{\chi_I}(d_1), \mu_{\chi_I}(d_2), ..., \mu_{\chi_I}(d_{i-1}), \mu_{\chi_I}(d_{i+1}), ..., \mu_{\chi_I}(d_n), \mu_{\chi_I}(b)\}.$$

Otherwise, $d_1^m, b \in I$ and I is an ideal of R. Hence, $x_i \in I$ which implies that $\mu_{\chi_I}(x_i) = 1$ and

$$\mu_{\chi_I}(x_i) \geq \min\{\mu_{\chi_I}(d_1), \mu_{\chi_I}(d_2), ..., \mu_{\chi_I}(d_{i-1}), \mu_{\chi_I}(d_{i+1}), ..., \mu_{\chi_I}(d_n), \mu_{\chi_I}(b)\},$$

(3) Let $w_1, w_2, ..., w_n \in R$. If there is $1 \leq i \leq n$ such that $w_i \notin I$, then

$$\min\{\mu(w_1), \mu(w_2), ..., \mu(w_n)\} = 0 \text{ and}$$
$$\mu_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) \geq \min\{\mu_{\chi_I}(w_1), \mu_{\chi_I}(w_2), ..., \mu_{\chi_I}(w_n)\}.$$

Otherwise, $w_1, w_2, ..., w_n \in I$ and I is an ideal of R, and so $g(w_1, w_2, ..., w_n) \in I$. This gives that

$$\mu_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) = 1,$$
$$\mu_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) \geq \min\{\mu_{\chi_I}(w_1), \mu_{\chi_I}(w_2), ..., \mu_{\chi_I}(w_n)\}.$$

(1) Let $w_1, w_2, ..., w_m \in R$. If there is $1 \leq i \leq m$ such that $w_i \notin I$, then

$$\max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_m)\} = 1,$$
$$\eta_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) \leq \max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_m)\}.$$

Otherwise, $w_1, w_2, ..., w_m \in I$, and I is an ideal of R so $f(w_1, w_2, ..., w_m) \in I$. Hence, we have

$$\eta_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) = 0,$$
$$\eta_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) \leq \max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_m)\}.$$

(2) Let $d_1^n, b \in R$. Since R is an (m, n)-near ring, it follows that for all $1 \leq i \leq n$ there is $x_i \in R$, $f(d_1^{i-1}, x_i, d_{i+1}^m) = b$. If there is $1 \leq i \leq m$ so that $d_i \notin I$ or $b \notin I$, then

$$\max\{\eta_{\chi_I}(d_1), \eta_{\chi_I}(d_2), ..., \eta_{\chi_I}(d_{i-1}), \eta_{\chi_I}(d_{i+1}), ..., \eta_{\chi_I}(d_n), \eta_{\chi_I}(b)\} = 1,$$
$$\eta_{\chi_I}(x_i) \leq \max\{\eta_{\chi_I}(d_1), \eta_{\chi_I}(d_2), ..., \eta_{\chi_I}(d_{i-1}), \eta_{\chi_I}(d_{i+1}), ..., \eta_{\chi_I}(d_n), \eta_{\chi_I}(b)\}.$$

Otherwise, $d_1^m, b \in I$ and I is an ideal of R so $x_i \in I$. Consequently, we get $\eta_{\chi_I}(x_i) = 0$ and

$$\eta_{\chi_I}(x_i) \leq \max\{\eta_{\chi_I}(d_1), \eta_{\chi_I}(d_2), ..., \eta_{\chi_I}(d_{i-1}), \eta_{\chi_I}(d_{i+1}), ..., \eta_{\chi_I}(d_n), \eta_{\chi_I}(b)\}.$$

(3) Let $w_1, w_2, ..., w_n \in R$. If there is $1 \leq i \leq n$ such that $w_i \notin I$, then

$$\max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_n)\} = 1 \text{ and}$$
$$\eta_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) \leq \max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_n)\}.$$

Otherwise, $w_1, w_2, ..., w_n \in I$ and I is an ideal of R so $g(w_1, w_2, ..., w_n) \in I$. This yields that hence

$$\eta_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) = 0,$$
$$\eta_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) \leq \max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_n)\}.$$

□

Definition 2.11. Let I be a non-empty subset of an (m, n)-near ring (R, f, g). Then I is called an ideal of R if

(1) I is a subgroup of m-ary group (R, f), (I, f) is an m-ary group,

(2) for every $q_1, q_2, ..., q_n \in R$, $g(q_1^{i-1}, I, q_{i+1}^n) \subseteq I$.

(3) for all $r_1^{j-1}, r_{j+1}^m, s_1^{j-1}, s_{j+1}^n \in R$ and $1 \leq k \leq n$, $d \in I$, there exists $l \in I$ so that
$g(s_1^{j-1}, f(r_1^{k-1}, d, r_{k+1}^m), s_{j+1}^n)$
$= f(g(s_1^{j-1}, r_1, s_{j+1}^n), g(s_1^{j-1}, r_2, s_{j+1}^n), ..., g(s_1^{j-1}, r_{k-1}, s_{j+1}^n), l, g(s_1^{j-1}, r_{k+1},$
$s_{j+1}^n), ..., g(s_1^{j-1}, r_n, s_{j+1}^n))$.
Note that I is called an i-ideal of R if it satisfies (1) and (2) and I is called a j-ideal of R for $j \neq i$ if it satisfies (1) and (3).

Also, if for every $1 \leq i \leq n$, I is an i-ideal, then I is called an ideal of R. In general, every ideal in an (m, n)-near ring is an (m, n)-subnear ring, but the converse is not true.

Example 2.12. In Example 1.12, the subset $2\mathbb{Z}$ is an n-ideal of $(\mathbb{Z}, h, k)$. Also, $(2\mathbb{Z}, h)$ is a normal subgroup of m-ary group $(\mathbb{Z}, h)$. For all $s_1, s_2, ..., s_n \in \mathbb{Z}$, $k(s_1^{n-1}, 2\mathbb{Z}) = 2\mathbb{Z} \subseteq 2\mathbb{Z}$.

Definition 2.13. Let (R, f, g) be an i-(m, n)-near ring and $A = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in X\}$ be an intuitionistic fuzzy set of R. We say that A is an intuitionistic fuzzy ideal of R, if

(1) for all $w_1, w_2, ..., w_m \in R$,

$$\mu\big(f(w_1, w_2, ..., w_m)\big) \geq \min\{\mu(w_1), \mu(w_2), ..., \mu(w_m)\},$$

(2) for all $a_1^m, b \in R$ and $1 \leq i \leq n$ there is $x_i \in R$ so that

$$f(a_1^{i-1}, x_i, a_{i+1}^m) = b,$$
$$\mu(x_i) \geq \min\{\mu(a_1), \mu(a_2), ..., \mu(a_{i-1}), \mu(a_{i+1}), ..., \mu(a_m), \mu(b)\},$$

(3) for all $w_1^{i-1}, w_{i+1}^m, w \in R$ and $1 \leq j \leq m$, there is $a \in R$ that

$$f(w_1^{j-1}, w, w_{j+1}^m) = f(w_1^{i-1}, a, w_{i+1}^m),\ \mu(a) \geq \mu(w),$$

(4) for all $w_1, w_2, ..., w_n \in R$,

$$\mu\big(g(w_1, w_2, ..., w_n)\big) \geq \min\{\mu(w_1), \mu(w_2), ..., \mu(w_n)\},$$

(5) for all $w_1^n, w \in R$ and $1 \leq i \leq n$,

$$\mu\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \geq \mu(w),$$

(6) for all $k \neq i$, $1 \leq k \leq m$, $d_1^{i-1}, d_{i+1}^n, w_1^m \in R$ and $w_k \in R$ there is $h_k \in R$ so that
$g\big(d_1^{i-1}, f(w_1, ..., w_m), d_{i+1}^n\big) = f\big(g(d_1^{i-1}, w_1, d_{i+1}^n), g(d_1^{i-1}, w_2, d_{i+1}^n), ..., g(d_1^{i-1},$ $w_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, w_{k+1}, d_{i+1}^n), ..., g(d_1^{i-1}, w_m, d_{i+1}^n)\big)$ and $\mu(h_k) \geq \mu(w_k)$,

(7) for all $w_1, w_2, ..., w_m \in R$,

$$\eta\big(f(w_1, w_2, ..., w_m)\big) \leq \max\{\eta(w_1), \eta(w_2), ..., \eta(w_m)\},$$

(8) for all $d_1^m, b \in R$ and $1 \leq i \leq n$ there is $w_i \in R$ so that

$$f(d_1^{i-1}, w_i, d_{i+1}^m) = b,$$
$$\eta(w_i) \leq \max\{\eta(d_1), \eta(d_2), ..., \eta(d_{i-1}), \eta(d_{i+1}), ..., \eta(d_m), \eta(b)\},$$

(9) for all $w_1^{i-1}, w_{i+1}^m, w \in R$ and $1 \leq j \leq m$, there is $a \in R$ so that

$$f(w_1^{j-1}, w, w_{j+1}^m) = f(w_1^{i-1}, a, w_{i+1}^m),\ \eta(a) \leq \eta(w),$$

(10) for all $w_1, w_2, ..., w_n \in R$,

$$\eta\big(g(w_1, w_2, ..., w_n)\big) \leq \max\{\eta(w_1), \eta(w_2), ..., \eta(w_n)\},$$

(11) for all $w_1^n, x \in R$ and $1 \leq i \leq n$,

$$\eta\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \leq \eta(w),$$

(12) for all $k \neq i$, $1 \leq k \leq m$, $d_1^{i-1}, d_{i+1}^n, w_1^m \in R$ and $w_k \in R$ there is $h_k \in R$ so that
$g\big(d_1^{i-1}, f(w_1, ..., w_m), d_{i+1}^n\big) = f\big(g(d_1^{i-1}, w_1, d_{i+1}^n), g(d_1^{i-1}, w_2, d_{i+1}^n), ..., g(d_1^{i-1}, w_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, w_{k+1}, d_{i+1}^n), ..., g(d_1^{i-1}, w_m, d_{i+1}^n)\big)$ and $\eta(h_k) \leq \eta(w_k)$.

Note that A is an intuitionistic fuzzy i-ideal of R if it satisfies $1, 2, ..., 5, 7, ..., 11$ and A is an intuitionistic fuzzy j-ideal, $j \neq i$, of R if it satisfies $1, 2, 3, 4, 6, ..., 10, 12$, $1 \leq i, j \leq n$.

