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Further Results on Generous Roman Domination

Seyed Mahmoud Sheikholeslami⋆, *Mustapha Chellali*
and Mariyeh Kor

Abstract

Let $G = (V(G), E(G))$ be a graph and h be a function defined from $V(G)$ to $\{0, 1, 2, 3\}$. A vertex x with $h(x) = 0$ is said to be undefended with respect to h if it has no neighbor assigned 2 or 3 under h. The function h is called a generous Roman dominating function (GRD-function) if for every vertex with $h(x) = 0$ there exists at least a vertex y with $h(y) \geq 2$ adjacent to x such that the function $\eta : V(G) \to \{0, 1, 2, 3\}$, defined by $\eta(x) = \alpha$, $\eta(y) = h(y) - \alpha$, where $\alpha \in \{1, 2\}$, and $\eta(z) = h(z)$ if $z \in V(G) - \{x, y\}$ has no undefended vertex. The weight of a GRD-function h is the value $\sum_{x \in V(G)} h(x)$, and the minimum weight of a GRD-function on G is the generous Roman domination number (GRD-number) of G. In this paper, we determine the exact value of the GRD-number for the ladder graphs, and we provide an upper bound on it for trees in terms of the order, the number of leaves and the number of stems. Moreover, we show that for every tree on at least three vertices, the GRD-number is bounded below by the domination number plus 2, and we characterize the extremal trees attaining this lower bound.

Keywords: Generous Roman domination, Weak double Roman domination number, Double Roman domination.

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⋆Corresponding author (E-mail: s.m.sheikholeslami@azaruniv.ac.ir)
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1. Introduction

All graphs in this paper are finite, undirected, and simple graphs G with vertex set $V := V(G)$ and edge set $E := E(G)$. The *order* $|V|$ of G is denoted by $p := p(G)$. The *open neighborhood* of a vertex $w \in V$ is the set $N_G(w) := N(w) = \{v \in V \mid wv \in E\}$, and its *closed neighborhood* is the set $N[w] := N(w) \cup \{w\}$. The *degree* $\deg_G(w)$ of w is $|N_G(w)|$. In a tree, a vertex of degree one is referred to as a *leaf*, while its adjacent vertex is known as a *stem*. A vertex adjacent to two or more leaves is called a *strong stem*. If $A \subseteq V$ and η is a function from V into $\mathbb{Z}^{\geq 0}$, then $\eta(A) = \sum_{x\in A} \eta(x)$, and the sum $\eta(V)$ is called the *weight* $\omega(\eta)$ of η. For any positive integer k, set $[k] := \{0, 1, \ldots, k\}$.

Typically, a *complete graph*, *cycle*, *path* and *star* on n vertices are represented by K_n, C_n, P_n and $K_{1,n-1}$, respectively, while $K_{r,s}$ and $S_{r,s}$ represent the *complete bipartite graph* and *double star* of order $r + s$ and $r + s + 2$, respectively.

A set D of vertices of G is considered as a *dominating set* of G if $N[D] = V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

Motivated by the defense strategies of the Roman Empire presented in [1] and [2], the concept of Roman domination was introduced in 2004 by Cockayne et al. [3]. A function f from V into the set [2] is a *Roman dominating function* (RD-function, for short) on G if every vertex $x \in V$ assigned 0 under f is adjacent to at least one vertex y with $f(y) = 2$. Since it was defined, a thorough study of Roman domination has given rise to a variety of concepts. For further information, we direct the reader to the relevant book chapters [4, 5] and surveys [6–8].

In 2003, Henning et al. [9] suggested a less restrictive version of Roman domination, but which still guaranteed the defense of the Roman Empire against a single attack, and which they called weak Roman domination (WRD). Formally, let $h : V \to [2]$ be a function on a graph G. A vertex x with $h(x) = 0$ is said to be *undefended* with respect to h if it is not adjacent to a vertex y with $h(y) > 0$. The function h is a *weak Roman dominating function* (WRD-function) if each vertex x with $h(x) = 0$ is adjacent to a vertex y with $h(y) > 0$ such that the function η defined by $\eta(x) = 1$, $\eta(y) = h(y) - 1$, and $\eta(z) = h(z)$ for all $z \in V \setminus \{x, y\}$, has no undefended vertex. The *weak Roman domination number* (WRD-number, for short) $\gamma_r(G)$ equals the minimum weight of a WRD-function in G. We direct the readers to [10–13] for more information on weak Roman domination.

In 2016, Beeler et al. [14] introduced a more robust concept of Roman domination, referred to as double Roman domination. A *double Roman dominating function* (DRD-function, for short) on a graph G is a function h from V into [3] that meets the following conditions: (i) If $h(x) = 0$, then x has at least two neighbors assigned 2 under h or one neighbor w with $h(w) = 3$; (ii) If $h(x) = 1$, then x must have at least one neighbor w with $h(w) \geq 2$.

Recently, two new variants of double Roman domination have been introduced, one called generous Roman domination due to Benatallah, Blidia and Ouldrabah [15], and the other is called weak double Roman domination due to Soltani et al.

[16]. Let h be a function defined from V to [3]. Benatallah et al. [15] defined a vertex x as being *undefended* with respect to h if $h(x) = 0$ and x has no neighbor y with $h(y) \geq 2$, while in [16], Soltani et al. defined a vertex x to be *doubly undefended* with respect to h if $h(N[x]) \leq 1$.

