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An Iterative Method for Numerically Solving a Class of Linear Volterra Delay Integral Equations

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Abstract

In this paper, a numerical method based on a recursive relation (sequence) is presented for numerically solving a class of linear Volterra delay integral equations (VDIEs), where the recursive relation is obtained from the considered integral equation itself. For this purpose, first, using the Banach fixed point theorem, the existence and uniqueness of the solution to the considered VDIEs are proven. It is also proven that the sequence mentioned above converges to the solution of the equation. Then, by considering a finite number of terms of the said sequence, an approximation to the solution of the equation is obtained. Finally, some numerical examples are given to verify the accuracy and efficiency of the proposed method.

Keywords: Delay integral equations, Volterra, Existence and uniqueness, Converges, Fixed point theorem.

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1. Introduction

Integral equations are a significant topics in applied mathematics. These equations are used to model various practical problems in fields such as physics, engineering,

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financial problems, and other applied sciences (see [1–5]).
A class of important integral equations that has many applications and has attracted the attention of many researchers is delay integral equations (DIEs). Many different methods have been extended to approximate solutions of DEIs. For example, in [6], the variational iteration method and in [7], a Taylor collocation method were extended to solve DIEs. The Haar wavelet method was proposed for numerical solution of a class of delay differential and delay partial differential equations in [8], and for delay Volterra-Fredholm integral equations in [9]. Sinc functions were applied for numerical solving of the pantograph Volterra delay integro-differential equations in [10]. A Tau-like numerical method was developed for solving fractional delay integro-differential equations in [11]. Delay Volterra integral equations on a half-line were investigated in [12]. In [13], the existence of solutions in nonlocal partial functional integro-differential equations with finite delay in nondense domains was investigated. In [14], solving differential equations with infinite delay via a coupled fixed point was considered. In [15], fixed point theorem was applied to prove the uniqueness and stability of solutions for a class of nonlinear integral equations. In [16], the existence, uniqueness, and numerical solutions of fractional crossover delay differential equations of the Mittag-Leffler kernel using the Galerkin algorithm based on shifted Legendre polynomials, were investigated.
Numerical solution of DIEs using operational matrices of a hybrid of block-pulse functions and Bernstein polynomials is examined in [17]. It was shown that the multistep collocation method for delay Volterra integral equations is superconvergence in [18]. In [19], a fitted mesh numerical scheme was extended for a singularly perturbed delay reaction diffusion problem with integral boundary conditions. And recently, in [20], the collocation method has been extended to solve delay Volterra integral equations with weakly singular kernels, and its convergence has been proven. For some other related work, see [21–24].
As mentioned previously, the subject of this paper is to study delay Volterra integral equations as [25]:

$$u(t) = g(t) + \int_{\theta(t)}^{t} k(t,s)u(s)ds, \quad t \in I = [0,T], \tag{1}$$

where $k \in C(D_\theta)$ and $f \in C(I)$ with

$$D_\theta = \{(t,s) : 0 \le \theta(t) \le s \le t < T\},$$

while u is unknown function of the equation.
Here, we consider the linear case, $\theta(t) = qt,\ 0 < q < 1$ in (1), that is

$$u(t) = g(t) + \int_{qt}^{t} k(t,s)u(s)ds, \quad t \in I = [0,T]. \tag{2}$$

Most of the available methods for solving integral and differential equations convert them to a system of algebraic equations. In the current paper, an iterative method is used to solve (2). The advantage of using iterative methods is that they do not require solving a system of algebraic equations. When the matrix is very big and sparse, iterative methods are preferred.

Definition 1.1. ([25]). The primary discontinuity points corresponding to the delay function $\theta(t) = t - \tau(t)$, $\{\xi_i : i \geq 0\}$ are defined as follows:

$$\theta(\xi_{i+1}) = \xi_{i+1} - \tau(\xi_{i+1}) = \xi_i, \quad i = 0, 1, \cdots, \quad \xi_0 = t_0. \tag{3}$$

At these points, as the name suggests, solutions to a delay equation will generally exhibit lower regularity, even if the initial functions are smooth. For instance, at $t = \xi_0 = t_0$, the solution remains continuous, but its derivative may be discontinuous.
It can be established that the DVIE (1) admits a unique continuous solution $u \in C(t_0, T]$, and since in this paper, $t_0 = 0$ (the equation has no primary discontinuity points), so, $u \in C[t_0, T]$, [25].
Further examination of the types of delays and the corresponding discontinuity points of each can be found in [26].