Example 2.14. In Example 1.12, let

$$\mu(w) = \begin{cases} 0.8, & \text{if } w \in 2\mathbb{Z}, \\ 0.2, & \text{if } w \notin 2\mathbb{Z}. \end{cases}$$

$$\eta(w) = \begin{cases} 0.1, & \text{if } w \in 2\mathbb{Z}, \\ 0.7, & \text{if } w \notin 2\mathbb{Z}. \end{cases}$$

In this case, we have $\mu_{0.8} = 2\mathbb{Z}$ and $\mu_{0.2} = \mathbb{Z}$ are both n-ideals of $(\mathbb{Z}, h, k)$ and $\eta_{0.1}^{\leq} = 2\mathbb{Z}$ and $\eta_{0.7}^{\leq} = \mathbb{Z}$ are both n-ideals of $(\mathbb{Z}, h, k)$, hence μ and $\eta^{\leq}$ are fuzzy n-ideals of $(\mathbb{Z}, h, k)$. So, according to Theorem 2.19, $A = \{\langle w, \mu(w), \eta(w)\rangle \mid w \in \mathbb{Z}\}$ is an intuitionistic fuzzy n-ideal of the (m, n)-near ring $(\mathbb{Z}, h, k)$. Using the above definition, we can prove that $A = \{\langle w, \mu(w), \eta(w)\rangle \mid w \in \mathbb{Z}\}$ is an intuitionistic fuzzy n-ideal of the (m, n)-near ring $(\mathbb{Z}, h, k)$.

Theorem 2.15. *Assume that $A = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in X\}$ is an intuitionistic fuzzy ideal of an (m, n)-near ring (R, f, g), then $A' = \{\langle x, \mu(x), \mu^c(x)\rangle \mid x \in X\}$ is an intuitionistic fuzzy ideal of R.*

Proof. Assume that μ is a fuzzy ideal of (R, f, g). Then, by Theorem 2.3, $A' = \{\langle x, \mu(x), \mu^c(x)\rangle \mid x \in X\}$ is an intuitionistic fuzzy subnear-ring of an (m, n)-near ring R and for all $w_1^{i-1}, w_{i+1}^m, w \in R$ and $1 \leq j \leq m$, there is $a \in R$ such that

$$f(w_1^{j-1}, w, w_{j+1}^m) = f(w_1^{i-1}, a, w_{i+1}^m),\ \mu(a) \geq \mu(w),$$

and so $\mu^c(a) = 1 - \mu(a) \leq 1 - \mu(w) = \mu^c(w)$.

For all $w_1^n, w \in R$ and $1 \leq i \leq n$,

$$\mu\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \geq \mu(w),$$

hence $\mu^c\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) = 1 - \mu\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \leq 1 - \mu(w) = \mu^c(w)$.

For all $k \neq i$, $1 \leq k \leq m$, $d_1^{i-1}, d_{i+1}^n, w_1^m \in R$ and $w_k \in R$ there is $h_k \in R$ so that

$g\big(d_1^{i-1}, f(w_1, ..., w_m), d_{i+1}^n\big) = f\big(g(d_1^{i-1}, w_1, d_{i+1}^n), g(d_1^{i-1}, w_2, d_{i+1}^n), ..., g(d_1^{i-1}, w_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, w_{k+1}, d_{i+1}^n), ..., g(d_1^{i-1}, w_m, d_{i+1}^n)\big)$ and $\mu(h_k) \geq \mu(x_k)$.

Therefore, we have $\mu^c(h_k) = 1 - \mu(h_k) \leq 1 - \mu(x_k) = \mu^c(w_k)$. □

Theorem 2.16. *Assume that $A = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in X\}$ is an intuitionistic fuzzy ideal of an (m, n)-near ring (R, f, g). Then $A'' = \{\langle x, \eta^c(x), \eta(x)\rangle \mid x \in X\}$ is an intuitionistic fuzzy ideal of R.*

Proof. Assume that $A = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in X\}$ is an intuitionistic fuzzy ideal of an (m, n)-near ring (R, f, g). Then $A'' = \{\langle x, \eta^c(x), \eta(x)\rangle \mid x \in X\}$ is an intuitionistic fuzzy subnear-ring of R.
For all $w_1^{i-1}, w_{i+1}^m, w \in R$ and $1 \leq j \leq m$, there is $a \in R$ such that

$$f(w_1^{j-1}, w, w_{j+1}^m) = f(w_1^{i-1}, a, w_{i+1}^m),\ \eta(a) \leq \eta(w),$$

and so $\eta^c(a) = 1 - \eta(a) \geq 1 - \eta(w) = \eta^c(w)$.
For all $w_1^n, w \in R$ and $1 \leq i \leq n$,

$$\eta\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \leq \eta(w).$$

This implies that, $\eta^c\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) = 1 - \eta\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \geq 1 - \eta(w) = \eta^c(w)$.

For all $k \neq i$, $1 \leq k \leq m$, $d_1^{i-1}, d_{i+1}^n, w_1^m \in R$ and $w_k \in R$ there is $h_k \in R$ so that

$g\big(d_1^{i-1}, f(w_1, ..., w_m), d_{i+1}^n\big) = f\big(g(d_1^{i-1}, w_1, d_{i+1}^n), g(d_1^{i-1}, w_2, d_{i+1}^n), ..., g(d_1^{i-1}, w_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, w_{k+1}, d_{i+1}^n), ..., g(d_1^{i-1}, w_m, d_{i+1}^n)\big)$ and $\eta(h_k) \leq \eta(w_k)$.

Thus, we get $\eta^c(h_k) = 1 - \eta(h_k) \geq 1 - \eta(w_k) = \eta^c(w_k)$. □

Lemma 2.17. *$A = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in X\}$ is an intuitionistic fuzzy ideal of an (m, n)-near ring (R, f, g) if and only if μ and η^c are fuzzy ideals of R.*

Proof. Let A be an intuitionistic fuzzy ideal of R. Clearly μ is a fuzzy ideal. Thus by Lemma 2.5, η^c is a fuzzy subnear-ring of R. For all $w_1^{i-1}, w_{i+1}^m, w \in R$ and $1 \leq j \leq m$, there is $a \in R$ such that

$$f(w_1^{j-1}, w, w_{j+1}^m) = f(w_1^{i-1}, a, w_{i+1}^m),\ \eta(a) \leq \eta(w),$$

and thus $\eta^c(a) = 1 - \eta(a) \geq 1 - \eta(w) = \eta^c(w)$.
For all $w_1^n, w \in R$ and $1 \leq i \leq n$,

$$\eta\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \leq \eta(w).$$

This gives that $\eta^c\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) = 1 - \eta\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \geq 1 - \eta(w) = \eta^c(w)$.
For all $k \neq i$, $1 \leq k \leq m$, $d_1^{i-1}, d_{i+1}^n, w_1^m \in R$ and $w_k \in R$ there is $h_k \in R$ so that

$g\big(d_1^{i-1}, f(w_1, w_2, ..., w_m), d_{i+1}^n\big) = f\big(g(d_1^{i-1}, w_1, d_{i+1}^n), g(d_1^{i-1}, w_2, d_{i+1}^n), ..., g(d_1^{i-1}, w_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, w_{k+1}, d_{i+1}^n), ..., g(d_1^{i-1}, w_m, d_{i+1}^n)\big)$ and $\eta(h_k) \leq \eta(w_k)$.

Thus $\eta^c(h_k) = 1 - \eta(h_k) \geq 1 - \eta(w_k) = \eta^c(w_k)$.
Therefore, we conclude that η^c is a fuzzy ideal of (R, f, g), $t \in [\mu(0), 1]$.

Conversely, if μ and η^c are fuzzy ideals of R, then by Lemma 2.5, A is an intuitionistic fuzzy subnear-ring of R. For all $w_1^{i-1}, w_{i+1}^m, w \in R$ and $1 \leq j \leq m$, there is $a \in R$ such that

$$f(w_1^{j-1}, w, w_{j+1}^m) = f(w_1^{i-1}, a, w_{i+1}^m),\ \eta(a) = 1 - \eta^c(a) \leq 1 - \eta^c(w) = \eta(w), \text{ so } \eta(a) \leq \eta(w).$$

For all $w_1^n, w \in R$ and $1 \leq i \leq n$,

$$\eta\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) = 1 - \eta^c\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \leq 1 - \eta^c(w) = \eta(w),$$

hence $\eta\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \leq \eta(w)$.
For all $k \neq i$, $1 \leq k \leq m$, $d_1^{i-1}, d_{i+1}^n, w_1^m \in R$ and $w_k \in R$ there is $h_k \in R$ so that
$g\big(d_1^{i-1}, f(w_1, w_2, ..., w_m), d_{i+1}^n\big) = f\big(g(d_1^{i-1}, w_1, d_{i+1}^n), g(d_1^{i-1}, w_2, d_{i+1}^n), ..., g(d_1^{i-1}, w_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, w_{k+1}, d_{i+1}^n), ..., g(d_1^{i-1}, w_m, d_{i+1}^n)\big)$
and
$\eta(h_k) = 1 - \eta^c(h_k) \leq \eta^c(w_k) = \eta(w_k)$. Thus, $\eta(h_k) \leq \eta(w_k)$.
Therefore, all the conditions of the definition are satisfied. Consequently, $A = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in X\}$ is an intuitionistic fuzzy ideal of (R, f, g). □

Theorem 2.18. *([20]). Assume that $A = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in X\}$ is an intuitionistic fuzzy set of an (m, n)-near ring (R, f, g). Then A is an intuitionistic fuzzy ideal of R if and only if $B = \{\langle x, \mu(x), \mu^c(x)\rangle \mid x \in X\}$ and $C = \{\langle x, \eta^c(x), \eta(x)\rangle \mid x \in X\}$ are intuitionistic fuzzy ideals of R.*

Proof. If A is an intuitionistic fuzzy ideal of R, then $\mu = (\mu^c)^c$ and η^c are fuzzy ideals of R by Lemma 2.17. Hence, B and C are intuitionistic fuzzy ideals of R. Conversely, if B and C are intuitionistic fuzzy ideals of R, then the fuzzy sets μ and η^c are fuzzy ideals of R. Therefore, A is an intuitionistic fuzzy ideal of R. □

Theorem 2.19. *The set $A = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in X\}$ is an intuitionistic fuzzy ideal of an (m, n)-near ring (R, f, g) if and only if for all $s, t \in [0, 1]$, the non-empty sets μ_t and $\eta_s^{\leq}$ are ideals of R.*

Proof. Suppose that the level subset μ_t, $t \in (0, 1]$, is an ideal of R. Then μ_t is a subnear ring of R. By using Theorem 2.6, μ is a fuzzy subnear ring of R.