So, according to [15], the function h is a *generous Roman dominating function* (GRD-function) if for each vertex with $h(x) = 0$ there exists at least a vertex y with $h(y) \geq 2$ adjacent to x such that the function $\eta : V \to [3]$ defined by $\eta(x) = \alpha$, $\eta(y) = h(y) - \alpha$ where $\alpha \in \{1, 2\}$, and $\eta(z) = h(z)$ if $z \in V - \{x, y\}$ has no undefended vertex. For ease of simplicity, such a vertex y will be called a moving neighbor for x. The weight of a GRD-function h is the value $\sum_{x\in V} h(x)$, and the minimum weight of a GRD-function on a graph G is called *the generous Roman domination number* (GRD-number, for short) of G, denoted by $\gamma_{gR}(G)$. For any GRD-function h of G, let $V_j = \{x \in V \mid h(x) = j\}$ for $j \in [3]$. Since these four sets determine h, we can write $h = (V_0, V_1, V_2, V_3)$ or $(V_0^h, V_1^h, V_2^h, V_3^h)$ to refer to h. Also, a $\gamma_{gR}(G)$-function h is a GRD-function of G with weight $\gamma_{gR}(G)$.

Furthermore, according to [16], the function h is a *weak double Roman dominating function* (WDRD-function) if for each vertex x with $h(x) \leq 1$ there is a neighbor y of x with $h(y) \geq 2$ such that the function η defined by $\eta(x) = h(x)+1$, $\eta(y) = h(y) - 1$ and $\eta(z) = h(z)$ for all $x \in V(G) - \{x, y\}$ has no doubly unprotected vertex. The *weak double Roman domination number* (WDRD-number, for short) $\gamma_{wdR}(G)$ equals the minimum weight of a WDRD-function on G.

We end this section by giving two results. The first one shows that the generous Roman domination number may be smaller or larger than the weak double Roman domination number, which reveals the distinction between these two variants of double Roman domination. The second result shows that for every graph the GRD-number is bounded above by twice the WRD-number.

Proposition 1.1. *There exist graphs G and H such that the differences $\gamma_{wdR}(G)-\gamma_{gR}(G)$ and $\gamma_{gR}(H) - \gamma_{wdR}(H)$ are arbitrarily large.*

Proof. Let G be a healthy spider S_t $(t \geq 3)$ obtained from star $K_{1,t}$ by subdividing each edge exactly once. It is easy to see that $\gamma_{gR}(G) = t+3$ and $\gamma_{wdR}(G) = 2t+1$, and therefore $\gamma_{wdR}(G) - \gamma_{gR}(G) = t - 2$ can be arbitrary large.

Now let $t \geq 2$ be an integer and let H_t be a connected graph formed from two disjoint stars $K_{1,2t}$, one with center vertex x and leaves $x_1, \ldots, x_{2t}$ and the other one with center vertex y and leaves $y_1, \ldots, y_{2t}$ by adding three internally disjoint paths $x_1u_1u_2u_3y_1$, $x_1v_1v_2v_3y_1$ and $x_1w_1w_2w_3y_1$. One can see that $\gamma_{wdR}(H_t) = 14$ while $\gamma_{gR}(H_t) = 15$. To obtain a graph H for which the difference $\gamma_{gR}(H) - \gamma_{wdR}(H)$ is large, it is enough to consider s copies of H_t by adding $s - 1$ edges between the leaves of these copies to ensure that the resulting graph is connected. So we will have $\gamma_{wdR}(H) = 14s$ and $\gamma_{gR}(H) = 15s$, leading to $\gamma_{gR}(H)-\gamma_{wdR}(H) = s$. □

Proposition 1.2. *For any graph G, $\gamma_{gR}(G) \leq 2\gamma_r(G)$.*

Proof. Let $h = (V_0^h, V_1^h, V_2^h)$ be a minimum WRD-function on G. Note that $\gamma_r(G) = |V_1^h| + 2|V_2^h|$. Define the function $\eta = (V_0^\eta, \emptyset, V_2^\eta, V_3^\eta)$ on G by $V_0^\eta = V_0^h, V_2^\eta = V_1^h$ and $V_3^\eta = V_2^h$. Note that the weight of η is $2|V_1^\eta| + 3|V_2^\eta| \le 2\gamma_r(G)$. We shall show that η is a GRD-function on G. Let x be an arbitrary vertex in V_0^η. Since $V_0^\eta = V_0^h$ and h is a WRD-function, x is adjacent to a vertex in $V_1^h \cup V_2^h$. Since $V_2^\eta \cup V_3^\eta = V_1^h \cup V_2^h$, let $v \in V_2^\eta \cup V_3^\eta$ be a neighbor of x chosen so that the function h' defined by $h'(x) = 1$, $h'(v) = h(v) - 1$ and $h'(z) = h(z)$ for all $z \in V(G) - \{x, v\}$ has no undefended vertex. If $v \in V_3^\eta$, then define the function η' on G by $\eta'(x) = 1$, $\eta'(v) = 2$ and $\eta'(z) = \eta(z)$ for all $z \in V(G) - \{x, v\}$. Clearly, every vertex in G assigned 0 under η' remains with at least one neighbor assigned 2 or 3 under η', and thus G has no undefended vertex with respect of η', leading that η is a GRD-function on G. Now, assume that $v \in V_2^\eta$, and define the function η' on G by $\eta'(x) = 2$, $\eta'(v) = 0$ and $\eta'(z) = \eta(z)$ for all $z \in V(G) - \{x, v\}$. To show that η is a GRD-function on G, assume, for a contradiction, that there is undefended vertex y with respect to η'. Thus $\eta'(y) = 0$ and y has no neighbor assigned 2 or 3 under η'. This means that y must be adjacent to v but not to x, and therefore y has no neighbor assigned 1 or 2 under h', that is y is also an undefended vertex under h', contradicting the fact that h is a WRD-function. Thus, every vertex is defended with respect to η' implying that η is indeed a GRD-function of G of weight at most $2\gamma_r(G)$. This completes the proof. □

2. Ladders

In this section, we determine the exact value of the GRD-number for ladder graphs. We remind the subsequent result.

Proposition 2.1 ([15])**.** *(i) For every path P_p, $\gamma_{gR}(P_p) = \lceil \frac{6p}{7} \rceil$.*
(ii) For $p \ge 4$, $\gamma_{gR}(C_p) = \gamma_{gR}(P_p)$.