2. Preliminary results

In this section, some basic concepts and tools which help us in the rest of the paper are given. They can be found in books on numerical analysis, such as [27, 28].

Definition 2.1. Suppose V is a Banach space with the norm $||.||_V$ and $W \subseteq V$. The operator $\Phi : V \to W$ is called contractive with contractivity constant $K \in [0, 1)$, if

$$||\Phi u - \Phi v||_V \leq K||u - v||_V, \quad \forall u, v \in V.$$

The Banach fixed point theorem, as noted in [28], is significant in establishing the existence and uniqueness of solutions of differential and integral equations.

Theorem 2.2. *Suppose that V is a Banach space and W is a nonempty closed subset of V and $\Phi : W \to W$ is a contractive mapping with contractivity constant $K \in [0, 1)$. Then the following results hold:*
(a) *There exists a unique $\bar{u} \in W$ such that $\Phi\bar{u} = \bar{u}$.*
(b) *For any $u_0 \in W$, the sequence $\{u_n\} \subset W$ generated by*

$$u_{n+1} = \Phi u_n, \quad n = 0, 1, \cdots,$$

converges to $\bar{u}$, that is

$$||u_n - \bar{u}|| \to 0, \quad as \quad n \to 0,$$

with error bounds as:

$$||u_n - \bar{u}||_V \leq \frac{K^n}{1 - K}||u_1 - u_0||_V,$$

$$||u_n - \bar{u}||_V \leq \frac{K}{1-K}||u_n - u_{n-1}||_V,$$
$$||u_n - \bar{u}||_V \leq K||u_{n-1} - \bar{u}||_V.$$

3. Main results

In view of Equation (2), we define the operator W as:

$$Wu(t) = g(t) + \int_{qt}^{t} k(t,s)u(s)ds, \tag{4}$$

so, Equation (2) is converted to

$$Wu(t) = u(t). \tag{5}$$

It is obvious that the fixed point of the operator W is the solution of Equation (2). Therefore, to solve Equation (2), it is sufficient to obtain the fixed point of the operator W.

Theorem 3.1. *Let $g \in C[0,T]$ and $k \in C(D_q)$. Then there exists some positive integer m for which W^m is a contraction mapping.*

Proof. Define
$$\bar{K} = max\{|k(t,s)| : (t,s) \in D_q\},$$
we prove by induction that

$$|W^n u(t) - W^n v(t)| \leq \frac{\bar{K}^n (1-q)(1-q^2)\cdots(1-q^n)}{n!} T^n ||u-v||. \tag{6}$$

To this end, we have

$$\begin{aligned}|W^n u(t) - W^n v(t)| &= \left| \int_{qt}^{t} k(t,s)(u(s)-v(s))ds \right| \\ &\leq \bar{K}||u-v||(1-q)t \leq \bar{K}||u-v||(1-q)T,\end{aligned}$$

and assuming (6) for n, we have

$$\begin{aligned}|W^{n+1}u(t) - W^{n+1}v(t)| &= \left| \int_{qt}^{t} k(t,s)(W^n u(s) - W^n v(s))ds \right| \\ &\leq \bar{K}\frac{\bar{K}^n(1-q)(1-q^2)\cdots(1-q^n)}{n!}||u-v|| \int_{qt}^{t} s^n ds \\ &= \frac{\bar{K}^{n+1}(1-q)(1-q^2)\cdots(1-q^{n+1})}{(n+1)!}||u-v||t^{n+1}\end{aligned}$$

$$\leq \frac{\bar{K}^{n+1}(1-q)(1-q^2)\cdots(1-q^{n+1})}{(n+1)!}T^{n+1}||u-v||,$$

so (6) is proved. Therefore, assuming

$$K_n = \frac{\bar{K}^n(1-q)(1-q^2)\cdots(1-q^n)T^n}{n!},$$

we have

$$|W^n u(t) - W^n v(t)| \leq K_n||u-v||, \ \forall t \in I,$$

which implies

$$||W^n u - W^n v|| \leq K_n||u-v||.$$

Hence, for sufficiently large n, it will be obtained $K_n < 1$, which means that W is a contractive mapping. □

Now, we give the following theorem from [29].