(1) Let $w_1^{i-1}, w_{i+1}^m \in R$, $w_i \in R$ and $t = \mu(w_i)$. Then $w_i \in \mu_t$ and μ_t is an ideal of R, and so for all $1 \leq j \leq m$ there is $a_j \in \mu_t$ so that $f(w_1^{i-1}, w_i, w_{i+1}^m) = f(w_1^{j-1}, a_j, w_{j+1}^m)$ and $\mu(a_j) \geq t = \mu(w_i)$. Therefore $\mu(a_j) \geq \mu(w_i)$.

(2) Let $w_1^n, w \in R$. Then there is $b \in (0, 1]$ such that $\mu(w) = b$. μ_b is an ideal of R, and so $g(w_1^{i-1}, \mu_b, w_{i+1}^n) \subseteq \mu_b$. This implies that $g(w_1^{i-1}, w, w_{i+1}^n) \in \mu_b$, and

so $\mu\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \geq b = \mu(w)$. Therefore, for all $w_1, w_2, ..., w_n, w \in R$ and $1 \leq i \leq n$, we have

$$\mu\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \geq \mu(w).$$

(3) Let $k \neq i$, $1 \leq k \leq n$, $d_1^{i-1}, d_{i+1}^n, w_1^m, \in R$ and $d \in R$. Then there is $b \in (0,1]$ so that $\mu(d) = b$. Since μ_b is an ideal of R, it follows that there exists $l \in \mu_b$ such that
$g\big(s_1^{i-1}, f(r_1^{k-1}, d, r_{k+1}^m), s_{i+1}^n\big) = f\big(g(s_1^{i-1}, r_1, s_{i+1}^n), ...,$
$g(s_1^{i-1}, r_{k-1}, s_{i+1}^n), l, g(s_1^{i-1}, r_{k+1}, s_{i+1}^n), ..., g(s_1^{i-1}, r_n, s_{i+1}^n)\big)$.
$l \in \mu_b$ so $\mu(l) \geq b = \mu(d)$.

Now, suppose that the lower level cut subset $\eta_t^{\leq}$, $t \in (0,1]$, is an ideal of R. Then $\eta_t^{\leq}$ is a subnear ring of R.

(1) Let $w_1^{i-1}, w_{i+1}^m \in R$, $w_i \in R$ and $t = \eta(w_i)$. Hence, $w_i \in \eta_t^{\leq}$ and $\eta_t^{\leq}$ is an ideal of R. Thus for all $1 \leq j \leq m$ there is $a_j \in \eta_t$ so that $f(w_1^{i-1}, w_i, w_{i+1}^m) = f(w_1^{j-1}, a_j, w_{j+1}^m)$ and $\eta(a_j) \leq t = \eta(w_i)$. Consequently, $\eta(a_j) \leq \eta(w_i)$.

(2) Let $w_1^n, w \in R$. Then, there is $b \in (0,1]$ so that $\mu(w) = b$. Since μ_b is an ideal of R, it follows that $g(w_1^{i-1}, \eta_b, w_{i+1}^n) \subseteq \eta_b$. Hence, $g(w_1^{i-1}, w, w_{i+1}^n) \in \eta_b$, and so $\eta\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \leq b = \eta(w)$. Therefore, for all $w_1, w_2, ..., w_n, x \in R$ and $1 \leq i \leq n$, we have

$$\eta\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \leq \eta(w).$$

(3) Let $k \neq i$, $1 \leq k \leq n$, $d_1^{i-1}, d_{i+1}^n, w_1^m, \in R$ and $d \in R$. Then, there is $b \in (0,1]$ such that $\eta(d) = b$. Since η_b is an ideal of R, it follows that there exists $l \in \eta_b$ such that
$g\big(s_1^{i-1}, f(r_1^{k-1}, d, r_{k+1}^m), s_{i+1}^n\big) = f\big(g(s_1^{i-1}, r_1, s_{i+1}^n), ...,$
$g(s_1^{i-1}, r_{k-1}, s_{i+1}^n), l, g(s_1^{i-1}, r_{k+1}, s_{i+1}^n), ..., g(s_1^{i-1}, r_n, s_{i+1}^n)\big)$.
$l \in \eta_b$ so $\eta(l) \leq b = \eta(d)$.

So $A = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in X\}$ is an intuitionistic fuzzy ideal of (R, f, g).

Conversely, suppose that $A = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in X\}$ is an intuitionistic fuzzy ideal of an (m,n)-near ring (R, f, g). Then μ is a fuzzy subnear-ring of R. By using Theorem 2.6, μ_t is a subnear-ring of R.

(1) Let $w_1^{i-1}, w_{i+1}^m \in R$, $w_i \in \mu_t$ and μ be an ideal of R. Then, for all $1 \leq j \leq m$, there is $a_j \in R$ so that $f(w_1^{i-1}, w_i, w_{i+1}^m) = f(w_1^{j-1}, a_j, w_{j+1}^m)$ and $\mu(a_j) \geq \mu(w_i) = t$. Hence $a_j \in \mu_t$.

(2) Let $w_1, w_2, ..., w_n \in R$ and $w \in \mu_t$. Then $\mu\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \geq \mu(w)$, and so $\mu\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \geq t$. Hence, $g(w_1^{i-1}, w, w_{i+1}^n) \in \mu_t$ which implies that $g(w_1^{i-1}, \mu_t, w_{i+1}^n) \subseteq \mu_t$.

(3) Let $k \neq i$, $1 \leq k \leq n$, $d_1^{i-1}, d_{i+1}^n, w_1^m \in R, w_k \in \mu_t$. Then, there is $h_k \in R$ so that
$g\big(d_1^{i-1}, f(w_1, w_2, ..., w_m), d_{i+1}^n\big)$
$= f\big(g(d_1^{i-1}, w_1, d_{i+1}^n), ..., g(d_1^{i-1}, w_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, w_{k+1}, d_{i+1}^n), ...,$
$g(d_1^{i-1}, w_m, d_{i+1}^n)\big)$,

and
$\mu(h_k) \geq \mu(w_k) \geq t$, so $h_k \in \mu_t$.
Therefore, all conditions in the definition of ideal are met, and consequently μ_t is an ideal.
By using Theorem 2.6, $\eta_t^{\leq}$ is a subnear-ring of R.

(1) Let $w_1^{i-1}, w_{i+1}^m \in R$, $w_i \in \eta_t^{\leq}$ and μ is an ideal of R. Then, for all $1 \leq j \leq m$, there is $a_j \in R$ so that $f(w_1^{i-1}, w_i, w_{i+1}^m) = f(w_1^{j-1}, a_j, w_{j+1}^m)$ and $\eta(a_j) \leq \eta(w_i) = t$. Hence $a_j \in \eta_t$.

(2) Let $w_1, w_2, ..., w_n \in R$ and $w \in \eta_t^{\leq}$. Then $\eta\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \leq \eta(w)$, and thus $\eta\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) \leq t$, which implies that $g(w_1^{i-1}, w, w_{i+1}^n) \in \mu_t^{\leq}$ and so $g(w_1^{i-1}, \eta_t^{\leq}, w_{i+1}^n) \subseteq \eta_t^{\leq}$.

(3) Let $k \neq i$, $1 \leq k \leq n$, $d_1^{i-1}, d_{i+1}^n, w_1^m \in R, w_k \in \eta_t^{\leq}$. Then there is $h_k \in R$ so that
$g\big(d_1^{i-1}, f(w_1, w_2, ..., w_m), d_{i+1}^n\big)$
$= f\big(g(d_1^{i-1}, w_1, d_{i+1}^n), ..., g(d_1^{i-1}, w_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, w_{k+1}, d_{i+1}^n), ...,$
$g(d_1^{i-1}, w_m, d_{i+1}^n)\big)$,
and
$\eta(h_k) \leq \eta(w_k) \leq t$, and so $h_k \in \eta_t^{\leq}$.
Therefore, all conditions in the definition of ideal are met, and consequently $\eta_t^{\leq}$ is an ideal. □

Formally, if $\{\mu_i | \ i \in \Delta\}$ is a family of fuzzy sets in an (m, n)-near ring R, then the union $\bigvee_{i\in\Delta} \mu_i$ of $\{\mu_i \mid i \in \Delta\}$ is defined by

$$(\bigvee_{i\in\Delta} \mu_i)(w) = \sup\{\mu_i(w) \mid i \in \Delta\},$$

for all $w \in R$.

Definition 2.20. ([2]). Assume that $A = \{\langle w, \mu(w), \eta(w)\rangle \mid w \in X\}$ is an intuitionistic fuzzy set. In this case, we define characteristic function of A by

$$\Omega_A(\langle w, a, b\rangle) = \begin{cases} 1, & \text{if } \mu(w) = a \text{ and } \eta(w) = b, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.21. ([3]). Assume that $A = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in X\}$ is an intuitionistic fuzzy set. In this case, we define complement of A by

$$A^c = \{\langle x, \eta^c(x), \mu^c(x)\rangle \mid x \in X\}$$

Definition 2.22. ([21]). Assume that $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. An intuitionistic fuzzy point, written as $w_{(\alpha,\beta)}$ is defined to be an intuitionistic fuzzy subset of R, given by

$$w_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta), & \text{if } w = y, \\ (0, 1), & \text{if } w \neq y. \end{cases}$$

An intuitionistic fuzzy point $x_{(\alpha,\beta)}$ is said to belong to an intuitionistic fuzzy set $\langle\mu,\eta\rangle$ denoted by $w_{(\alpha,\beta)}$ if $\mu(w)\geq\alpha$ and $\eta(w)\leq\beta$ and for $w,y\in R$ we have

$$f(w_{1_{(t_1,s_1)}},...,w_{m_{(t_m,s_m)}})=f(w_1,w_2,...,w_m)_{(t_1\wedge...\wedge t_m,s_1\vee...\vee s_m)}),$$

$$g(w_{1_{(t_1,s_1)}},...,w_{n_{(t_n,s_n)}})=g(w_1,w_2,...,w_n)_{(t_1\wedge...\wedge t_n,s_1\vee...\vee s_n)}).$$

Definition 2.23. An intuitionistic fuzzy ideal of an (m,n)-near ring (R,f,g), not necessarily non-constant, is called intuitionistic fuzzy prime ideal, if for any intuitionistic fuzzy ideals $W_1,W_2,...,W_n$ of R ($W_i=\{\langle x,\mu_i(x),\eta_i(x)\rangle\mid x\in R\}$, $i\in\{1,2,...,n\}$) the condition $g(W_1,W_2,...,W_n)\subseteq P$ implies that either $W_1\subseteq P$ or $W_2\subseteq P$ or ... or $W_n\subseteq P$.