Let the vertices of the i-th copy of P_2 in the ladder $P_2 \Box P_p$ are u_i, v_i for $i = 1, 2, \ldots, p$. Observe that by Proposition 2.1, we have $\gamma_{gR}(P_2 \Box P_1) = 2 = \lceil \frac{4}{2} \rceil$ and $\gamma_{gR}(P_2 \Box P_2) = 4 = \lceil \frac{7}{2} \rceil$, while it can be easily seen that $\gamma_{gR}(P_2 \Box P_3) = 5 = \lceil \frac{10}{2} \rceil$. In the following, to simplify our notation, we will write $G_{2,p}$ instead of $P_2 \Box P_p$ and Ind-Hyp instead of induction hypothesis.

Theorem 2.2. *For $p \ge 1$, $\gamma_{gR}(G_{2,p}) = \lceil \frac{3p+1}{2} \rceil$.*

Proof. The result is clear for $p = 1$. Assume that $p \ge 2$ and consider the function h defined on $V(G_{2,p})$ as follows:
(i) If $p \equiv 0 \pmod 4$, then let $h(u_{4i+2}) = h(v_{4i+4}) = 3$ for $0 \le i \le \frac{p}{4} - 1$, $h(v_1) = 1$ and $h(y) = 0$ for other vertices.
(ii) If $p \equiv 1 \pmod 4$, then let $h(u_{4i+2}) = h(v_{4i+4}) = 3$ for $0 \le i \le \frac{p-1}{4} - 1$, $h(v_1) = h(u_p) = 1$ and $h(y) = 0$ for other vertices.
(iii) If $p \equiv 2 \pmod 4$, then let $h(u_{4i+2}) = h(v_{4j+4}) = 3$ for $0 \le i \le \frac{p-2}{4}, 0 \le j \le \frac{p-2}{4} - 1$, $h(v_1) = 1$ and $h(y) = 0$ for other vertices.

(iv) If $p \equiv 3 \pmod 4$, then let $h(u_{4i+2}) = h(v_{4j+4}) = 3$ for $0 \leq i \leq \frac{p-3}{4}, 0 \leq j \leq \frac{p-3}{4} - 1$, $h(v_1) = h(v_p) = 1$ and $h(y) = 0$ for other vertices.
It is not hard to verify that h is a GRD-function of $G_{2,p}$ of weight $\lceil \frac{3p+1}{2} \rceil$, and consequently, $\gamma_{gR}(G_{2,p}) \leq \lceil \frac{3p+1}{2} \rceil$.

Now we prove that for any integer $p \geq 1$, $\gamma_{gR}(G_{2,p}) \geq \lceil \frac{3p+1}{2} \rceil$, and to do this, we proceed by induction on p. The inequality is valid for $p \in \{1, 2, 3, 4\}$. Therefore, the base case has been established. Let $p \geq 5$, and consider a $\gamma_{gR}(G_{2,p})$-function h such that $\ell = h(u_p) + h(v_p) + h(u_{p-1}) + h(v_{p-1})$ is minimized and let h_i be the restriction of h on the ladder $G_{2,p} - \{u_j, v_j \mid i + 1 \leq j \leq p\}$. Observe that $\ell \geq 2$. If $\ell \geq 5$, then the function η defined on $V(G_{2,p}) - \{u_p, u_{p-1}, v_p, v_{p-1}\}$ by $\eta(u_{p-2}) = \min\{3, 1 + h(u_{p-2})\}$, $\eta(v_{p-2}) = \min\{3, 1 + h(v_{p-2})\}$, and $\eta(x) = h(x)$ for other vertices is a GRD-function of $G_{2,p-2}$ with weight at most $\omega(h) - 3$. By the Ind-Hyp on $G_{2,p-2} = G_{2,p} - \{u_{p-1}, v_{p-1}, u_p, v_p\}$, we get $\omega(h) \geq \omega(\eta) + 3 \geq \lceil \frac{3(p-2)+1}{2} \rceil + 3 = \lceil \frac{3p+1}{2} \rceil$. Thus, in the following we can suppose that $\ell \in \{2, 3, 4\}$.

First, assume that $\ell = 2$. Then either $h(u_p) = 2$ or $h(v_p) = 2$ or $h(u_p) = h(v_p) = 1$. If $h(u_p) = h(v_p) = 1$, then the restriction of h on $G_{2,p-1} = G_{2,p} - \{u_p, v_p\}$ is a GRD-function and the Ind-Hyp leads to $\omega(h) \geq \lceil \frac{3(p-1)+1}{2} \rceil + 2 = \lceil \frac{3p+2}{2} \rceil$. Now, wlog, assume that $h(u_p) = 2$. Thus $h(u_{p-1}) = h(v_p) = h(v_{p-1}) = 0$, and clearly u_p is a moving neighbor for only v_p. It follows that u_{p-2} is a moving neighbor of u_{p-1} and likewise v_{p-2} is a moving neighbor of v_{p-1} leading to $h(u_{p-2}) \geq 2$ and $h(v_{p-2}) \geq 2$. Now, if $h(u_{p-2}) = 3$ (the case $h(v_{p-2}) = 3$ is similar), then the function η defined on $V(G_{2,p}) - \{u_{p-1}, v_{p-1}, u_p, v_p\}$ by $\eta(u_{p-2}) = 2$ and $\eta(z) = h(z)$ for the remaining vertices, is a GRD-function of the ladder $G_{2,p-2}$ of weight $\omega(h) - 3$. Using the Ind-Hyp we get $\omega(h) \geq \lceil \frac{3(p-2)+1}{2} \rceil + 3 = \lceil \frac{3p+1}{2} \rceil$. Hence, we can assume that $h(u_{p-2}) = h(v_{p-2}) = 2$. If $h(u_{p-3}) = h(v_{p-3}) = 0$, then the function h_{p-4} on $G_{2,p-4} = G_{2,p} - \{u_i, v_i \mid p - 3 \leq i \leq p\}$ is a GRD-function of weight $\omega(h) - 6$. Using the Ind-Hyp we get $\omega(h) \geq \lceil \frac{3(p-4)+1}{2} \rceil + 6 = \lceil \frac{3p+1}{2} \rceil$. Henceforth, we assume that $\max\{h(u_{p-3}), h(v_{p-3})\} \geq 1$. Wlog, assume that $h(u_{p-3}) \geq 1$. Then, the function η defined on $V(G_{2,p}) - \{u_p, u_{p-1}, u_{p-2}, v_p, v_{p-1}, v_{p-2}\}$ by $\eta(v_{p-3}) = \min\{3, 1 + h(v_{p-3})\}$ and $\eta(z) = h(z)$ for the remaining vertices, is a GRD-function of $G_{2,p-3}$ of weight at most $\omega(h) - 5$, and by the Ind-Hyp we get $\omega(h) \geq \omega(\eta) + 5 \geq \lceil \frac{3(p-3)+1}{2} \rceil + 5 \geq \lceil \frac{3p+1}{2} \rceil$.