Theorem 3.2. *Suppose V is a Banach space and W is a nonempty closed set of it, and $\Phi : W \to W$ is continuous. Also, suppose Φ^m is a contraction for some positive integer m. Then Φ has a unique fixed point in W. Futhermore, the iteration method*

$$v_{n+1} = \Phi v_n, \quad n = 0, 1, \cdots,$$

is convergent.

Remark 1. In view of Theorems 3.1 and 3.2, the sequence

$$u_{n+1}(t) = g(t) + \int_{qt}^{t} k(t,s)u_n(s)ds, \quad n = 0, 1, \cdots, \tag{7}$$

will be converge to the solution u of (2). Therefore, for each N, the Nth term of this sequence $(u_N(t))$ is an approximate solution of (2).

Remark 2. A similar result to Theorem 3.1 also holds for the following integral equation

$$u(t) = g(t) + \int_{0}^{qt} k(t,s)u(s)ds, \quad t \in I = [0,T]. \tag{8}$$

To prove it, we proceed similarly to Theorem 3.1, except that in this case we take K_n as follows

$$K_n = \frac{\bar{K}^n q^{n(n+1)/2}T^n}{n!},$$

and $\bar{K}$ is defined similarly to the proof of Theorem 3.1.
In this case, we also have a similar recursive relationship to (7) as:

$$u_{n+1}(t) = g(t) + \int_{0}^{qt} k(t,s)u_n(s)ds, \quad n = 0, 1, \cdots. \tag{9}$$

4. Numerical experiments

In this section, we solve some examples by the proposed method. All numerical results have been obtained by programming in Maple Software 2019 on a personal computer with a 64-bit Windows 7 operating system, 4/0 GB of RAM, and an Intel Core 2 Duo @ 2.80GHz processor.

Table 1: Numerical results of Example 4.1.

t	$q=\frac{1}{3}$		$q=\frac{1}{2}$		$q=\frac{2}{3}$	
	$N=5$	$N=7$	$N=5$	$N=7$	$N=5$	$N=7$
0.1	$0.17e-14$	$0.13e-17$	$0.13e-14$	$0.10e-17$	$0.72e-15$	$0.40e-18$
0.2	$0.25e-11$	$0.18e-16$	$0.19e-11$	$0.15e-16$	$0.11e-11$	$0.69e-17$
0.3	$0.16e-9$	$0.51e-14$	$0.12e-9$	$0.39e-14$	$0.66e-10$	$0.21e-14$
0.4	$0.27e-8$	$0.25e-12$	$0.21e-8$	$0.19e-12$	$0.11e-8$	$0.98e-13$
0.5	$0.24e-7$	$0.46e-11$	$0.18e-7$	$0.35e-11$	$0.93e-8$	$0.18e-11$
0.6	$0.13e-6$	$0.45e-10$	$0.98e-7$	$0.34e-10$	$0.50e-7$	$0.17e-10$
0.7	$0.52e-6$	$0.30e-9$	$0.39e-6$	$0.22e-9$	$0.20e-6$	$0.11e-9$
0.8	$0.171e-5$	$0.14e-8$	$0.13e-5$	$0.11e-8$	$0.62e-6$	$0.53e-9$
0.9	$0.45e-5$	$0.54e-8$	$0.34e-5$	$0.41e-8$	$0.16e-5$	$0.20e-8$
1.0	$0.11e-4$	$0.18e-7$	$0.80e-5$	$0.13e-7$	$0.38e-5$	$0.61e-8$
CPU times	$10.94s$	$102.98s$	$4.38s$	$39.48s$	$8.61s$	$82.90s$

Example 4.1. Consider the DVIE

$$u(t) = \mathrm{e}^t + (1-q)t^2 - \int_{qt}^{t} t\mathrm{e}^{-s}u(s)ds, \quad t \in [0,1], \tag{10}$$

with the exact solution $u(t) = e^t$. We compute the sequence (7) in truncated form ($\{u_n(t)\}_{n=1}^N$) for some N as an approximate solution of the Equation (10). Absolute errors $|u(t) - u_N(t)|$ are reported in Table 1 and Figure 1, in which u and u_N denote the exact and approximate solution, respectively.