Definition 2.24. ([6]). Assume that W is intuitionistic fuzzy set of a universe set X. Then (α,β)-cut of A is a crisp subset $C_{\alpha,\beta}(W)$ of the intuitionistic fuzzy set W is given by

$$C_{\alpha,\beta}(W)=\{w\mid w\in W,\mu(w)\geq\alpha\ and\ \eta(w)\leq\beta\}\text{ for }\alpha,\beta\in[0,1]\text{ with }\alpha+\beta\leq 1.$$

Theorem 2.25. *If I is an i-ideal of an (m,n)-near ring (R,f,g), then for any $t,s\in(0,1)$ there exists an intuitionistic fuzzy i-ideal $x_{(t,s)}$ of R such that $C_{(t,s)}=I$.*

Proof. Let $\mu:R\longrightarrow[0,1]$ and $\eta^c:R\longrightarrow[0,1]$ be fuzzy sets defined by:

$$x_{(t,s)}(y)=\begin{cases}(t,s), & \text{if } y\in I,\\ (0,1), & \text{if } y\notin I,\end{cases}$$

where t,s are fixed numbers in $(0,1)$. Then clearly $C_{(t,s)}=I$.

(1) Let $w_1,w_2,...,w_m\in R$. If there is $1\leq i\leq m$ such that $w_i\notin I$, then

$$\min\{\mu(w_1),\mu(w_2),...,\mu(w_m)\}=0,$$
$$\mu\big(f(w_1,w_2,...,w_m)\big)\geq\min\{\mu(w_1),\mu(w_2),...,\mu(w_m)\}.$$

Otherwise, $w_1,w_2,...,w_m\in I$ and I is an ideal of R. Then $f(w_1,w_2,...,w_m)\in I$, and hence

$$\mu\big(f(w_1,w_2,...,w_m)\big)=t,$$
$$\mu\big(f(w_1,w_2,...,w_m)\big)\geq\min\{\mu(w_1),\mu(w_2),...,\mu(w_m)\}.$$

(2) Let $d_1^n,b\in R$. Since R is an (m,n)-near ring, it follows that for all $1\leq i\leq n$ there is $w_i\in R$ so that $f(d_1^{i-1},w_i,d_{i+1}^m)=b$. If there is $1\leq j\leq m$ so that $d_j\notin I$ or $b\notin I$, then

$$\min\{\mu(d_1),\mu(d_2),...,\mu(d_{i-1}),\mu(d_{i+1}),...,\mu(d_n),\mu(b)\}=0,$$
$$\mu(w_i)\geq\min\{\mu(d_1),\mu(d_2),...,\mu(d_{i-1}),\mu(d_{i+1}),...,\mu(d_n),\mu(b)\}.$$

Otherwise, $d_1^m,b\in I$ and I is an ideal of R, then $w_i\in I$ and hence $\mu(w_i)=t$,

$$\mu(w_i) \geq \min\{\mu(d_1), \mu(d_2), ..., \mu(d_{i-1}), \mu(d_{i+1}), ..., \mu(d_n), \mu(b)\}.$$

(3) Let $w_1^{i-1}, w_{i+1}^m \in R$, $w \in R$. If $w \in I$, because I is an ideal of R, then for all $1 \leq j \leq m$ there is $a \in I$ so that

$$f(w_1^{i-1}, w, w_{i+1}^m) = f(w_1^{i-1}, a, w_{i+1}^m),$$

and so $t = \mu(a) \geq \mu(w)$. If $w \notin I$, $b = f(w_1^{i-1}, w, w_{i+1}^m)$ and $w_1 = y_1$, $w_2 = y_2$, ..., $w_{i-1} = y_{i-1}, w_{i+1} = y_i, ..., w_m = y_{m-1}$, then by the definition of (m, n)-near ring for $y_1^{m-1}, b \in R$ there is $a \in R$ so that $b = f(y_1^{j-1}, a, y_j^{m-1})$. Hence, we obtain

$$f(w_1^{i-1}, w, w_{i+1}^m) = f(y_1^{j-1}, a, y_j^{m-1}),\ \mu(a) \geq 0 = \mu(w).$$

(4) Let $w_1, w_2, ..., w_n \in R$. If there is $1 \leq i \leq n$ so that $w_i \notin I$, then

$$\min\{\mu(w_1), \mu(w_2), ..., \mu(w_n)\} = 0,$$
$$\mu\big(g(w_1, w_2, ..., w_n)\big) \geq \min\{\mu(w_1), \mu(w_2), ..., \mu(w_n)\}.$$

Otherwise, $w_1, w_2, ..., w_n \in I$ and I is an ideal of R so $g(w_1, w_2, ..., w_n) \in I$. Hence, we have

$$\mu\big(g(w_1, w_2, ..., w_n)\big) = t,$$
$$\mu\big(g(w_1, w_2, ..., w_n)\big) \geq \min\{\mu(w_1), \mu(w_2), ..., \mu(w_n)\}.$$

(5) Let $w_1^n, w \in R$ and $1 \leq i \leq n$. If $w \in I$, then $g(w_1^{i-1}, w, w_{i+1}^n) \in I$ and so $\mu(w) = \mu\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) = t$. If $w \notin I$, then $0 = \mu(w) \leq \mu\big(g(w_1^{i-1}, w, w_{i+1}^n)\big)$.

(6) Let $k \neq i$, $1 \leq k \leq n$, $d_1^{i-1}, d_{i+1}^n, w_1^m \in R$ and $w_k \in I$. Then, there is $h_k \in I$ so that

$g\big(d_1^{i-1}, f(w_1, w_2, ..., w_m), d_{i+1}^n\big) = f\big(g(d_1^{i-1}, w_1, d_{i+1}^n), ...,$
$g(d_1^{i-1}, w_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, w_{k+1}, d_{i+1}^n), ..., g(d_1^{i-1}, w_m, d_{i+1}^n)\big).$

So $\mu(h_k) \geq \mu(w_k)$.

(1) Let $w_1, w_2, ..., w_m \in R$. If there is $1 \leq i \leq m$ that $w_i \notin I$, then

$$\max\{\eta(w_1), \eta(w_2), ..., \eta(w_m)\} = 1,$$
$$\eta\big(f(w_1, w_2, ..., w_m)\big) \leq \max\{\eta(w_1), \eta(w_2), ..., \eta(w_m)\}.$$

Otherwise, $w_1, w_2, ..., w_m \in I$ and I is an ideal of R, then $f(w_1, w_2, ..., w_m) \in I$ and hence

$$\eta\big(f(w_1, w_2, ..., w_m)\big) = t,$$
$$\eta\big(f(w_1, w_2, ..., w_m)\big) \leq \max\{\eta(w_1), \eta(w_2), ..., \eta(w_m)\}.$$

(2) Let $d_1^n, b \in R$. Since R is an (m, n)-near ring, it follows that for all $1 \leq i \leq n$ there is $w_i \in R$ so that $f(d_1^{i-1}, w_i, d_{i+1}^m) = b$. If there is $1 \leq j \leq m$ so that $d_j \notin I$ or $b \notin I$, then

$$\max\{\eta(d_1), \eta(d_2), ..., \eta(d_{i-1}), \eta(d_{i+1}), ..., \eta(d_n), \eta(b)\} = 1,$$
$$\eta(w_i) \leq \max\{\eta(d_1), \eta(d_2), ..., \eta(d_{i-1}), \eta(d_{i+1}), ..., \eta(d_n), \eta(b)\}.$$

Otherwise, $d_1^m, b \in I$ and I is an ideal of R and so $w_i \in I$. Thus $\eta(w_i) = t$,

$$\eta(w_i) \leq \max\{\eta(d_1), \eta(d_2), ..., \eta(d_{i-1}), \eta(d_{i+1}), ..., \eta(d_n), \eta(b)\}.$$

(3) Let $w_1^{i-1}, w_{i+1}^m \in R$, $w \in R$. If $w \in I$, since I is an ideal of R, it follows that for all $1 \leq j \leq m$ there is $a \in I$ so that

$$f(w_1^{i-1}, w, w_{i+1}^m) = f(w_1^{i-1}, a, w_{i+1}^m),$$

and so $t = \eta(a) \leq \eta(w)$. If $w \notin I$, $b = f(w_1^{i-1}, w, w_{i+1}^m)$ and $w_1 = y_1$, $w_2 = y_2$, ..., $w_{i-1} = y_{i-1}, w_{i+1} = y_i, ..., w_m = y_{m-1}$, then by the definition of (m,n)-near ring, for $y_1^{m-1}, b \in R$ there is $a \in R$ so that $b = f(y_1^{j-1}, a, y_j^{m-1})$. Hence, we obtain

$$f(w_1^{i-1}, w, w_{i+1}^m) = f(y_1^{j-1}, a, y_j^{m-1}),\ \eta(a) \leq 1 = \eta(w).$$

(4) Let $w_1, w_2, ..., w_n \in R$. If there is $1 \leq i \leq n$ such that $w_i \notin I$, then

$$\max\{\eta(w_1), \eta(w_2), ..., \eta(w_n)\} = 1,$$
$$\eta\big(g(w_1, w_2, ..., w_n)\big) \leq \max\{\eta(w_1), \eta(w_2), ..., \eta(w_n)\}.$$

Otherwise, $w_1, w_2, ..., w_n \in I$ and I is an ideal of R and so $g(w_1, w_2, ..., w_n) \in I$. This implies that

$$\eta\big(g(w_1, w_2, ..., w_n)\big) = t,$$
$$\eta\big(g(w_1, w_2, ..., w_n)\big) \geq \max\{\eta(w_1), \eta(w_2), ..., \eta(w_n)\}.$$

(5) Let $w_1^n, w \in R$ and $1 \leq i \leq n$. If $x \in I$, then $g(w_1^{i-1}, w, w_{i+1}^n) \in I$ and so $\eta(w) = \eta\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) = t$. If $w \notin I$, then $1 = \eta(w) \geq \eta\big(g(w_1^{i-1}, w, w_{i+1}^n)\big)$.