Suppose now that $\ell = 3$. Obviously, $h(u_p) + h(v_p) \geq 1$. If $h(u_p) + h(v_p) \geq 2$, then $h(u_{p-1}) + h(v_{p-1}) \leq 1$ and thus the function h_{p-2} on the ladder $G_{2,p-2}$ induced by $V(G_{2,p}) - \{u_{p-1}, v_{p-1}, u_p, v_p\}$ is a GRD-function with weight $\omega(h) - 3$, leading as before to the intended inequality. Thus, assume that $h(u_p) + h(v_p) = 1$. Wlog, let $h(u_p) = 1$ and $h(v_p) = 0$. It follows from $\ell = 3$ that $h(u_{p-1}) = 0$ and $h(v_{p-1}) = 2$. Moreover, v_{p-1} is a moving neighbor for only v_p. Hence, u_{p-2} would be a moving neighbor of u_{p-1} and thus $h(u_{p-2}) \geq 2$. If $h(u_{p-2}) = 3$ or $h(v_{p-2}) \geq 1$, then the function h_{p-2} on the ladder $G_{2,p-2}$ induced by $V(G_{2,p}) - \{u_{p-1}, v_{p-1}, u_p, v_p\}$ is a GRD-function with weight $\omega(h) - 3$, leading as before to the intended inequality. Henceforth, we suppose that $h(u_{p-2}) =$

2 and $h(v_{p-2}) = 0$. If $h(u_{p-3}) \geq 1$, then the function h_{p-3} on the ladder $G_{2,p-3}$ induced by $V(G_{2,p}) - \{u_p, u_{p-1}, u_{p-2}, v_p, v_{p-1}, v_{p-2}\}$ is a GRD-function on $G_{2,p-3}$ with weight at most $\omega(h) - 5$, and by the Ind-Hyp we get $\omega(h) \geq \omega(h_{p-3}) + 5 \geq \lceil \frac{3(p-3)+1}{2} \rceil + 5 \geq \lceil \frac{3p+1}{2} \rceil$. Hence we suppose that $h(u_{p-3}) = 0$. Observe that if $h(v_{p-3}) = 1$, then the function h_{p-4} on the ladder $G_{2,p-4}$ induced by $V(G_{2,p}) - \{u_p, u_{p-1}, u_{p-2}, u_{p-3}, v_p, v_{p-1}, v_{p-2}, v_{p-3}\}$ is a GRD-function of $G_{2,n-p}$ with weight at most $\omega(h) - 6$, and by the Ind-Hyp we get $\omega(h) \geq \omega(h_{p-4}) + 6 \geq \lceil \frac{3(p-4)+1}{2} \rceil + 6 = \lceil \frac{3p+1}{2} \rceil$. Moreover, if $h(v_{p-3}) \geq 2$, then the function η defined on $V(G_{2,p}) - \{u_p, u_{p-1}, u_{p-2}, u_{p-3}, v_p, v_{p-1}, v_{p-2}, v_{p-3},\}$ by $\eta(v_{p-4}) = \min\{3, 1 + h(v_{p-4})\}$ and $\eta(x) = h(x)$ for other vertices, is a GRD-function of $G_{2,p-4}$ with weight at most $\omega(h) - 6$, leading as before to the intended inequality. Thus, we may assume for the next that $h(u_{p-3}) = h(v_{p-3}) = 0$. Since u_{p-2} is the only moving neighbor for u_{p-1}, we must have $h(u_{p-4}) \geq 2$ and since v_{p-4} is the only candidate to be a moving neighbor for v_{p-3}, we have $h(v_{p-4}) \geq 2$. If $p = 5$, then $\omega(h) = 9 > \lceil \frac{3p+1}{2} \rceil = 8$, and thus the result is valid. Hence assume that $p \geq 6$. If $h(u_{p-5}) \geq 1$, then the function η defined on $V(G_{2,p}) - \{u_p, u_{p-1}, u_{p-2}, u_{p-3}, u_{p-4}, v_p, v_{p-1}, v_{p-2}, v_{p-3}, v_{p-4}\}$ by $\eta(v_{p-5}) = \min\{3, 1 + h(v_{p-5})\}$ and $\eta(x) = h(x)$ for other vertices, is a GRD-function of the ladder $G_{2,p-5}$ of weight at most $\omega(h) - 8$, and using the induction the intended inequality follows. Hence suppose that $h(u_{p-5}) = 0$. Likewise, we may assume that $h(v_{p-5}) = 0$. Then, the function h_{p-6} on $G_{2,p-6}$ induced by $V(G_{2,p}) - \{u_p, u_{p-1}, u_{p-2}, u_{p-3}, u_{p-4}, u_{p-5}, v_p, v_{p-1}, v_{p-2}, v_{p-3}, v_{p-4}, v_{p-5}\}$ is a GRD-function of $G_{2,p-6}$ with weight at most $\omega(h) - 9$, and the Ind-Hyp leads to $\omega(h) \geq \omega(h_{p-6}) + 9 \geq \lceil \frac{3(p-6)+1}{2} \rceil + 9 \geq \lceil \frac{3p+1}{2} \rceil$.