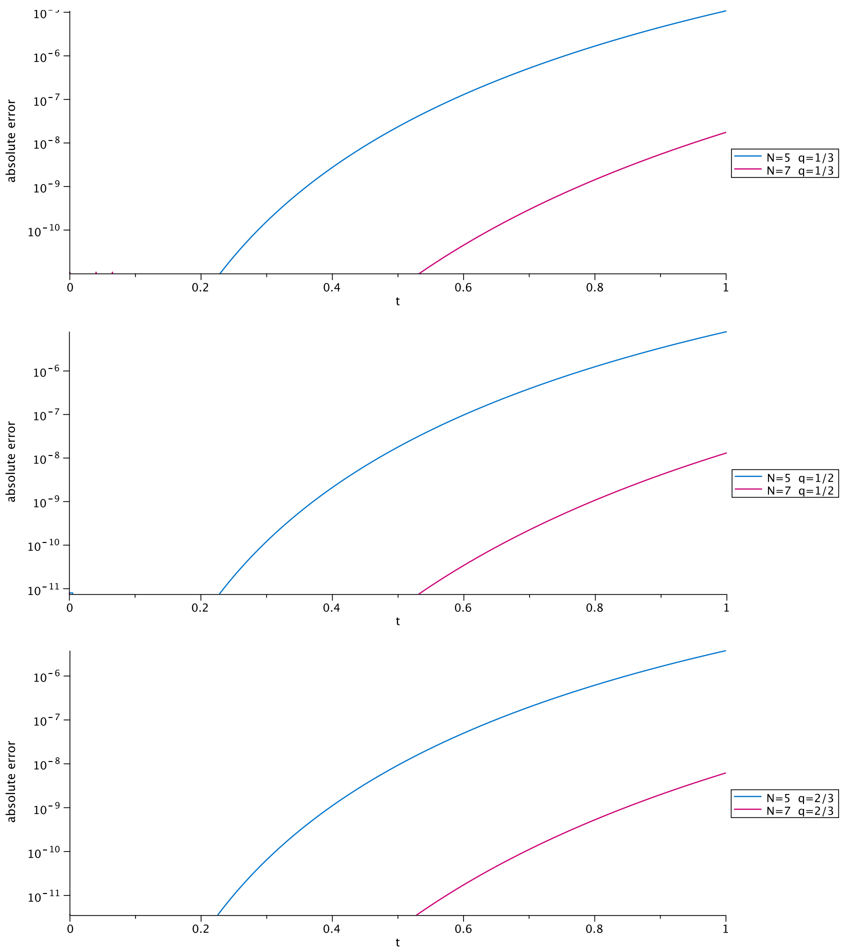

Figure 1: Comparison of absolute errors between $N = 5$ and $N = 7$ for $q = 1/3$, $q = 1/2$ and $q = 2/3$, in Example 4.1.

Example 4.2. In this example, we apply the proposed method to solve the following DVIE

$$u(t) = sin(t) - \frac{1}{2}sin^2(qt) + \int_0^{qt} cos(s)u(s)ds, \quad t \in [0, 1], \tag{11}$$

where the exact solution is $u(t) = sin(t)$. We proceed as in the previous example and report absolute errors in Table 2 and Figure 2.

Table 2: Numerical results of Example 4.2.

t	$q = 0.4$		$q = 0.8$	
	$N = 3$	$N = 6$	$N = 3$	$N = 6$
0.2	$0.11e-8$	$0.2e-19$	$0.55e-6$	$0.17e-9$
0.4	$0.70e-7$	$0.8e-19$	$0.33e-4$	$0.31e-10$
0.6	$0.79e-6$	$0.4e-19$	$0.33e-3$	$0.14e-8$
0.8	$0.43e-5$	$0.19e-18$	$0.16e-2$	$0.21e-8$
1.0	$0.16e-4$	$0.11e-17$	$0.53e-2$	$0.15e-7$
1.2	$0.46e-4$	$0.36e-17$	$0.10e-1$	$0.10e-6$
1.4	$0.11e-3$	$0.22e-16$	$0.26e-1$	$0.51e-6$
1.6	$0.24e-3$	$0.10e-15$	$0.43e-1$	$0.15e-5$
1.8	$0.46e-3$	$0.40e-15$	$0.61e-1$	$0.34e-5$
2.0	$0.82e-3$	$0.13e-14$	$0.72e-1$	$0.47e-5$
CPU times	$78.25s$	$2490s$	$83.35s$	$2610s$

