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Some Results of Ricci Bi-Conformal Vector Fields

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Abstract

The investigation of Ricci bi-conformal vector fields and their associated outcomes is crucial for gaining insights into the geometric and topological characteristics of the underlying manifolds. The study of conformal vector fields and their extensions is highly valuable in the realms of geometry and physics. In this manuscript, we study the topological properties of the Ricci bi-conformal vector field. The goal of this paper is to find some results of the Ricci bi-conformal vector fields. We prove that a complete manifold admits the Ricci bi-conformal vector fields has a finite fundamental group. For this purpose, we first state the definition and lemma, and then use them to prove our theorems.

Keywords: Ricci bi-conformal vector field, Fundamental group, Complete manifold.

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1. Introduction

Researching conformal vector fields (CVFs) and their generalizations holds significant importance in the fields of geometry and physics. In this manuscript, we study the topological properties of the Ricci bi-conformal vector field (RBCVF). De et al. [1] defined RBCVFs as follows:

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A Riemannian manifold (M, g) is characterized by a vector field X being classified as an RBCVF if it satisfies certain equations that are defined in terms of non-zero smooth functions α and β, applicable to any vector fields Y and Z:

$$(\mathcal{L}_X g)(Y, Z) = \alpha g(Y, Z) + \beta S(Y, Z), \tag{1}$$

and

$$(\mathcal{L}_X S)(Y, Z) = \alpha S(Y, Z) + \beta g(Y, Z), \tag{2}$$

where the Ricci tensor S of the manifold M is associated with the metric tensor g, while $\mathcal{L}_X$ denotes the Lie derivative taken in the direction of the vector field X. In [2] and [3], Ricci solitons and RBCVFs on the model space Sol_1^4 and RBCVFs on Lorentzian Walker manifolds of low dimension have been studied, respectively.

Now, in this paper, the study of complete connected Riemannian manifolds (M, g) with a vector field X is considered as:

$$\mathcal{L}_X g - \beta S \leq \alpha g, \tag{3}$$

where smooth functions such as α and β are exist, and the expression $\mathcal{L}_X g$ represents the Lie derivative of the function g along the vector field X.

In this paper, we apply Myer's theorem, which states that any two points in the manifold M can be connected by a geodesic segment with a maximum length of πr (refer to [4]). This implies that the manifold is geodesically complete, meaning that geodesics can be extended indefinitely. The theorem emphasizes the connection between curvature and the overarching characteristics of manifolds. In particular, the theorem concludes that the diameter of M is finite. Thus, M needs to be compact, as a closed and compact ball of finite radius within any tangent space is mapped onto all of M through an exponential map. Thus, the outcome indicates that any compact manifold meeting the criteria outlined in (3) possesses a finite fundamental group. Numerous authors have explored this subject. For example, Azami investigated complete Ricci-Bourguignon solitons on Finsler manifolds [5], complete shrinking general Ricci flow soliton systems [6], and complete shrinking Ricci-Bourguignon harmonic solitons [7]. This paper is a generalization of William's work in [8], which in special cases where we can prove it, yields a Ricci soliton. We present our findings, beginning with the following definition.

Definition 1.1. Let (M, g) represent a Riemannian manifold. For every point a located in M, we establish a definition

$$D_a = max\{0, sup\{S_y(\xi, \xi) : y \in B(a, 1), ||\xi|| = 1\}\}, \tag{4}$$

where S_y is the Ricci tensor of M.
The following theorems are stated

Theorem 1.2. *Let (M, g) be a complete Riemannian manifold that satisfies the condition stated in (3). For any points a and b within the manifold M, we can derive the following result*

$$d(a,b) \leq max\left\{1, \frac{k_2}{k_1}\left(2(n-1)+D_a+D_b\right)+\frac{2}{k_1}\left(||X_a||+||X_b||\right)\right\}, \tag{5}$$

where $\alpha \geq k_1$ and $\beta = -k_2$. (In (3), α and β were introduced).