Finally assume that $\ell = 4$. If $h(u_{p-2}) \geq 1$ (the case $h(v_{p-2}) \geq 1$ is similar), then the function η defined on $V(G_{2,p}) - \{u_p, u_{p-1}, v_p, v_{p-1}\}$ by $\eta(v_{p-2}) = \min\{3, 1 + h(v_{p-2})\}$ and $\eta(x) = h(x)$ for other vertices, is a GRD-function of $G_{2,p-2}$ with weight at most $\omega(h) - 3$, leading, as in the previous situations, to the intended inequality. Henceforth, we assume that $h(u_{p-2}) = h(v_{p-2}) = 0$. If $h(u_p) + h(v_p) \geq 3$ and so $h(u_{p-1}) + h(v_{p-1}) \leq 1$ or $h(u_p) + h(v_p) = 2$ and $h(u_{p-1}) = h(v_{p-1}) = 1$, then the function h_{p-2} on the ladder $G_{2,p-2}$ induced by $V(G_{2,p}) - \{u_{p-1}, v_{p-1}, u_p, v_p\}$ is a GRD-function of weight $\omega(h) - 4$, leading again the result. If $h(u_p) + h(v_p) = 2$ and $h(u_{p-1}) = 2$ (the case $h(u_p) + h(v_p) = 2$ and $h(v_{p-1}) = 2$ is similar), then define the function η on $V(G_{2,p}) - \{u_p, u_{p-1}, v_p, v_{p-1}\}$ by $\eta(u_{p-2}) = \min\{3, 1 + h(u_{p-2})\}$ and $\eta(x) = h(x)$ for other vertices. Clearly, η is a GRD-function of $G_{2,p-2}$ with weight at most $\omega(h) - 3$, and the result follows by the Ind-Hyp. Hence we suppose that $h(u_p) + h(v_p) \leq 1$. First let $h(u_p) + h(v_p) = 1$. Wlog, suppose that $h(u_p) = 1$ and $h(v_p) = 0$. Then v_{p-1} is a moving neighbor of v_p and so $h(v_{p-1}) \geq 2$. In this case, the function η defined on $V(G_{2,p}) - \{u_p, u_{p-1}, v_p, v_{p-1}\}$ by $\eta(v_{p-2}) = \min\{3, 1 + h(v_{p-2})\}$ and $\eta(x) = h(x)$ for other vertices, is a GRD-function of $G_{2,p-2}$ with weight at most $\omega(h) - 3$, and as before the result is obtained. Thus assume that $h(u_p) = h(v_p) = 0$. Since h is a GRD-function of $G_{2,h}$ and since $\ell = 4$, we deduce that

$h(u_{p-1}) = h(v_{p-1}) = 2$. Since u_{p-1} is not a moving neighbor of u_{p-2} and likewise v_{p-1} is not a moving neighbor of v_{p-2}, we must have $h(u_{p-3}) \geq 2$ and $h(v_{p-3}) \geq 2$ (since we already assumed that $h(u_{p-2}) = h(v_{p-2}) = 0$). Then, the function η defined on $V(G_{2,p}) - \{u_p, u_{p-1}, u_{p-2}, u_{p-3}, v_p, v_{p-1}, v_{p-2}, u_{p-3}\}$ by $\eta(u_{p-4}) = \min\{3, 1+h(u_{p-4})\}$, $\eta(v_{p-4}) = \min\{3, 1+h(v_{p-4})\}$ and $\eta(x) = h(x)$ for other vertices, is a GRD-function of the ladder $G_{2,p-4}$ of weight at most $\omega(h) - 6$, and clearly the inequality will be obtained. All in all, we have $\gamma_{gR}(G_{2,p}) \geq \lceil \frac{3p+1}{2} \rceil$ and thus $\gamma_{gR}(G_{2,p}) = \lceil \frac{3p+1}{2} \rceil$. This completes the proof. □

3. Trees

We now focus on trees for which we mainly establish a lower bound on the GRD-number as a function of the domination number. Furthermore, we provide a characterization of extremal trees attaining this lower bound. Subsequently, we present an upper bound on the GRD-number expressed as a function of the order and the numbers of leaves and the stems. For some supplementary notation and definitions, we refer the reader to [16].

For any integers $m \geq 1$ and $\ell \geq 0$ with $m \geq \ell$, let $SP_{\ell,m}$ be the tree formed from the star $K_{1,m}$ by subdividing ℓ edges of the star exactly once. The tree $SP_{\ell,m}$ is called a *wounded spider* if $\ell \leq m-1$ and a *healthy spider* when $\ell = m$. We will sometimes say that we have a spider regardless of its condition whether it is wounded or healthy. Let $\mathcal{F}$ be the family of all trees $SP_{\ell,m}$. Before stating the result, it is important to mention that if T is a trivial tree, then $\gamma_{gR}(T) = \gamma(T) = 1$ while if T has order 2, then $\gamma_{gR}(T) = 2 = \gamma(T) + 1$.