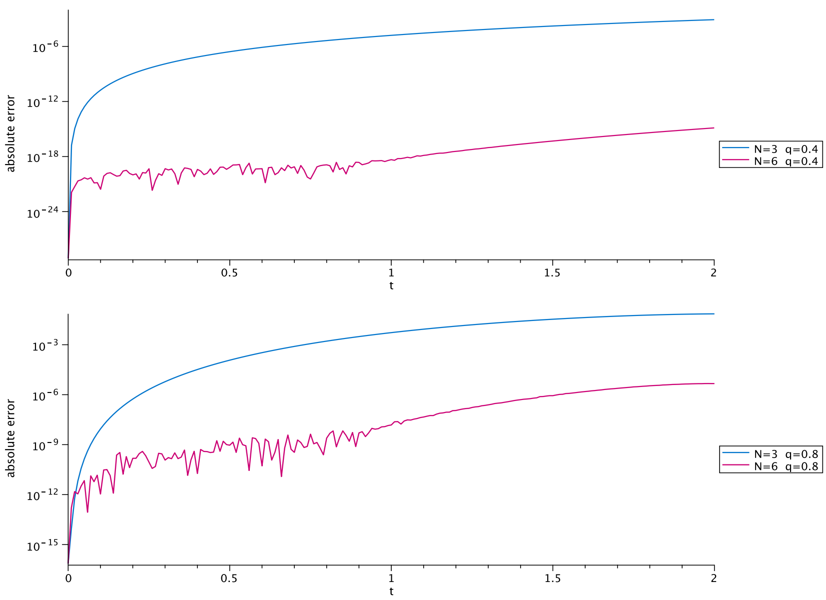

Figure 2: Comparison of absolute errors between $N = 3$ and $N = 3$ for $q = 0.4$ and $q = 0.8$, in Example 4.2.

Example 4.3. In this example, we consider the following DVIE

$$u(t) = \frac{1}{t+1} - (1-q)t^2 + \int_{qt}^{t} t(s+1)u(s)ds, \quad t \in [0,1], \tag{12}$$

where the exact solution is $u(t) = \frac{1}{t+1}$. For this example, in Table 3 and Figure 3, we report the maximum norm of the absolute error, which is defined as:

$$\max_{t\in[0,1]} |u(t) - u_N(t)|.$$

Table 3: Numerical results of Example 4.3.

	$q = 0.2$	$q = 0.4$	$q = 0.6$	$q = 0.8$
$N = 6$	$0.3268e-3$	$0.2797e-3$	$0.1847e-3$	$0.4546e-4$
CPU times	$4.04s$	$2.29s$	$3.21s$	$2.12s$
$N = 8$	$0.4174e-5$	$0.3571e-5$	$0.2351e-5$	$0.5501e-6$
CPU times	$3.60s$	$7.08s$	$5.87s$	$7.97s$
$N = 10$	$0.3291e-7$	$0.2814e-7$	$0.1850e-7$	$0.4230e-8$
CPU times	$15.08s$	$16.52s$	$14.02s$	$18.27s$
$N = 12$	$0.1761e-9$	$0.1505e-9$	$0.9878e-10$	$0.2230e-10$
CPU times	$92.10s$	$2430s$	$2950s$	$2450s$

Example 4.4. Consider the DVIE

$$u(t) = (-t^2 + t + 1)\mathrm{e}^t + t(qt-1)\mathrm{e}^{qt} + \int_{qt}^{t} tsu(s)\,ds. \tag{13}$$

We compute the sequence $(\{u_n(t)\}_{n=1}^N)$ for some N as an approximate solution of the Equation (13). In other words

$$u_n(t) \simeq \underbrace{(-t^2 + t + 1)\mathrm{e}^t + t(qt-1)\mathrm{e}^{qt} + \int_{qt}^{t} tsu_n(s)\,ds}_{=Wu_n(t)}, \tag{14}$$

and let $r_n(t) = |u_n(t) - Wu_n(t)|$ be the residual function. The value $\|r_n(t)\|_\infty$ quantifies the accuracy of the approximate solution. A smaller infinity norm indicates a better approximation. Values of $r_N(t)$ for $N = 2, 3$ are reported in Table 4 and Figure 4.