The second theorem is

Theorem 1.3. *If (M, g) represents a complete connected Riemannian manifold that meets the conditions outlined in (3), then it follows that the fundamental group of M is finite.*

2. Proofs of our main results

To continue, the following main lemma is needed.

Lemma 2.1. *Let (M, g) denote a complete Riemannian manifold, and consider points a and b within M such that the distance $r = d(a, b) > 1$. Additionally, let δ represent the shortest geodesic connecting points a and b, which is parametrized by arclength, then*

$$\int_0^r S(\delta^{'}(s),\delta^{'}(s))ds \leq 2(n-1)+D_a+D_b. \tag{6}$$

This lemma was proved in [4]. Now, by using Myer's theorem and the above lemma, according to the following theorem, the upper bound on the distance between points a and b that depends only on $||X||$ and D is obtained. Next, we have the proof of the first theorem as follows:

Proof of Theorem 1.2. Assume δ be the minimal geodesic between a and b and $d(a, b) > 1$, according to (3), we get

$$\textstyle\int_0^r S(\delta^{'}(s),\delta^{'}(s))ds \geq \int_0^r \frac{1}{\beta}(-\alpha g(\delta^{'}(s),\delta^{'}(s))+(\mathcal{L}_X g)(\delta^{'}(s),\delta^{'}(s)))ds. \tag{7}$$

Using identity $\mathcal{L}_X g(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)$, the Lie derivative of g, we have

$$\mathcal{L}_X g(\delta^{'}(s),\delta^{'}(s)) = 2\frac{d}{ds}g(X,\delta^{'}(s)). \tag{8}$$

Now, by substituting (8) in (7), we infer

$$\textstyle\int_0^r S(\delta^{'}(s),\delta^{'}(s))ds \geq \int_0^r \frac{1}{\beta}(-\alpha g(\delta^{'}(s),\delta^{'}(s))+2g(\nabla_{\delta^{'}(s)}X,\delta^{'}(s)))ds. \tag{9}$$

So, we get

$$\int_0^r S(\delta^{'}(s),\delta^{'}(s))ds \geq \int_0^r \frac{1}{\beta}(-\alpha g(\delta^{'}(s),\delta^{'}(s))+2\frac{d}{ds}g(X,\delta^{'}(s)))ds, \tag{10}$$

now, assume that $\alpha \geq k_1$ and $\beta = -k_2$, we obtain

$$\begin{aligned} \textstyle\int_0^r S(\delta'(s), \delta'(s))ds &\geq \tfrac{k_1}{k_2} d(a,b) + \tfrac{2}{k_2} g(X, \delta'(r)) - \tfrac{2}{k_2} g(X, \delta'(0)) \\ &\geq \tfrac{k_1}{k_2} d(a,b) + \tfrac{2}{k_2} ||X_b|| - \tfrac{2}{k_2} 2||X_a|| \\ &\geq \tfrac{k_1}{k_2} d(a,b) - \tfrac{2}{k_2} ||X_b|| - \tfrac{2}{k_2} 2||X_a||. \end{aligned} \tag{11}$$

By using Lemma 2.1, we conclude

$$d(a,b) \leq max\{1, \frac{k_2}{k_1}(2(n-1) + D_a + D_b) + \frac{2}{k_1}(||X_a|| + ||X_b||)\}.$$

□

Proof of Theorem 1.3. Consider $(\tilde{M}, \tilde{g})$ is the Riemannian universal cover of (M, g), and the lift of X is $\tilde{X}$, so suppose $p : \tilde{M} \to M$ is the universal covering map of M. Since p is a local isometry, for each $x = p(\tilde{x})$, $p^{-1}(x)$ is bounbed discrete and closed set. Every closed and bounded set is a compact set. Therefore, $p^{-1}(x)$ is finite. Let $\tilde{a}$ in $\tilde{M}$, and assume $\gamma \in \pi_1(M)$ is a deck transformation on $\tilde{M}$. We know $B(\tilde{a}, 1)$, and $B(\gamma(\tilde{a}), 1)$ are isometric, thus $D_a = D_{\gamma(a)}$. Also, $||\tilde{X}_{\tilde{a}}|| = ||\tilde{X}_{\gamma}(\tilde{a})||$, and by using Theorem 1.2 to the point $\tilde{a}$ and $\gamma(\tilde{a})$, we can write

$$d(\tilde{a}, \gamma(\tilde{a})) \leq max\{1, \frac{2k_2}{k_1}(\gamma - 1 + D_{\tilde{a}}) + \frac{4}{k_1}||\tilde{X}_{\tilde{a}}||\}, \quad \forall \gamma \in \pi_1(M). \tag{12}$$