Theorem 3.1. *Let T be a tree of order $p \geq 3$. Then $\gamma_{gR}(T) \geq \gamma(T) + 2$. The equality holds if and only if $T \in \mathcal{F}$.*

Proof. We use induction on the order p. If $p = 3$, then T is a path $P_3 \in \mathcal{F}$, where $\gamma(P_3) = 1$ and $\gamma_{gR}(T) = 3$ leading that $\gamma_{gR}(T) = \gamma(T) + 2$. This establishes the base case. Let $p \geq 4$ and assume that the result holds for any tree T' of order $p' < p$. Let T be a tree of order p. If $\mathrm{diam}(T) = 2$, then T is a star, where $T \in \mathcal{F}$ and $\gamma_{gR}(T) = \gamma(T) + 2$. Moreover, if $\mathrm{diam}(T) = 3$, then T is a double star $S_{r,s}$ for some integers $s \geq r \geq 1$, where $\gamma_{gR}(T) \geq 4 = \gamma(T) + 2$ and equality holds if and only if $r = 1$, that is T is a path P_4 and clearly $P_4 = S_{1,1} \in \mathcal{F}$. Therefore, in the subsequent discussion, we assume that $\mathrm{diam}(T) \geq 4$. Among all $\gamma_{gR}(G)$-functions, let $h = (V_0, V_1, V_2, V_3)$ be one such that the value $\sum_{v \in L(T)} h(v)$ is as small as possible, where $L(T)$ denote the set of leaves of T. Observe that $V_1 \cup V_2 \cup V_3$ is a dominating set of T. We distingusih the following two cases.

Case 1. $V_3 \neq \emptyset$.
If $|V_3| \geq 2$ or $V_2 \neq \emptyset$, then $\gamma_{gR}(T) = |V_1| + 2|V_2| + 3|V_3| \geq |V_1| + |V_2| + |V_3| + 3 \geq \gamma(T) + 3$. Hence we can assume that $|V_3| = 1$ and $V_2 = \emptyset$. As before, one can see that

$$\gamma_{gR}(T) = |V_1| + 3|V_3| = |V_1| + |V_3| + 2 \geq \gamma(T) + 2. \tag{1}$$

Further, if $\gamma_{gR}(T) = \gamma(G) + 2$, then we achieve equality along the entire inequality chain (1). Specifically, $V_1 \cup V_3$ is a minimum dominating set of T, where V_3 contains a single vertex say v. Clearly, in that situation, each vertex in V_0 is adjacent to v. Moreover, since T is a tree, no two vertices of V_0 can have common neighbor in V_1. Now, if some vertex $u \in V_0$ has at least two neighbors in V_1, then $\{u\} \cup V_3 \cup V_1 - (N(u) \cap V_1)$ is a dominating set of T smaller than $V_1 \cup V_3$, a contradiction. Hence every vertex of V_0 has at most one neighbor in V_1. On the other hand, if some vertex $u \in V_1$ has a neighbor in V_1, then $(V_1 \cup V_3) - \{u\}$ is a dominating set of T smaller than $V_1 \cup V_3$, a contradiction too. Therefore, $V_1 \cup V_3$ is an independent set, and since each vertex of V_0 is adjacent to v and has at most one neighbor in V_1, we conclude that T is a spider. Observe that if v is not a stem, then T is a healthy spider. But then $\gamma_{gR}(T) = \deg_T(v) + 3 = \gamma(T) + 3 > \gamma(T) + 2$, a contradiction. As a result, T is a wounded spider and thus $T \in \mathcal{F}$.

Case 2. $V_3 = \emptyset$.
Let $u_1u_2\dots u_t$ be a longest path in T and let T be rooted at u_t. Observe that $t \geq 5$, because $\mathrm{diam}(T) \geq 4$. Also, u_1 is a leaf, and thus u_2 is a stem. It is worth noting that no stem in T has three or more leaf neighbors, for otherwise we can reassign such a stem the value 3 and any of its leaf the value 0, which contradicts the assumption that $V_3 = \emptyset$. Thus, any stem has one or two leaf neighbors. First, assume that u_2 is a strong stem and let w be a second leaf neighbor of u_2. Clearly, since $V_3 = \emptyset$, we have $h(u_1) + h(u_2) + h(w) \leq 2$ leading that $h(u_1) = h(w) = 1$ and $h(u_2) = 0$. Then the function h restricted to the tree T^1 formed from T by removing u_1, u_2 and w is GRD-function of T^1 of weight $\gamma_{gR}(T) - 2$. Hence $\gamma_{gR}(T) \geq \gamma_{gR}(T^1) + 2$. Moreover, since the order of T^1 is at least three, by the Ind-Hyp, $\gamma_{gR}(T^1) \geq \gamma(T^1) + 2$. Now, using the fact that every dominating set of T^1 can be expanded to a dominating of T by adding to it u_2, we obtain that $\gamma(T) \leq \gamma(T^1) + 1$. Now all these together lead to

$$\begin{aligned}\gamma_{gR}(T) &\geq \gamma_{gR}(T^1) + 2 \geq (\gamma(T^1) + 2) + 2\\ &\geq (\gamma(T) - 1 + 2) + 2 = \gamma(T) + 3.\end{aligned}$$

Therefore, in the following $\deg(u_2) = 2$. To complete the proof, we distinguish three more subcases.

Subcase 2.1. $\deg(u_3) = 2$.
Let $T^1 = T - \{u_1, u_2, u_3\}$. Obviously, $\gamma(T) = \gamma(T^1) + 1$, and $p(T^1) \geq 2$. If $p(T^1) = 2$, then $T = P_5$ and by Proposition 2.1 $\gamma_{gR}(T) = 5 > \gamma(T) + 2$. Hence, let $p(T^1) \geq 3$. Obviously, $h(u_1) + h(u_2) + h(u_3) \geq 2$. If $h(u_1) + h(u_2) + h(u_3) \geq 3$, then the function η defined on T^1 by $\eta(u_4) = \min\{3, h(u_4) + 1\}$ and $\eta(x) = h(x)$ otherwise, is a GRD-function of T^1 with weight at most $\omega(h) - 2$. Applying the Ind-Hyp on T^1, we have

$$\begin{aligned}\gamma_{gR}(T) &= \omega(h) \geq \omega(\eta) + 2 \geq \gamma_{gR}(T^1) + 2\\ &\geq \gamma(T^1) + 2 + 2 = \gamma(T) + 3.\end{aligned}$$

Hence, we assume that $h(u_1)+h(u_2)+h(u_3)=2$. It follows that $h(u_1)+h(u_2)=2$ and $h(u_3)=0$. Then the restriction of f on T^1 is a GRD-function and as above we have $\gamma_{gR}(T)\geq\gamma(T)+3$.