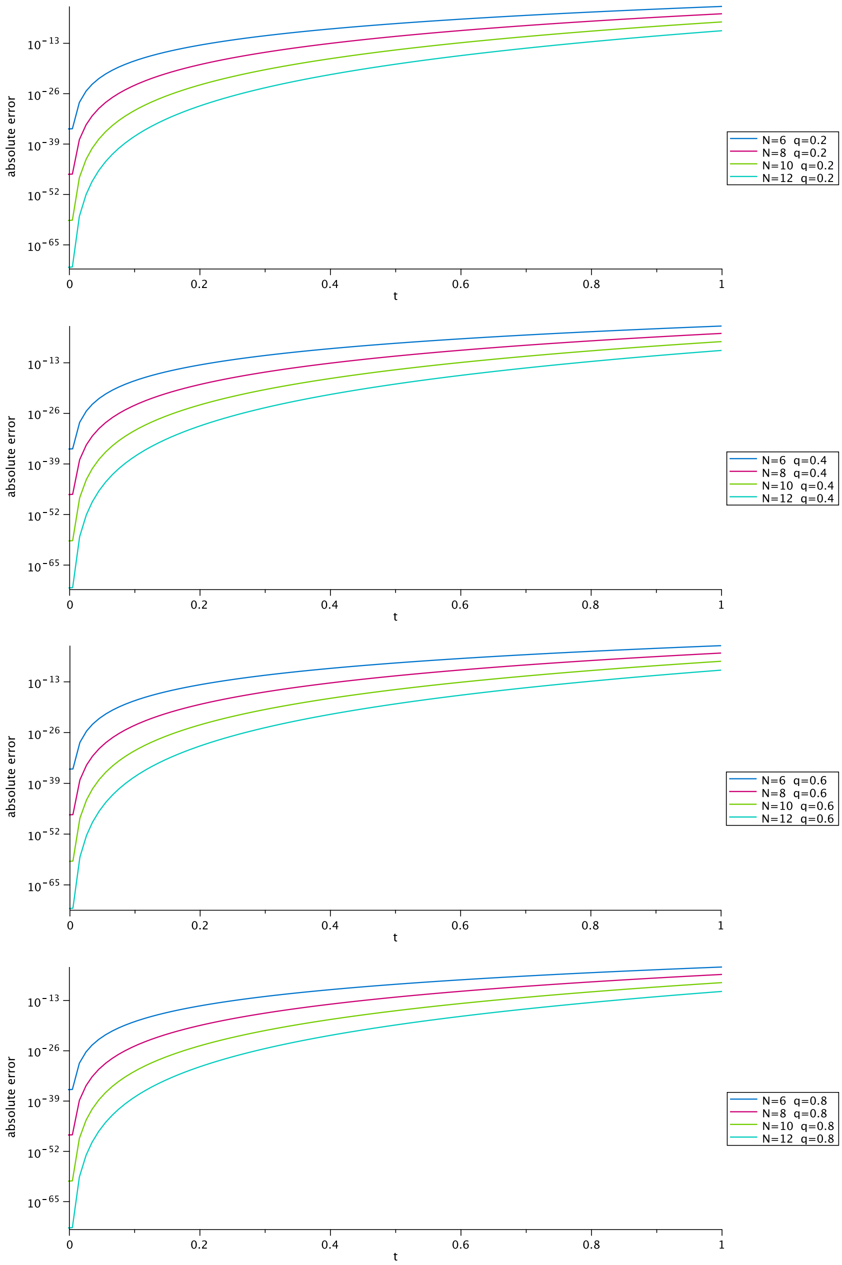

Figure 3: Comparison of absolute errors between $N = 6$, $N = 8$, $N = 10$, and $N = 12$ for $q = 0.2$, $q = 0.4$, $q = 0.6$ and $q = 0.8$, in Example 4.3.