(6) Let $k \neq i$, $1 \leq k \leq n$, $d_1^{i-1}, d_{i+1}^n, w_1^m \in R$ and $w_k \in I$. Then, there is $h_k \in I$ so that

$$g\big(d_1^{i-1}, f(w_1, w_2, ..., w_m), d_{i+1}^n\big)$$
$$= f\big(g(d_1^{i-1}, w_1, d_{i+1}^n), ..., g(d_1^{i-1}, w_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, w_{k+1}, d_{i+1}^n), ...,$$
$$g(d_1^{i-1}, w_m, d_{i+1}^n)\big).$$

Therefore, we get $\eta(h_k) \leq \eta(w_k)$. □

Lemma 2.26. *If I is a subset of an (m,n)-near ring (R, f, g), then I is an ideal of R if and only if the intuitionistic characteristic function $\chi_I = \langle \mu_{\chi_I}, \eta_{\chi_I} \rangle$ of I is an intuitionistic fuzzy ideal of R.*

Proof. Assume that the intuitionistic characteristic function $\chi_I = \langle \mu_{\chi_I}, \eta_{\chi_I} \rangle$ of I is an intuitionistic fuzzy ideal of R.

(1) Let $w_1, w_2, ..., w_m \in I$. Then $\mu_{\chi_I}(w_1) = \mu_{\chi_I}(w_2) = ... = \mu_{\chi_I}(w_m) = 1$, and thus $\mu_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) \geq \min\{\mu_{\chi_I}(w_1), \mu_{\chi_I}(w_2), ..., \mu_{\chi_I}(w_m)\} = 1$, and so $f(w_1, w_2, ..., w_m) \in I$.

(2) Let $d_1^m, b \in I$. Then there is $w_i \in R$, $1 \leq i \leq m$, so that $f(d_1^{i-1}, w_i, d_{i+1}^m) = b$ and $\mu_{\chi_I}(w_i) \geq \min\{\mu_{\chi_I}(d_1), ..., \mu_{\chi_I}(d_{i-1}), \mu_{\chi_I}(d_{i+1}), ..., \mu_{\chi_I}(d_m), \mu_{\chi_I}(b)\} = 1$, and so $w_i \in I$.

(3) Since μ_{χ_I} is a fuzzy ideal of R so for all $w_1^{i-1}, w_{i+1}^m \in R$, $w \in I$ and $1 \leq j \leq m$, there is $a \in R$ so that $f(w_1^{j-1}, a, w_{j+1}^m) = f(w_1^{i-1}, w, w_{i+1}^m)$ and

$\mu_{\chi_I}(a) \geq \mu_{\chi_I}(w)$, and hence $a \in I$. Thus, for all $w_1^{i-1}, w_{i+1}^m \in R$ and $w \in I$ there is $a \in I$ so that

$$f(w_1^{j-1}, a, w_{j+1}^m) = f(w_1^{i-1}, w, w_{i+1}^m) \text{ and } \mu_{\chi_I}(a) \geq \mu_{\chi_I}(w).$$

(4) Let $w_1, w_2, ..., w_n \in I$. Then $\mu_{\chi_I}(w_1) = \mu_{\chi_I}(w_2) = ... = \mu_{\chi_I}(w_m) = 1$, and thus

$$\mu_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) \geq \min\{\mu_{\chi_I}(w_1), \mu_{\chi_I}(w_2), ..., \mu_{\chi_I}(w_n)\} = 1.$$

Hence, $g(w_1, w_2, ..., w_n) \in I$.

(5) For every $d_1^n, w \in I$ and $1 \leq i \leq n$, $1 = \mu_{\chi_I}(w) \leq \mu_{\chi_I}\big(g(d_1^{i-1}, w, d_{i+1}^n)\big)$, and so

$$g(d_1^{i-1}, w, d_{i+1}^n) \in I.$$

(6) For all $k \neq i$, $1 \leq k \leq m$, $d_1^{i-1}, d_{i+1}^n, w_1^m \in R$ and $w_k \in I$ there is $h_k \in R$ so that
$g\big(d_1^{i-1}, f(w_1, w_2, ..., w_m), d_{i+1}^n\big)$
$= f\big(g(d_1^{i-1}, w_1, d_{i+1}^n), ..., g(d_1^{i-1}, w_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, w_{k+1}, d_{i+1}^n), ...,$
$g(d_1^{i-1}, w_m, d_{i+1}^n)\big)$ and $\mu_{\chi_I}(h_k) \geq \mu_{\chi_I}(w_k)$, thus $h_k \in I$.

(1) Let $w_1, w_2, ..., w_m \in I$. Then $\eta_{\chi_I}(w_1) = \eta_{\chi_I}(w_2) = ... = \eta_{\chi_I}(w_m) = 0$, and so $\eta_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) \leq \max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_m)\} = 0$ so $f(w_1, w_2, ..., w_m) \in I$.

(2) Let $d_1^m, b \in I$. Then there is $x_i \in R$, $1 \leq i \leq m$, so that $f(d_1^{i-1}, x_i, d_{i+1}^m) = b$ and $\eta_{\chi_I}(x_i) \leq \max\{\eta_{\chi_I}(d_1), \eta_{\chi_I}(d_2), ..., \eta_{\chi_I}(d_{i-1}), \eta_{\chi_I}(d_{i+1}), ..., \eta_{\chi_I}(d_m), \eta_{\chi_I}(b)\} = 0$, and so $x_i \in I$.

(3) For all $w_1^{i-1}, w_{i+1}^m \in R$, $w \in I$ and $1 \leq j \leq m$, there is $a \in R$ so that $f(w_1^{j-1}, a, w_{j+1}^m) = f(w_1^{i-1}, w, w_{i+1}^m)$ and $\eta_{\chi_I}(a) \leq \eta_{\chi_I}(w)$, hence $a \in I$.
Thus, for all $w_1^{i-1}, w_{i+1}^m \in R$ and $x \in I$ there is $a \in I$ so that

$$f(w_1^{j-1}, a, w_{j+1}^m) = f(w_1^{i-1}, w, w_{i+1}^m) \text{ and } \eta_{\chi_I}(a) \leq \eta_{\chi_I}(w).$$

(4) Let $w_1, w_2, ..., w_n \in I$. Then $\eta_{\chi_I}(w_1) = \eta_{\chi_I}(w_2) = ... = \eta_{\chi_I}(w_m) = 0$, thus

$$\eta_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) \leq \max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_n)\} = 0,$$

and so $g(w_1, w_2, ..., w_n) \in I$.

(5) For every $d_1^n, x \in I$ and $1 \leq i \leq n$, $0 = \eta_{\chi_I}(x) \geq \eta_{\chi_I}\big(g(d_1^{i-1}, x, d_{i+1}^n)\big)$, we have

$$g(d_1^{i-1}, x, d_{i+1}^n) \in I.$$

(6) For all $k \neq i$, $1 \leq k \leq m$, $d_1^{i-1}, d_{i+1}^n, w_1^m \in R$ and $w_k \in I$ there is $h_k \in R$ so that
$g\big(d_1^{i-1}, f(w_1, w_2, ..., w_m), d_{i+1}^n\big)$
$= f\big(g(d_1^{i-1}, w_1, d_{i+1}^n), ..., g(d_1^{i-1}, w_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, w_{k+1}, d_{i+1}^n), ...,$

$g(d_1^{i-1}, w_m, d_{i+1}^n))$
and $\eta_{\chi_I}(h_k) \leq \eta_{\chi_I}(w_k)$. Thus $h_k \in I$.

Conversely, assume that I is an ideal of R.

(1) Let $w_1, w_2, ..., w_m \in R$. If there is $1 \leq i \leq m$ that $w_i \notin I$, then

$$\min\{\mu_{\chi_I}(w_1), \mu_{\chi_I}(w_2), ..., \mu_{\chi_I}(w_m)\} = 0,$$
$$\mu_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) \geq \min\{\mu_{\chi_I}(w_1), \mu_{\chi_I}(w_2), ..., \mu_{\chi_I}(w_m)\}.$$

Otherwise, $w_1, w_2, ..., w_m \in I$ and I is an ideal of R and so $f(w_1, w_2, ..., w_m) \in I$. Hence

$$\mu_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) = 1,$$
$$\mu_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) \geq \min\{\mu_{\chi_I}(w_1), \mu_{\chi_I}(w_2), ..., \mu_{\chi_I}(w_m)\}.$$

(2) Let $d_1^m, b \in R$. Since R is an (m,n)-near ring, it follows that for all $1 \leq i \leq m$ there is $w_i \in R$, $f(d_1^{i-1}, w_i, d_{i+1}^m) = b$. If there is $1 \leq i \leq m$ such that $d_i \notin I$ or $b \notin I$, then

$$\min\{\mu_{\chi_I}(d_1), \mu_{\chi_I}(d_2), ..., \mu_{\chi_I}(d_{i-1}), \mu_{\chi_I}(d_{i+1}), ..., \mu_{\chi_I}(d_m), \mu_{\chi_I}(b)\} = 0,$$
$$\mu_{\chi_I}(w_i) \geq \min\{\mu_{\chi_I}(d_1), \mu_{\chi_I}(d_2), ..., \mu_{\chi_I}(d_{i-1}), \mu_{\chi_I}(d_{i+1}), ..., \mu_{\chi_I}(d_m), \mu_{\chi_I}(b)\}.$$

Otherwise, $d_1^m, b \in I$ and I is an ideal of R so $w_i \in I$ hence $\mu_{\chi_I}(w_i) = 1$ and

$$\mu_{\chi_I}(w_i) \geq \min\{\mu_{\chi_I}(d_1), \mu_{\chi_I}(d_2), ..., \mu_{\chi_I}(d_{i-1}), \mu_{\chi_I}(d_{i+1}), ..., \mu_{\chi_I}(d_n), \mu_{\chi_I}(b)\}.$$

(3) Let $w_1^{i-1}, w_{i+1}^m, w \in R$. If $w \in I$, because I is an ideal of R, then for all $1 \leq j \leq m$ there is $a \in I$ so that $f(w_1^{i-1}, w, w_{i+1}^m) = f(w_1^{i-1}, a, w_{i+1}^m)$, $a \in I$ and so $1 = \mu_{\chi_I}(a) \geq \mu_{\chi_I}(w)$.