Subcase 2.2. $\deg(u_3)\geq 3$ and u_3 is not a stem.

Let $u_2=w_1,w_2,\ldots,w_k$ be the children of u_3 and let z_i be a leaf neighbor of w_i, for each i. Note that $k\geq 2$. Since each w_i plays the same role as u_2, we have $\deg(w_i)=2$ for each i. Let $T^1=T-T_{u_3}$. Clearly, $p(T')\geq 2$. If $p(T^1)=2$, then T is a healthy spider, where $\gamma_{gR}(T)=3+k=\gamma(T)+3$. Hence assume that $p(T^1)\geq 3$. Since any $\gamma(T^1)$-set may be expanded to a dominating set of T by adding to it vertices $w_1,\ldots,w_k$, we have $\gamma(T)\leq\gamma(T^1)+k$. On the other hand, since $V_3=\emptyset$, we must have $h(z_i)+h(w_i)\geq 2$ for each i, except possibly for one index j for which we can have $h(z_j)+h(u_j)=1$. But then u_3 must be assigned 2. In any case, one can see that $h(V(T_{u_3}))\geq k+2$. Then we consider the function η defined on T^1 by $\eta(u_4)=\min\{3,h(u_4)+1\}$ and $\eta(x)=h(x)$ otherwise. Clearly η is a GRD-function of T^1 with weight at most $\omega(h)-k-1$. Applying the Ind-Hyp on T^1, we have

$$\begin{aligned}\gamma_{gR}(T)&=\omega(h)\geq\omega(\eta)+k+1\geq\gamma_{gR}(T^1)+k+1\\&=\gamma(T^1)+2+k+1=\gamma(T)+3.\end{aligned}$$

Subcase 2.3. $\deg(u_3)\geq 3$ and u_3 is a stem.

Observe that u_3 can have at most two leaf neighbors, for otherwise it is possible to reassign u_3 the value 3 contradicting $V_3=\emptyset$. Let ℓ_{u_3} denote the number of leaf neighbor of u_3, where $\ell_{u_3}\in\{1,2\}$. Also, as in Subcase 2.2, let $u_2=w_1,w_2,\ldots,w_k$ be the children of u_3 that are not leaves, where each w_i has degree 2. For each i, let z_i denote the unique leaf neighbor of w_i. Now, consider the tree $T^1=T-T_{u_3}$ and observe that T^1 has order $p(T^1)\geq 2$. Assume that $p(T^1)=2$. Then T is a tree isomorphic to $SP_{k+1,\deg_T(u_3)}\in\mathcal{F}$, where $\deg_T(u_3)=k+1+\ell_{u_3}$. In this case, one can easily see that $\gamma(T)=k+2$ and $\gamma_{gR}(T)=(k+1)+3=\gamma(T)+2$, and thus the result is valid. Hence, we can assume in the following that $p(T^1)\geq 3$. Since any dominating set of T^1 may be extended to a dominating set of T by adding to it the vertices $u_3,w_1,w_2,\ldots,w_k$, we have $\gamma(T)\leq\gamma(T^1)+k+1$. Moreover, the choice of h together with the facts u_3 is a stem and $V_3=\emptyset$ imply that $h(w_i)=2$ and $h(z_i)=0$ for each $i\in\{1,2,\ldots,k\}$. Furthermore, each w_i is a moving vertex for its own leaf neighbor. Again since $V_3=\emptyset$, the total of values assigned to u_3 and its leaf neighbors is at most two and clearly at least equal to ℓ_{u_3}. Therefore, $h(V(T_{u_3}))\geq 2k+\ell_{u_3}$. In this case, consider the function η defined on T^1 by $\eta(u_4)=\min\{3,h(u_4)+1\}$ and $\eta(x)=h(x)$ otherwise. Clearly, η is a GRD-function of T^1 of weight at most $\omega(h)-2k-\ell_{u_3}+1$. Applying the Ind-Hyp on T^1, we have

$$\begin{aligned}\gamma_{gR}(T)&=\omega(h)\geq\omega(\eta)+2k+\ell_{u_3}-1\geq\gamma_{gR}(T^1)+2t+\ell_{u_3}-1\\&=\gamma(T^1)+2+2k+\ell_{u_3}-1\geq(\gamma(T)-k-1)+2+2k+\ell_{v_3}-1\\&=\gamma(T)+k+\ell_{u_3}>\gamma(T)+2,\end{aligned}$$

since $k \geq 2$ and $\ell_{u_3} \in \{1, 2\}$. This completes the proof. □

Theorem 3.2. *Let T be a tree of order $p \geq 3$ with ℓ leaves and s stems. Then*

$$\gamma_{gR}(T) \leq \left\lfloor \frac{6}{7}(p + 2\ell - s) \right\rfloor.$$

Proof. It is enough to show that $\gamma_{gR}(T) \leq \frac{6}{7}(p + 2\ell - s)$ because $\gamma_{gR}(T)$ is an integer. We use induction on p. If $p = 3$, then $T = P_3$, and clearly $\gamma_{gR}(P_3) = 3 < \frac{6}{7}(p + 2\ell - s)$. Thus, the base case has been established. Let $p \geq 4$, and suppose that any tree T^1 of order p^1, with $3 \leq p^1 < p$, having ℓ^1 leaves and s^1 stems satisfies $\gamma_{gR}(T^1) \leq \frac{6}{7}(p^1 + 2\ell^1 - s^1)$. Let T be a tree of order p with ℓ leaves and s stems. If $\operatorname{diam}(T) = 2$, then T is a star, where $\gamma_{gR}(T) = 3 < \frac{6}{7}(p + 2\ell - s)$. If $\operatorname{diam}(T) = 3$, then T is a double star $S_{r,q}$ where $\gamma_{gR}(T) \in \{4, 5, 6\}$ and in either case we have $\gamma_{gR}(T) < \frac{6}{7}(p + 2\ell - s)$. Thus, we can assume that the diameter of T is at least four.