Table 4: Numerical results of Example 4.4.

t	$q = 0.3$		$q = 0.5$		$q = 0.7$	
	$N = 2$	$N = 3$	$N = 2$	$N = 3$	$N = 2$	$N = 3$
0.1	$0.56e-8$	$0.84e-6$	$0.49e-8$	$0.33e-10$	$0.33e-8$	$0.33e-12$
0.2	$0.74e-6$	$0.12e-5$	$0.66e-6$	$0.86e-9$	$0.45e-6$	$0.38e-9$
0.3	$0.13e-4$	$0.20e-5$	$0.12e-4$	$0.34e-7$	$0.79e-5$	$0.23e-7$
0.4	$0.10e-3$	$0.38e-5$	$0.90e-4$	$0.63e-6$	$0.61e-4$	$0.41e-6$
0.5	$0.49e-3$	$0.65e-5$	$0.44e-3$	$0.60e-5$	$0.30e-3$	$0.40e-5$
0.6	$0.18e-2$	$0.40e-4$	$0.16e-2$	$0.38e-4$	$0.11e-2$	$0.25e-4$
0.7	$0.54e-2$	$0.20e-3$	$0.49e-2$	$0.18e-3$	$0.34e-2$	$0.12e-3$
0.8	$0.14e-1$	$0.79e-3$	$0.13e-1$	$0.71e-3$	$0.89e-2$	$0.48e-3$
0.9	$0.33e-1$	$0.26e-2$	$0.29e-1$	$0.23e-2$	$0.21e-1$	$0.16e-2$
1.0	$0.69e-1$	$0.76e-2$	$0.62e-1$	$0.68e-2$	$0.44e-1$	$0.46e-2$
CPU times	$0.402s$	$0.578s$	$0.499s$	$0.703s$	$0.879s$	$0.889s$

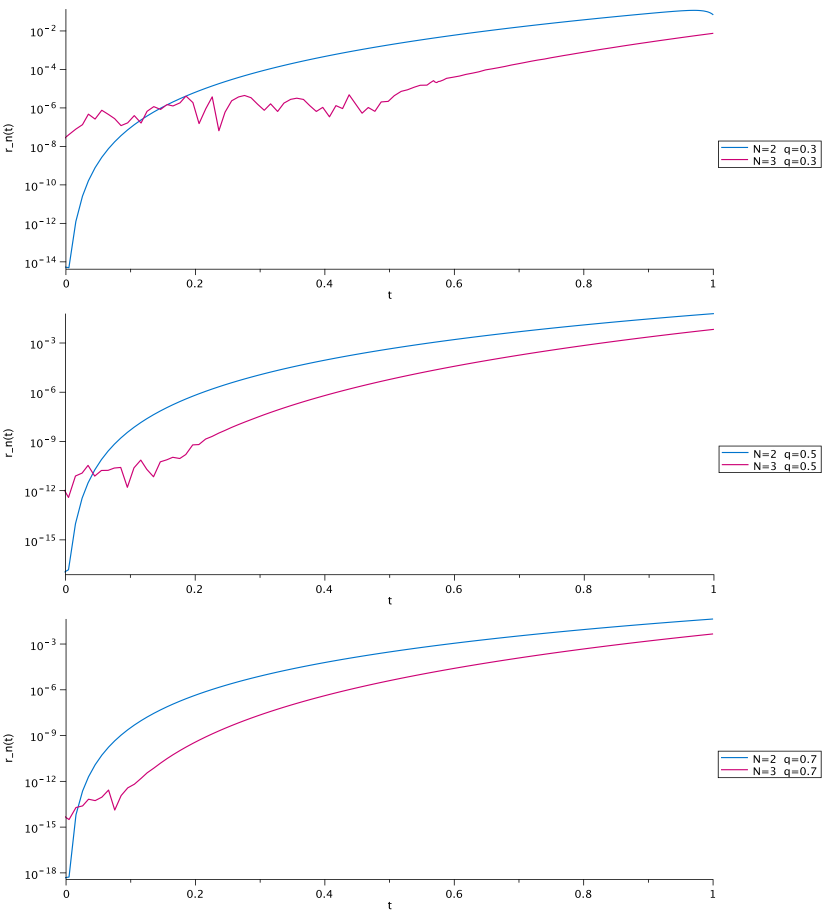

Figure 4: Comparison of residual function between $N = 2$ and $N = 3$ for $q = 0.3$, $q = 0.5$ and $q = 0.7$, in Example 4.4.

5. Conclusion

In this paper, an iterative method for numerically solving a class of delay Volterra integral equations was presented. The existence and uniqueness of the solution and the convergence of the proposed method were also proven. The accuracy of the proposed method was demonstrated through numerical experiments. Numerical results also confirmed the convergence.
It seems that the proposed method of this paper can be used for other types of delay integral equations, such as delay Volterra integro-differential equations and fractional delay Volterra integro-differential equations.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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