If $x \notin I$, $b = f(w_1^{i-1}, w, w_{i+1}^m)$ and $w_1 = y_1$, $w_2 = y_2$, ..., $w_{i-1} = y_{i-1}$, $w_{i+1} = y_i$, ..., $w_m = y_{m-1}$, then by the definition of (m,n)-near ring, for $y_1^{m-1}, b \in R$ there is $a \in R$ so that $b = f(y_1^{j-1}, a, y_j^{m-1})$. Hence, we have

$$f(w_1^{i-1}, w, w_{i+1}^m) = f(y_1^{j-1}, a, y_j^{m-1}),\ \mu_{\chi_I}(a) \geq 0 = \mu_{\chi_I}(w).$$

(4) Let $w_1, w_2, ..., w_n \in R$. If there is $1 \leq i \leq n$ such that $w_i \notin I$, then

$$\min\{\mu_{\chi_I}(w_1), \mu_{\chi_I}(w_2), ..., \mu_{\chi_I}(w_n)\} = 0 \text{ and}$$
$$\mu_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) \geq \min\{\mu_{\chi_I}(w_1), \mu_{\chi_I}(w_2), ..., \mu_{\chi_I}(w_n)\}.$$

Otherwise, $w_1, w_2, ..., w_n \in I$ and I is an ideal of R so $g(w_1, w_2, ..., w_n) \in I$. Hence, we obtain

$$\mu_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) = 1,$$
$$\mu_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) \geq \min\{\mu_{\chi_I}(w_1), \mu_{\chi_I}(w_2), ..., \mu_{\chi_I}(w_n)\}.$$

(5) Let $w_1^n, w \in R$ and $1 \leq i \leq n$. If $w \in I$, then $g(w_1^{i-1}, w, w_{i+1}^n) \in I$ and so $\mu_{\chi_I}(w) = \mu_{\chi_I}\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) = 1$. If $w \notin I$, then $0 = \mu_{\chi_I}(w) \leq \mu_{\chi_I}\big(g(w_1^{i-1}, w, w_{i+1}^n)\big)$.

(6) Let $k \neq i$, $1 \leq k \leq n$, $d_1^{i-1}, d_{i+1}^n, w_1^m \in R$ and $w_k \in I$. Then there is $h_k \in I$ so that

$g\big(d_1^{i-1}, f(w_1, w_2, ..., w_m), d_{i+1}^n\big) =$
$f\big(g(d_1^{i-1}, w_1, d_{i+1}^n), ..., g(d_1^{i-1}, w_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, w_{k+1}, d_{i+1}^n), ...,$
$g(d_1^{i-1}, w_m, d_{i+1}^n)\big)$,
and $\mu_{\chi_I}(h_k) \geq \mu_{\chi_I}(w_k)$.

(1) Let $w_1, w_2, ..., w_m \in R$. If there is $1 \leq i \leq m$ that $w_i \notin I$, then

$$\max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_m)\} = 1,$$
$$\eta_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) \leq \max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_m)\}.$$

Otherwise, $w_1, w_2, ..., w_m \in I$ and I is an ideal of R so $f(w_1, w_2, ..., w_m) \in I$. This implies that

$$\eta_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) = 0,$$
$$\eta_{\chi_I}\big(f(w_1, w_2, ..., w_m)\big) \leq \max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_m)\}.$$

(2) Let $d_1^m, b \in R$. Since R is an (m, n)-near ring, it follows that for all $1 \leq i \leq m$ there is $x_i \in R$, $f(d_1^{i-1}, x_i, d_{i+1}^m) = b$. If there is $1 \leq i \leq m$ so that $d_i \notin I$ or $b \notin I$, then

$$\max\{\eta_{\chi_I}(d_1), \eta_{\chi_I}(d_2), ..., \eta_{\chi_I}(d_{i-1}), \eta_{\chi_I}(d_{i+1}), ..., \eta_{\chi_I}(d_m), \eta_{\chi_I}(b)\} = 1,$$
$$\eta_{\chi_I}(x_i) \leq \max\{\eta_{\chi_I}(d_1), \eta_{\chi_I}(d_2), ..., \eta_{\chi_I}(d_{i-1}), \eta_{\chi_I}(d_{i+1}), ..., \eta_{\chi_I}(d_m), \eta_{\chi_I}(b)\}.$$

Otherwise, $d_1^m, b \in I$ and I is an ideal of R so $w_i \in I$. Hence $\eta(w_i) = 1$ and

$$\eta(w_i) \leq \max\{\eta(d_1), \eta(d_2), ..., \eta(d_{i-1}), \eta(d_{i+1}), ..., \eta(d_m), \eta(b)\}.$$

(3) Let $w_1^{i-1}, w_{i+1}^m, w \in R$. If $w \in I$, because I is an ideal of R, then for all $1 \leq j \leq m$ there is $a \in I$ so that $f(w_1^{i-1}, w, w_{i+1}^m) = f(w_1^{i-1}, a, w_{i+1}^m)$, $a \in I$ so $0 = \eta_{\chi_I}(a) \leq \eta_{\chi_I}(w)$.

If $w \notin I$, $b = f(w_1^{i-1}, w, w_{i+1}^m)$ and $w_1 = y_1$, $w_2 = y_2$, ..., $w_{i-1} = y_{i-1}$, $w_{i+1} = y_i$, ..., $w_m = y_{m-1}$, then by the definition of (m, n)-near ring, for $y_1^{m-1}, b \in R$ there is $a \in R$ so that $b = f(y_1^{j-1}, a, y_j^{m-1})$. Hence, we get

$$f(w_1^{i-1}, w, w_{i+1}^m) = f(y_1^{j-1}, a, y_j^{m-1}),\ \eta_{\chi_I}(a) \leq 1 = \eta_{\chi_I}(w).$$

(4) Let $w_1, w_2, ..., w_n \in R$. If there is $1 \leq i \leq n$ such that $w_i \notin I$, then

$$\max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_n)\} = 1 \text{ and}$$
$$\eta_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) \leq \max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_n)\}.$$

Otherwise, $w_1, w_2, ..., w_n \in I$ and I is an ideal of R and so $g(w_1, w_2, ..., w_n) \in I$. This yields that

$$\eta_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) = 0,$$
$$\eta_{\chi_I}\big(g(w_1, w_2, ..., w_n)\big) \leq \max\{\eta_{\chi_I}(w_1), \eta_{\chi_I}(w_2), ..., \eta_{\chi_I}(w_n)\}.$$

(5) Let $w_1^n, w \in R$ and $1 \leq i \leq n$. If $x \in I$, then $g(w_1^{i-1}, w, w_{i+1}^n) \in I$ and so $\eta(w) = \eta\big(g(w_1^{i-1}, w, w_{i+1}^n)\big) = 0$. If $w \notin I$, then $1 = \eta_{\chi_I}(w) \geq \eta_{\chi_I}\big(g(w_1^{i-1}, w, w_{i+1}^n)\big)$.

(6) Let $k \neq i$, $1 \leq k \leq n$, $d_1^{i-1}, d_{i+1}^n, w_1^m \in R$ and $w_k \in I$. Then there is $h_k \in I$ so that
$g\big(d_1^{i-1}, f(w_1, w_2, ..., w_m), d_{i+1}^n\big) =$
$f\big(g(d_1^{i-1}, w_1, d_{i+1}^n), ..., g(d_1^{i-1}, w_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, w_{k+1}, d_{i+1}^n), ...,$
$g(d_1^{i-1}, w_m, d_{i+1}^n)\big)$,
and $\eta_{\chi_I}(h_k) \leq \eta_{\chi_I}(w_k)$. □

3 Direct product of (m, n)-near rings

If (R_1, f, g) and (R_2, f, g) are (m, n)-near rings, then direct product $R_1 \times R_2$ of R_1 and R_2 is an (m, n)-near ring with F and G defined as

$$F\big((s_1, k_1), (s_2, k_2), ..., (s_m, k_m)\big) = \big(f(s_1, s_2, ..., s_m), f(k_1, k_2, ..., k_m)\big),$$
$$G\big((c_1, d_1), (c_2, d_2), ..., (c_n, d_n)\big) = \big(g(c_1, c_2, ..., c_n), g(d_1, d_2, ..., d_n)\big),$$

respectively, for every $(s_i, k_i), (c_j, d_j) \in R_1 \times R_2$ so that $1 \leq i \leq m$ and $1 \leq j \leq n$. Likewise the direct product $R = \times_{i \in \omega} R_i$ of a family of (m, n)-near rings $\{R_i \mid i \in \omega\}$ has the structure of an (m, n)-near ring with the operations of F and G defined as

$$\begin{aligned} F(w_1, w_2, ..., w_m) &= F\big((w_{1_1}, w_{1_2}, ...), (w_{2_1}, w_{2_2}, ...), ..., (w_{m_1}, w_{m_2}, ...)\big) \\ &= \big(f(w_{1_1}, w_{2_1}, ..., w_{m_1}), f(w_{1_2}, w_{2_2}, ..., w_{m_2}), ...\big) \\ G(l_1, l_2, ..., l_n) &= G\big((l_{1_1}, l_{1_2}, ...), (l_{2_1}, l_{2_2}, ...), ..., (l_{n_1}, l_{n_2}, ...)\big) \\ &= \big(g(l_{1_1}, l_{2_1}, ..., l_{n_1}), g(l_{1_2}, l_{2_2}, ..., l_{n_2}), ...\big), \end{aligned}$$

for all $w_1^m, l_1^n \in R$.

Lemma 3.1. *If A and B are two subnear-rings of (m, n)-near rings R_1 and R_2 respectively, then $A \times B$ is also a subnear-ring of $R_1 \times R_2$ under the same operations defined as in $R_1 \times R_2$.*

Proof. It is straightforward. □

Let $A = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in X\}$ and $B = \{\langle x, \mu'(x), \eta'(x)\rangle \mid x \in X\}$ be two intuitionistic fuzzy subsets of (m, n)-near rings R_1 and R_2, respectively. The direct product of A and B, is denoted by $A \times B$ and is defined as follows:

$$A \times B = \{\big((w, l), \mu''\big((w, l)\big), \eta''\big((w, l)\big)\big) \mid \textit{ for all } w \in R_1 \textit{ and } l \in R_2\},$$

where $\mu''\big((w, l)\big) = \min\{\mu(w), \mu'(l)\}$ and $\eta''\big((w, l)\big) = \max\{\eta(w), \eta'(l)\}$.

Theorem 3.2. *If A and B are two subnear-rings of (m,n)-near rings R_1 and R_2 respectively, then $A\times B$ is a subnear-ring of $R_1\times R_2$ if and only if the intuitionistic characteristic function $\chi_c = \langle \mu_{\chi_c}, \eta_{\chi_c}\rangle$ of $C = A\times B$ is an intuitionistic fuzzy subnear ring of $R_1 \times R_2$.*

Proof. Assume that $C = A\times B$ is a subnear-ring of $R_1\times R_2$ and $a_1^m, b_1^n \in R_1\times R_2$. If $a_1^m, b_1^n \in A\times B = C$, then by the definition of characteristic function $\mu_{\chi_C}(a_i) = 1 = \mu_{\chi_C}(b_j)$ and $\eta_{\chi_C}(a_i) = 0 = \eta_{\chi_C}(b_j)$ for $1 \le i \le m$ and $1 \le j \le n$.