The space $\tilde{M}$ serves as a universal covering space for M, and there exists a bijective correspondence between the fundamental group $\pi_1(M, x)$, and the preimage $p^{-1}(x)$. Then $\pi_1(M, x)$ is finite. Since M is a connected space, the fundamental group $\pi_1(M, x)$ is isomorphic for all points X within M. This indicates that the fundamental group of M is finite.

□

We now present two examples that apply to the conditions of the proven theorems.

Example 2.2. Let $M = \mathbb{R}^n$ be Euclidean space equipped with the standard flat Riemannian metric:

$$g = \sum_{i=1}^{n} dx^i \otimes dx^i,$$

this metric is globally defined, smooth, symmetric and positive-definite. Let X be the vector field with constant coefficients:

$$X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i},$$

where $a^i \in \mathbb{R}$. This defines a smooth vector field on $\mathbb{R}^n$ with constant components. Therefore, the squared norm of X with respect to the Euclidean metric is:

$$||X||^2 = g(X, X) = \sum_{i=1}^{n} (a^i)^2 = constant,$$

therefore, $||X||$ is bounded globally. Also, the Lie derivative of the metric g with respect to a constant vector field X vanishes $\mathcal{L}_X g = 0$, because partial derivatives of the metric coefficients (which are constant) vanish, and X has constant coefficients. Hence, X is a Killing vector field. Now, since $\mathbb{R}^n$ with the standard metric is flat, $Ric = 0$, thus, for any α, β:

$$\mathcal{L}_X Ric = 0 = \alpha Ric + \beta g.$$

So, X is a RBCVF with arbitrary α, β (e.g. $\alpha = \beta = 0$). Also, the manifold $\mathbb{R}^n$ is simply connected, thus $\pi_1(\mathbb{R}^n) = 0$.

Example 2.3. Suppose $\mathbb{S}^2$ in spherical coordinates $(\theta, \phi) \in (0, \pi) \times (0, 2\pi)$ be equipped with the standard Riemannian metric:

$$g = d\theta^2 + sin^2\theta d\phi^2.$$

Let us consider the vector field:

$$X = \frac{\partial}{\partial \phi}.$$

This vector field is a Killing vector field because the metric g does not depend on the coordinate ϕ. Therefore, $\mathcal{L}_X g = 0$. Also, the squared norm of X with respect to g is:

$$||X||^2 = g(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi}) = sin^2\theta,$$

since $sin^2\theta \in [0, 1]$, the norm $||X||$ is bounded on $\mathbb{S}^2$. Hence, X has finite norm. Also, the manifold $\mathbb{S}^2$ is simply connected, so its fundamental group is trivial $\pi_1(\mathbb{S}^2) = 0$. This satisfies the hypothesis of many comparison theorems involving curvature and vector fields. Now, the Ricci tensor on $\mathbb{S}^2$ is given by $Ric_g = g$, because $\mathbb{S}^2$ has constant sectional curvature $K = +1$, and the Ricci tensor satisfies:

$$Ric = (n-1)K \cdot g = 1 \cdot g.$$

Also, since $\mathcal{L}_X g = 0$, and $Ric_g = g$, we have:

$$Ric_g + \mathcal{L}_X g = g \geq \lambda g,$$

for $\lambda \in (0, 1]$. Hence, the vector field X satisfies the inequality:

$$Ric_g + \mathcal{L}_X g \geq \lambda g,$$

for some $\lambda > 0$, which means X is an RBCVF.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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