Let $P = u_1 u_2 \ldots u_t$ be a longest path in T and let T is rooted at v_t. Observe that $t \geq 5$, because diam(T) ≥ 4. We also observe that if T is a path P_p, then the result follows from Proposition 2.1. Hence we assume that $\Delta(T) \geq 3$. Consider the following situations.

Case 1. $\deg_T(u_2) \geq 3$.
Assume first that $\deg_T(u_3) \geq 3$, and let $T^1 = T - T_{u_2}$. Then $3 \leq p^1 \leq p - 3$, $\ell^1 \leq \ell - 2$ and $s^1 = s - 1$. Since any $\gamma_{gR}(T^1)$-function can be expanded to a GRD-function of T by assigning 3 to u_2 and 0 to every child of u_2, $\gamma_{gR}(T) \leq \gamma_{gR}(T^1) + 3$. Applying the Ind-Hyp on T^1, it follows that

$$\begin{aligned}
\gamma_{gR}(T) &\leq \gamma_{gR}(T^1) + 3 \leq \frac{6}{7}(p^1 + 2\ell^1 - s^1) + 3 \\
&\leq \frac{6}{7}((p - 3) + 2(\ell - 2) - s + 1) + 3 \\
&= \frac{6}{7}(p + 2\ell - s).
\end{aligned}$$

Assume now that $\deg_T(u_3) = 2$, and let $T^1 = T - T_{u_3}$. Observe that T^1 has order p^1 such that $2 \leq p^1 \leq p - 4$. If $p^1 = 2$, then one can see that $\gamma_{gR}(T) = 5$ (for instance, assign 3 to u_2, 2 to u_4 and 0 elsewhere) and thus $\gamma_{gR}(T) < \frac{6}{7}(p + 2\ell - s)$. Hence, assume that $p^1 \geq 3$. Then, $\ell^1 \leq \ell - 1$ and $s^1 \geq s - 1$. Since any $\gamma_{gR}(T^1)$-function can be expanded to a GRD-function of T by assigning 3 to u_2 and 0 to every neighbor of u_2, $\gamma_{gR}(T) \leq \gamma_{gR}(T^1) + 3$. By induction of T^1, we have

$$\begin{aligned}
\gamma_{gR}(T) &\leq \gamma_{gR}(T^1) + 3 \leq \frac{6}{7}(p^1 + 2\ell^1 - s^1) + 3 \\
&\leq \frac{6}{7}((p - 4) + 2(\ell - 1) - s + 1) + 3 < \frac{6}{7}(p + 2\ell - s).
\end{aligned}$$

Case 2. From now on, we can assume that $\deg_T(u_2) = 2$.
Let r be the smallest index such that for every $i \in \{2, \ldots, r\}$, all vertices u_i have

degree two and $\deg_T(u_{r+1}) \geq 3$. Clearly, such an index exists since we assumed T is different from a path. Also, since $r \geq 2$ because of $\deg_T(v_2) = 2$. Let the tree T_r and T_{r+1} be the components of T obtained from the deletion of the edge $u_r u_{r+1}$, where $u_r \in V(T_r)$ and $u_{r+1} \in T_{r+1}$. Clearly, T_r is a path of order r and T_{r+1} is a tree of order $p(T_{r+1}) = p - r \geq 3$ with $\ell(T_{r+1}) = \ell - 1$ leaves and $s(T_{r+1}) = s - 1$ stems. Let f_r and f_{r+1} be two minimum GRD-functions on T_r and T_{r+1}, respectively. By Proposition 2.1, $\omega(f_r) = \lceil \frac{6r}{7} \rceil$ and by the Ind-Hyp on T_{r+1}, we have $\omega(f_{r+1}) \leq \frac{6}{7}(p(T_{r+1}) + 2\ell(T_{r+1}) - s(T_{r+1}))$. In this case, the function η defined on T by $\eta(x) = f_r(x)$ if $x \in V(T_r)$ and $\eta(x) = f_{r+1}(x)$ if $x \in V(T_{r+1})$ is a GRD-function on T, yielding $\gamma_{gR}(T) \leq \omega(f_r) + \omega(f_{r+1})$. It follows that

$$\begin{aligned}\gamma_{gR}(T) &\leq \frac{6}{7}(p(T_{r+1}) + 2\ell(T_{r+1}) - s(T_{r+1})) + \left\lceil \frac{6r}{7} \right\rceil \\ &= \frac{6}{7}(p + 2\ell - s) - \frac{6}{7}(r+1) + \left\lceil \frac{6r}{7} \right\rceil \leq \frac{6}{7}(p + 2\ell - s).\end{aligned}$$

This completes the proof. □

We close this paper by two problems proposed by this research. But first, it is worth mentioning that in [15] the authors demonstrated that the decision problem related to the problem of computing the GRD-number is NP-complete for bipartite graphs. We have looked at their proof and we believe that same proof can be applied for chordal graphs, by simply adding all edges between vertices y_j's (see [15]).

Problem 1. For which graphs G we have $\gamma_{wdR}(G) \geq \gamma_{gR}(G)$ or $\gamma_{wdR}(G) \leq \gamma_{gR}(G)$.

Problem 2. Characterize the trees T such that $\gamma_{gR}(T) = 2\gamma_r(T)$.

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Seyed Mahmoud Sheikholeslami
Department of Mathematics,
Azarbaijan Shahid Madani University,
Tabriz, I. R. Iran
e-mail: s.m.sheikholeslami@azaruniv.ac.ir

Mustapha Chellali
LAMDA-RO Laboratory,
Department of Mathematics,
University of Blida,
B.P. 270, Blida, Algeria
e-mail: m_chellali@yahoo.com

Mariyeh Kor
Department of Mathematics,
Azarbaijan Shahid Madani University,
Tabriz, I. R. Iran
e-mail: mariekor@yahoo.com