(1) For all $d_1^m \in A\times B$, $f(d_1, d_2, ..., d_m) \in C$ then $\mu_{\chi_C}\big(f(d_1, d_2, ..., d_m)\big) = 1$, and so $\mu_{\chi_C}\big(f(d_1, d_2, ..., d_m)\big) \ge \min\{\mu_{\chi_C}(d_1), \mu_{\chi_C}(d_2), ..., \mu_{\chi_C}(d_m)\}$.

(2) For all $d_1^m, b \in A\times B$ and $1 \le i \le n$ there is $w_i \in A\times B$ so that

$$f(d_1^{i-1}, w_i, d_{i+1}^m) = b \text{ and } \mu_{\chi_C}(d_1) = \mu_{\chi_C}(d_2) = ... = \mu_{\chi_C}(d_{i-1}) = \mu_{\chi_C}(d_{i+1}) = ... = \mu_{\chi_C}(d_m) = \mu_{\chi_C}(b) = 1, \text{ so } 1 = \mu_{\chi_C}(w_i) \ge \min\{\mu_{\chi_C}(d_1), \mu_{\chi_C}(d_2), ..., \mu_{\chi_C}(d_{i-1}), \mu_{\chi_C}(d_{i+1}), ..., \mu_{\chi_C}(d_m), \mu_{\chi_C}(b)\},$$

(3) For all $w_1, w_2, ..., w_n \in A\times B$, $g(w_1, w_2, ..., w_n) \in A\times B$ then

$$1 = \mu\big(g(w_1, w_2, ..., w_n)\big) = \mu(w_1) = \mu(w_2) = ... = \mu(w_n),$$
$$\mu\big(g(w_1, w_2, ..., w_n)\big) \ge \min\{\mu(w_1), \mu(w_2), ..., \mu(w_n)\}.$$

(4) For all $w_1, w_2, ..., w_m \in A\times B$ we have $f(w_1, w_2, ..., w_n) \in A\times B$. Thus, we have

$$0 = \eta\big(f(w_1, w_2, ..., w_m)\big) = \eta(w_1) = \eta(w_2) = ... = \eta(w_m),$$
$$\eta\big(f(w_1, w_2, ..., w_m)\big) \le \max\{\eta(w_1), \eta(w_2), ..., \eta(w_m)\},$$

(5) For all $d_1^m, b \in A\times B$ and $1 \le i \le n$ there is $w_i \in A\times B$ so that

$$f(d_1^{i-1}, w_i, d_{i+1}^m) = b,$$
$$0 = \eta(w_i) = \eta(d_1) = \eta(d_2) = ... = \eta(d_{i-1}) = \eta(d_{i+1}) = ... = \eta(d_m) = \eta(b) \text{ so}$$
$$\eta(w_i) \le \max\{\eta(d_1), \eta(d_2), ..., \eta(d_{i-1}), \eta(d_{i+1}), ..., \eta(d_m), \eta(b)\},$$

(6) For all $w_1, w_2, ..., w_n \in A\times B$ we have $g(w_1, w_2, ..., w_n) \in A\times B$, and so

$$0 = \eta\big(g(w_1, w_2, ..., w_n)\big) = \eta(w_1) = \eta(w_2) = ... = \eta(w_n),$$
$$\eta\big(g(w_1, w_2, ..., w_n)\big) \le \max\{\eta(w_1), \eta(w_2), ..., \eta(w_n)\}.$$

Conversely, assume that the intuitionistic characteristic function $\chi_c = \langle \mu_{\chi_c}, \eta_{\chi_c}\rangle$ of $C = A\times B$ is an intuitionistic fuzzy subnear ring of $R_1 \times R_2$.

(1) For all $d_1^m \in A\times B$,

$$\mu_{\chi_C}\big(f(d_1, d_2, ..., d_m)\big) \ge \min\{\mu_{\chi_C}(d_1), \mu_{\chi_C}(d_2), ..., \mu_{\chi_C}(d_m)\} = 1,$$
$$\mu_{\chi_C}\big(f(d_1, d_2, ..., d_m)\big) = 1,$$

for all $d_1, d_2, ..., d_m \in A\times B$ we have $0 = \eta_{\chi_C}(d_1) = \eta_{\chi_C}(d_2) = ... = \eta_{\chi_C}(d_m)$

$$\eta_{\chi_C}\big(f(d_1, d_2, ..., d_m)\big) \le \max\{\eta_{\chi_C}(d_1), \eta_{\chi_C}(d_2), ..., \eta_{\chi_C}(d_m)\} = 0, \text{ and so}$$
$$\eta_{\chi_C}\big(f(d_1, d_2, ..., d_m)\big) = 0.$$

Thus $f(d_1, d_2, ..., d_m)\big) \in A \times B$.

(2) For all $d_1^m, b \in A \times B$ and $1 \leq i \leq n$ there is $w_i \in R_1 \times R_2$ so that

$$f(d_1^{i-1}, w_i, d_{i+1}^m) = b \text{ and } \mu_{\chi_C}(d_1) = \mu_{\chi_C}(d_2) = ... = \mu_{\chi_C}(d_{i-1}) = \mu_{\chi_C}(d_{i+1}) = \\ ... = \mu_{\chi_C}(d_m) = \mu_{\chi_C}(b) = 1, \text{ so } \mu_{\chi_C}(w_i) \geq \\ \min\{\mu_{\chi_C}(d_1), \mu_{\chi_C}(d_2), ..., \mu_{\chi_C}(d_{i-1}), \mu_{\chi_C}(d_{i+1}), ..., \mu_{\chi_C}(d_m), \mu_{\chi_C}(b)\} = 1,$$

and so $\mu_{\chi_C}(w_i) \geq 1$. This implies that $\mu_{\chi_C}(w_i) = 1$.

$$f(d_1^{i-1}, w_i, d_{i+1}^m) = b, \\ 0 = \eta_{\chi_C}(d_1) = \eta_{\chi_C}(d_2) = ... = \eta_{\chi_C}(d_{i-1}) = \eta_{\chi_C}(d_{i+1}) = ... = \eta_{\chi_C}(d_m) = \eta_{\chi_C}(b) \\ \text{so } \eta_{\chi_C}(w_i) \leq \\ \max\{\eta_{\chi_C}(d_1), \eta_{\chi_C}(d_2), ..., \eta_{\chi_C}(d_{i-1}), \eta_{\chi_C}(d_{i+1}), ..., \eta_{\chi_C}(d_m), \eta_{\chi_C}(b)\} = 0,$$

and so $\eta_{\chi_C}(w_i) = 0$. Consequently, $w_i \in A \times B$.

(3) For all $w_1, w_2, ..., w_n \in A \times B$, we have $1 = \mu_{\chi_C}(w_1) = \mu_{\chi_C}(w_2) = ... = \mu_{\chi_C}(w_n)$,

$$\mu_{\chi_C}\big(g(w_1, w_2, ..., w_n)\big) \geq \min\{\mu_{\chi_C}(w_1), \mu_{\chi_C}(w_2), ..., \mu_{\chi_C}(w_n)\} = 1.$$

So $\mu_{\chi_C}\big(g(w_1, w_2, ..., w_n)\big) = 1$,

$0 = \eta_{\chi_C}(w_1) = \eta_{\chi_C}(w_2) = ... = \eta_{\chi_C}(w_n)$. Thus, we have

$$\eta_{\chi_C}\big(g(w_1, w_2, ..., w_n)\big) \leq \max\{\eta_{\chi_C}(w_1), \eta_{\chi_C}(w_2), ..., \eta_{\chi_C}(w_n)\} = 0.$$

So $\eta_{\chi_C}\big(g(w_1, w_2, ..., w_n)\big) = 0$. Consequently, we get $g(w_1, w_2, ..., w_n) \in A \times B$.

Therefore, all conditions of the definition of an (m, n)-near ring are satisfied, and so $A \times B$ is an (m, n)-near ring. □

Theorem 3.3. *If A and B are two intuitionistic fuzzy subnear-rings of (m, n)-near rings R_1 and R_2 respectively, then $A \times B$ is an intuitionistic fuzzy subnear-ring of $R_1 \times R_2$.*

Proof. Let $A = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in R_1\}$ and $B = \{\langle y, \mu'(y), \eta'(y)\rangle \mid y \in R_2\}$ be intuitionistic fuzzy subnear-rings of (m, n)-near rings R_1 and R_2, respectively. Now $A \times B = \{\langle (x, y), \mu''\big((x, y)\big), \eta''\big((x, y)\big)\rangle \mid x \in R_1,\ y \in R_2\}$, where $\mu''\big((x, y)\big) = \min\{\mu(x), \mu'(y)\}$ and $\eta''\big((x, y)\big) = \max\{\eta(x), \eta'(y)\}$. We have to show that $A \times B$ is an intuitionistic fuzzy subnear-ring of $R_1 \times R_2$.

(1) Let $(d_i, l_i) \in R_1 \times R_2$ for $1 \leq i \leq m$,

$$\begin{aligned} &\mu''\big(F((d_1, l_1), (d_2, l_2), ..., (d_m, l_m))\big) \\ &= \mu''\big((f(d_1, d_2, ..., d_m), f(l_1, l_2, ..., l_m))\big) \\ &= \min\{\mu\big(f(d_1, d_2, ..., d_m)\big), \mu'\big(f(l_1, l_2, ..., l_m)\big)\} \\ &\geq \min\{\mu(d_1), \mu(d_2), ..., \mu(d_m), \mu'(l_1), \mu'(l_2), ..., \mu'(l_m)\} \\ &= \min\{\min\{\mu(d_1), \mu'(l_1)\}, \min\{\mu(d_2), \mu'(l_2)\}, ..., \min\{\mu(d_m), \mu'(l_m)\}\} \\ &= \min\{\mu''\big((d_1, l_1)\big), \mu''\big((d_2, l_2)\big), ..., \mu''\big((d_m, l_m)\big)\}, \end{aligned}$$

so we get

$$\mu''\big(F((d_1,l_1),(d_2,l_2),...,(d_m,l_m))\big) \geq$$
$$\min\{\mu''\big((d_1,l_1)\big),\mu''\big((d_2,l_2)\big),...,\mu''\big((d_m,l_m)\big)\}.$$

(2) For all $d_1^m, c \in A$ and $1 \leq i \leq n$ there is $w_i \in A$ so that

$$f(d_1^{i-1},w_i,d_{i+1}^m) = c,$$
$$\mu(w_i) \geq \min\{\mu(d_1),\mu(d_2),...,\mu(d_{i-1}),\mu(d_{i+1}),...,\mu(d_m),\mu(c)\}.$$

For all $d_1^m, d \in B$ and $1 \leq i \leq n$ there is $y_i \in B$ so that

$$f(d_1^{i-1},y_i,d_{i+1}^m) = d,$$
$$\mu'(y_i) \geq \min\{\mu'(d_1),\mu'(d_2),...,\mu'(d_{i-1}),\mu'(d_{i+1}),...,\mu'(d_m),\mu'(d)\}.$$

Let $(d_i,l_i),(c,d) \in A \times B$ for $1 \leq i \leq m$. Then

$$\begin{aligned}(c,d) &= \big(f(d_1^{i-1},x_i,d_{i+1}^m),f(l_1^{i-1},y_i,l_{i+1}^m)\big)\\ &= F\big((d_1,l_1),...,(d_{i-1},l_{i-1}),(x_i,y_i),(d_{i+1},l_{i+1}),...,(d_m,l_m)\big).\end{aligned}$$

Hence, there is $(x_i,y_i) \in A \times B$ such that

$$(c,d) = F\big((d_1,l_1),...,(d_{i-1},l_{i-1}),(x_i,y_i),(d_{i+1},l_{i+1}),...,(d_m,l_m)\big)$$

$\mu''\big((x_i,y_i)\big) = \min\{\mu(x_i),\mu'(y_i)\} \geq \min\{\min\{\mu(d_1),\mu(d_2),...,\mu(d_{i-1}),$ $\mu(d_{i+1}),...,\mu(d_m),\mu(c)\},\min\{\mu'(l_1),\mu'(l_2),...,\mu'(l_{i-1}),\mu'(l_{i+1}),...,\mu'(l_m),$ $\mu'(d)\}\} = \min\{\mu(d_1),\mu(d_2),...,\mu(d_{i-1}),\mu(d_{i+1}),...,\mu(d_m),\mu(c),\mu'(l_1),$ $\mu'(l_2),...,\mu'(l_{i-1}),\mu'(l_{i+1}),...,\mu'(l_m),\mu'(d)\} = \min\{\min\{\mu(d_1),\mu'(l_1)\},...,$ $\min\{\mu(d_{i-1}),\mu'(l_{i-1})\},\min\{\mu(c),\mu'(d)\},\min\{\mu(d_{i+1}),\mu'(l_{i+1})\},...,$ $\min\{\mu(d_m),\mu'(l_m)\}\} = \min\{\mu''\big((d_1,l_1)\big),...,\mu''\big((d_{i-1},l_{i-1})\big),\mu''\big((c,d)\big),$ $\mu''\big((d_{i+1},l_{i+1})\big),...,\mu''\big((d_m,l_m)\big)\}.$
Therefore, we get

$$\mu''\big((x_i,y_i)\big) \geq$$
$$\min\{\mu''\big((d_1,l_1)\big),...,\mu''\big((d_{i-1},l_{i-1})\big),\mu''\big((c,d)\big),\mu''\big((d_{i+1},l_{i+1})\big),...,\mu''\big((d_m,l_m)\big)\},$$

(3) Let $(d_i,l_i) \in R_1 \times R_2$ for $1 \leq i \leq n$. Then

$$\begin{aligned}&\mu''\big(G((d_1,l_1),(d_2,l_2),...,(d_n,l_n))\big)\\ &= \mu''\big((g(d_1,d_2,...,d_n),g(l_1,l_2,...,l_n))\big)\\ &= \min\{\mu\big(g(d_1,d_2,...,d_n)\big),\mu'(g(l_1,l_2,...,l_n))\}\\ &\geq \min\{\mu(d_1),\mu(d_2),...,\mu(d_n),\mu'(l_1),\mu'(l_2),...,\mu'(l_n)\}\\ &= \min\{\min\{\mu(d_1),\mu'(l_1)\},\min\{\mu(d_2),\mu'(l_2)\},...,\min\{\mu(d_n),\mu'(l_n)\}\}\\ &= \min\{\mu''\big((d_1,l_1)\big),\mu''\big((d_2,l_2)\big),...,\mu''\big((d_n,l_n)\big)\},\end{aligned}$$

and so

$$\mu''\big(G((d_1,l_1),(d_2,l_2),...,(d_n,l_n))\big) \geq$$
$$\min\{\mu''\big((d_1,l_1)\big),\mu''\big((d_2,l_2)\big),...,\mu''\big((d_n,l_n)\big)\},$$

(4) Let $(d_i, l_i) \in R_1 \times R_2$ for $1 \le i \le m$. Then

$$\begin{aligned}
&\eta''\big(F((d_1,l_1),(d_2,l_2),...,(d_m,l_m))\big)\\
&= \eta''\big((f(d_1,d_2,...,d_m), f(l_1,l_2,...,l_m))\big)\\
&= \max\{\eta\big(f(d_1,d_2,...,d_m)\big), \eta'\big(f(l_1,l_2,...,l_m)\big)\}\\
&\le \max\{\eta(d_1),\eta(d_2),...,\eta(d_m),\eta'(l_1),\eta'(l_2),...,\eta'(l_m)\}\\
&= \max\{\max\{\eta(d_1),\eta'(l_1)\},\max\{\eta(d_2),\eta'(l_2)\},...,\max\{\eta(d_m),\eta'(l_m)\}\}\\
&= \max\{\eta''\big((d_1,l_1)\big),\eta''\big((d_2,l_2)\big),...,\eta''\big((d_m,l_m)\big)\},
\end{aligned}$$

and so

$$\eta''\big(F((d_1,l_1),(d_2,l_2),...,(d_m,l_m))\big) \le \max\{\eta''\big((d_1,l_1)\big),\eta''\big((d_2,l_2)\big),...,\eta''\big((d_m,l_m)\big)\},$$

(5) For all $d_1^m, c \in A$ and $1 \le i \le n$ there is $x_i \in A$ so that

$$f(d_1^{i-1}, x_i, d_{i+1}^m) = c,$$
$$\eta(x_i) \le \max\{\eta(d_1),\eta(d_2),...,\eta(d_{i-1}),\eta(d_{i+1}),...,\eta(d_m),\eta(c)\}.$$

For all $l_1^m, d \in B$ and $1 \le i \le n$ there is $y_i \in B$ so that

$$f(l_1^{i-1}, y_i, l_{i+1}^m) = d,$$
$$\eta'(y_i) \le \max\{\eta'(l_1),\eta'(l_2),...,\eta'(l_{i-1}),\eta'(l_{i+1}),...,\eta'(l_m),\eta'(d)\}.$$

Let $(d_i, l_i), (c, d) \in A \times B$ for $1 \le i \le m$

$$\begin{aligned}
(c,d) = \ &\big(f(d_1^{i-1},x_i,d_{i+1}^m), f(l_1^{i-1},y_i,l_{i+1}^m)\big)\\
&= F\big((d_1,l_1),...,(d_{i-1},l_{i-1}),(x_i,y_i),(d_{i+1},l_{i+1}),...,(d_m,l_m)\big).
\end{aligned}$$

Then there is $(x_i, y_i) \in A \times B$ that

$$(c,d) = F\big((d_1,l_1),...,(d_{i-1},l_{i-1}),(x_i,y_i),(d_{i+1},l_{i+1}),...,(d_m,l_m)\big),$$

$\eta''\big((x_i,y_i)\big) = \max\{\eta(x_i),\eta'(y_i)\} \le \max\{\max\{\eta(d_1),\eta(d_2),...,\eta(d_{i-1}),$ $\eta(d_{i+1}),...,\eta(d_m),\eta(c)\},\max\{\eta'(l_1),\eta'(l_2),...,\eta'(l_{i-1}),\eta'(l_{i+1}),...,$ $\eta'(l_m),\eta'(d)\}\} = \max\{\eta(d_1),\eta(d_2),...,\eta(d_{i-1}),\eta(d_{i+1}),...,\eta(d_m),\eta(c),$ $\eta'(l_1),\eta'(l_2),...,\eta'(l_{i-1}),\eta'(l_{i+1}),...,\eta'(l_m),\eta'(d)\} = \max\{\max\{\eta(d_1),$ $\eta'(l_1)\},...,\max\{\eta(d_{i-1}),\eta'(l_{i-1})\},\max\{\eta(c),\eta'(d)\},\max\{\eta(d_{i+1}),$ $\eta'(l_{i+1})\},...,\max\{\eta(d_m),\eta'(l_m)\}\} = \max\{\eta''\big((d_1,l_1)\big),...,\eta''\big((d_{i-1},l_{i-1})\big),$ $\eta''\big((c,d)\big),\eta''\big((d_{i+1},l_{i+1})\big),...,\eta''\big((d_m,l_m)\big)\}$. Therefore, we get

$$\eta''\big((x_i,y_i)\big) \le \max\{\eta''\big((d_1,l_1)\big),...,\eta''\big((d_{i-1},l_{i-1})\big),\eta''\big((c,d)\big),\eta''\big((d_{i+1},l_{i+1})\big),...,\eta''\big((d_m,l_m)\big)\}$$

(6) Let $(d_i, l_i) \in R_1 \times R_2$ for $1 \le i \le n$. Then, we have $\eta''\big(G((d_1,l_1),(d_2,l_2),...,(d_n,l_n))\big) = \eta''\big((g(d_1,d_2,...,d_n), g(l_1,l_2,...,l_n))\big)$ $= \max\{\eta\big(g(d_1,d_2,...,d_n)\big),\eta'\big(g(l_1,l_2,...,l_n)\big)\} \le \max\{\eta(d_1),\eta(d_2),...,\eta(d_n),\eta'(l_1),$ $\eta'(l_2),...,\eta'(l_n)\} = \max\{\max\{\eta(d_1),\eta'(l_1)\},\max\{\eta(d_2),\eta'(l_2)\},...,\max\{\eta(d_n),$ $\eta'(l_n)\}\} = \max\{\eta''\big((d_1,l_1)\big),\eta''\big((d_2,l_2)\big),...,\eta''\big((d_n,l_n)\big)\}$, so

$$\eta''\big(G((d_1,l_1),(d_2,l_2),...,(d_n,l_n))\big) \leq$$
$$\max\{\eta''\big((d_1,l_1)\big),\eta''\big((d_2,l_2)\big),...,\eta''\big((d_n,l_n)\big)\}.$$

□

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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